# *Constructivist Learning Design: Classroom Tasks for Deeper Learning*

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# *Constructivist Learning Design: Classroom Tasks for Deeper Learning*

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For more information about the Constructivist Learning Design research, please refer to: https://constructivistld.rdc.nie.edu.sg/wordpress/
Alternatively, you can write to us at the National Institute of Education.

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# PREFACE

***Constructivist Learning Design (Classroom Tasks for Deeper Learning)*** is an eBook written for mathematics teachers, key personnel, teacher leaders, mathematics educators, researchers, and practitioners of curriculum. It introduces the Constructivist Learning Design (CLD) as one interpretation of how teaching *through* problem solving can be enacted in Singapore mathematics classrooms to attain the main goal of mathematical problem solving in the Singapore school mathematics curriculum framework.

Many teachers may be familiar with the Direct Instruction approach in the teaching of mathematical concepts where concepts are formally taught at the first instance, often with teacher-directed inputs before students apply the concepts to solve problems. While such teaching approaches have served us well thus far, the future requires our students to have competencies beyond the mastery of content. Teachers need to adopt a range of pedagogical approaches to serve students' needs.

CLD offers a pedagogical approach supporting students to develop mathematical competencies and dispositions akin to the 21st century competencies, preparing them to be future-ready. In CLD, students first perform *inquiries* into a problem with an embedded concept, new to the learner. Then, students' intuitive preliminary constructions of the concept provide the springboards for construction and learning of the concept during the formal, purposeful instruction that follows.

CLD has been trialled and tested in Singapore mathematics classrooms in secondary schools since 2018 by a collaborative research group consisting of representatives from the National Institute of Education and the Singapore Ministry of Education. CLD units have provided rich learning platforms for r*obust constructions of mathematical concepts* and fostered *deep learning of the concepts*. As the first of its kind, this eBook presents the key ideas of CLD, outlines the roots of CLD as part of constructivist teaching and learning, and unpacks the intricacies of CLD enactment using simple language supported by diagrams and actual student responses to CLD units. Rarely done in many other teaching materials for Singapore secondary mathematics, this eBook discusses the targeted concepts to be constructed by students during the CLD units in depth; but in a concise manner so that teachers will understand what a robust construction of the concept entails and what deep learning of the concept refers to.

This eBook is thoughtfully put together to facilitate a clear understanding of CLD for readers. It consists of five main sections.

- The first section is an introduction to CLD, structured in a question-answer format. This section not only draws helpful connections between theory, practice, and curriculum focuses, but also summarises the key ideas of CLD in a succinct, easy-to-understand manner using diagrams and elaborations with reference to examples. Most importantly, the section also outlines *how construction of mathematical concepts and deep learning can be nurtured* in a CLD environment.

- The second section presents an exemplar CLD unit. Here, a student-task is presented and the embedded concept within the task unpacked. Next, we showcase valuable student work from the task: their actual constructions of the concept during the task and the range of mental interpretations of the concept. Lastly but most importantly, we discuss and suggest how teacher facilitation can be done to help students refine their construction of the concept towards a canonical, desired mathematical representation. Proposed teacher prompts for both generic facilitation and targeted support in reaction to different student responses to the problem are also outlined with explanations of the intentions behind the facilitation moves.
- The third to fifth sections discuss three more CLD units with selected concepts from the Mathematics and Additional Mathematics syllabuses.

There will be subsequent updates of the eBook as more CLD units are co-developed and enacted by the research team and school members of the CLD Networked Learning Community at the Academy of Singapore Teachers. We invite teachers in Singapore secondary schools to try the CLD units with their mathematics classrooms and engage in practitioner-reflection on how CLD also addresses elements of *inquiry-based learning*. It is hoped that this eBook also becomes a platform for Singapore teachers to be *metacognitive* about their pedagogical approaches and the student learning which takes place as they attain educational goals in the Singapore school mathematics curriculum framework in preparation for students to be future-ready.

Dawn Ng and Cynthia Seto

# ACKNOWLEDGEMENTS

The Constructivist Learning Design (CLD) research project started in February 2018 with an aim to share lesson design packages in secondary mathematics focusing on deeper learning of mathematical concepts. To date, a total of 8 CLD units were developed and enacted in 10 schools, with more than 40 teachers and 1300 students.

The research was shared in various international and local platforms. These include a keynote address for the 2018 *Mathematics Educators Conference* in Argentina (Lee, 2018b) and invited talks at the *Asian Centre for Mathematics Education* in China (Chua, 2018) and the *Mathematics Teachers' Conference* in Singapore (Lee, 2018a). A presentation on CLD was done during the 2019 *Research Translation Standing Committee* meeting at the Singapore Ministry of Education. The *CLD Seminar* was also held on 16 September 2020 in conjunction with the Centre for Teaching and Learning Excellence (CTLE) @ Yusof Ishak Secondary School with overwhelming responses from over 200 teachers. In addition, CLD was featured in the *2020 MOE Research Forum for Research and Practice* attended by 400 participants, including MOE master teachers and specialists. As part of on-going efforts to share theory-informed practice from CLD, a book chapter on how constructivism was realised in Singapore mathematics classrooms was published in 2020 (Pang, Zhu, & Muhammad, 2020). More publications on CLD units and their impact on classroom practice will be released in the future.

We would like to thank Singapore MOE for supporting the CLD research project; particularly the Curriculum Planning and Development Division (CPDD) and the Academy of Singapore Teachers (AST) for rendering the expertise whenever it is needed. We would also like to thank the members of the CLD Networked Learning Community for their contributions.

# CONTENTS

# SECTION 1:

## INTRODUCING CONSTRUCTIVIST LEARNING DESIGN

Mathematical problem solving has always been the main goal of the Singapore school mathematics curriculum framework since 1992. Generally, there are three approaches to mathematical problem solving (Schroeder and Lester, 1989). They are Teaching *for* Problem Solving, Teaching *about* Problem Solving and Teaching *through* Problem Solving. Constructivist Learning Design (CLD) is one interpretation of how teaching through problem solving can be enacted in Singapore mathematics classrooms.

**What is teaching *through* problem solving?**

The teaching *through* problem solving approach perceives problem solving as an integral part of mathematics learning (see Lambdin, 2003). Aptly put by Schroeder and Lester (1989), in this approach, “problems are valued not only as a purpose for learning mathematics but also as a primary means of doing so” (p. 33).

In other words, problem solving is not viewed as a separate activity done after mathematics concepts and skills are taught. Instead, problems with embedded relevant new mathematics concepts and skills are carefully chosen or designed to provide a platform for learning during lessons. Students learn mathematical content *through* the process of *solving a problem* (see Lester & Charles, 2003).


*Figure 1.* The symbiotic relationship between problem solving and deep understanding of mathematical concepts.

Lester and Cai (2016) also bring out the symbiotic relationship between problem solving and deep understanding of mathematical concepts as depicted in Figure 1. This is because teaching through problem solving brings about an improved performance in problem solving due to students’ deep, conceptual understanding of mathematics. Reciprocally, as students’ competencies in problem solving develop, their understanding of mathematical concepts also deepens (Lambdin, 2003).

**What does a typical lesson with *teaching through problem solving* look like?**

In this approach, students typically embark on a quest to find solutions to a problem before a new mathematical topic or concept is formally taught. Drawing upon their existing repertoire of knowledge and strategies, students can work on a problem individually before moving on to group discussions. In many cases, students can construct different mathematical arguments based on various interpretations of the problem and subsequently justify their mathematical conclusions.

Working in groups allow for sharing of solutions, co-construction of ideas arising from negotiation of meaning, and establishing consensus in understanding of the problem and its context (see Lester & Cai, 2016). Fruitful group discussions purposefully facilitated by the teacher can result in robust construction of mathematical concepts, thus strengthening existing conceptual understanding of mathematical knowledge. This is because during productive group discussions, students engage each other to clarify and compare ideas, analyse approaches critically, and develop increasingly connected and complex system of knowledge (see Cai, 2003). Hence, when the teaching through problem solving approach is well-executed, students construct, refine, and connect mathematical knowledge and ideas as they work towards more sophisticated solution pathways.

**What is the role of the teacher during teaching through problem solving?**

During students' attempts at problem solving, the teacher takes on the role of a facilitator, observing students' strategies and asking questions at appropriate junctures to encourage students to reflect on their mathematical thinking, strategy choices and decisions. Active listening is key when facilitating students' problem solving (see Ng et al., 2015). This is because active listening sets aside time for the teacher to understand student's mathematical (and contextual) interpretations of the problem which will form the basis of analysing the appropriateness of prior knowledge drawn upon, and subsequent mathematical solution pathways. Teacher demonstrations of active listening together with purposeful questioning for critical comparisons of mathematical ideas allow for students to build upon each other's mathematical thinking. In cases where the problem becomes a platform for construction of new mathematical ideas (see English, 2010), teachers can also elicit students' intuitive mental constructions of new mathematical concepts which may emerge from the process of problem solving. Students can be encouraged to communicate or show representations of their intuitive constructions of new mathematical concepts. Lim (2020) provides a comprehensive discussion about the role of the teacher in planning for and in implementing teaching through problem solving against the backdrop of theory-informed practices.

**What is the Constructivist Learning Design and Why?**

Constructivist Learning Design (CLD) presents one interpretation of how teaching through problem solving can be realized in Singapore mathematics classrooms. CLD draws upon three effective learning designs which were already tested in Asian classrooms: Productive Failure (see Kapur, 2008, 2010), the Open-Ended Approach (Becker & Shimada, 1997), and the Post-Tea House Teaching Approach (Tan, 2013). Central to these learning designs are two related phases in structuring a lesson or a series of lessons. The two phases can be developmental, with the second phase drawing upon student responses from first towards deeper learning. The first phase allows for activation of prior knowledge and experiences as students engage in carefully designed problems, tasks, or activities aimed at eliciting a diversity of knowledge structures during the construction of concept, and ***Representations and Solution Methods*** (RSMs; Kapur & Bielaczyc, 2012). A RSM can be a combination of diagrams and mathematical argument. The second phase serves as a purposeful platform for focused discussions where students are encouraged to analyse and compare the variety of solution pathways or RSMs created earlier, often with teacher facilitation.

Likewise, CLD also proposes a framework which outlines two essential and *sequential phases* in structuring *consecutive lessons*: (a) Problem Solving Phase and (b) Instruction Phase. Collectively, the phases aim to foster a robust construction of the mathematical concept (i.e., the ***canonical solution***) in conjunction with deep learning about the concept. A strength of the CLD is the affordances of CLD tasks by virtue of their design. Evident in the CLD unit "Gradient of Linear Graph" (see Section 2) are carefully selected slopes with deliberate measurements provided. Together with the real-world context, these provide *stimuli* for students to generate a rich variety of RSMs, drawing upon different prior knowledge. The open-ended nature of the CLD task also allows for a wide range of RSMs generated. Another strength of the CLD is targeted teacher consolidation which promotes mathematical thinking and reasoning as students work towards the canonical solution from a critical analysis of different RSMs. Compared to Direct Instruction where teacher instruction of the new concept is usually conducted before students apply what is learnt at problem solving, CLD uses the problem task as a platform for construction of the concept first. Figure 2 shows the framework with its phases and the actions students engage in during these phases.

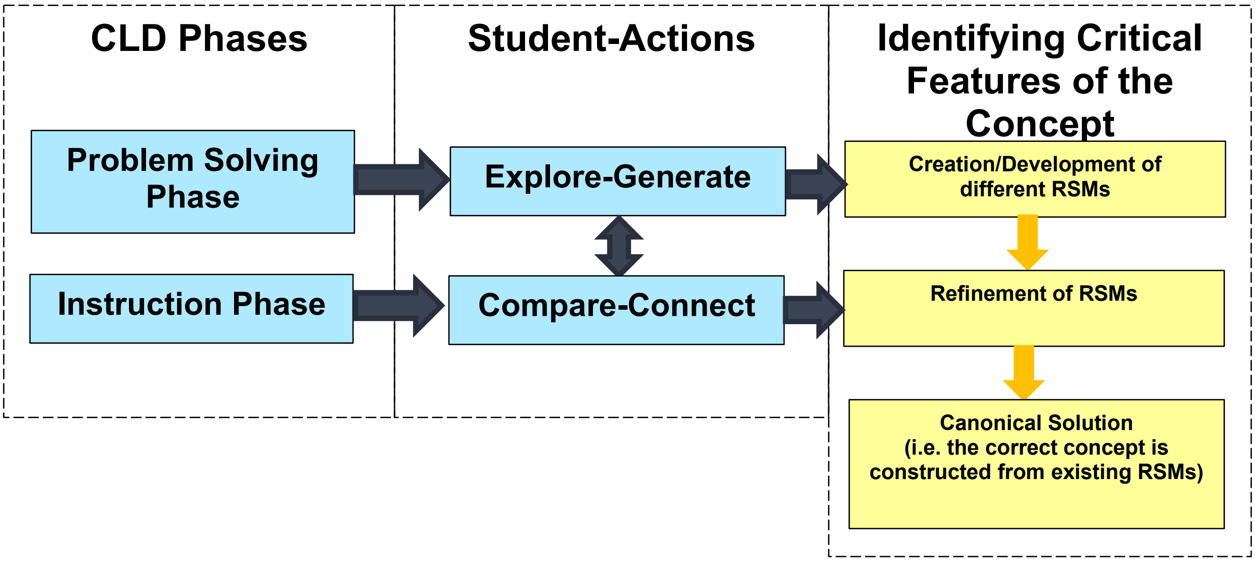



*Figure 2.* CLD phases and student-actions during these phases.

We will unpack the framework in Figure 2 starting from ***critical features*** of a concept because this is the first cornerstone of CLD. The critical features of a concept refer to the crucial elements or building blocks which define the concept. For instance, "gradient" as a concept refers to steepness or inclination. Mathematically, the gradient of a straight line is how steep the straight line is. The concept of gradient will involve crucial elements of "measurement", "steepness", "variables", "direction" and "magnitude". A robust construction of the concept of gradient will involve assessment of varied cases (i.e., different types of slopes, different directions, different elevations, or inclinations) to determine key variables for measurement of steepness. Likely variables for consideration may be presented in the carefully designed task or problem as part of the stimuli. In order for generalisation to all cases, there is a need to decide whether a single variable for measurement for gradient (e.g., height of slope) is sufficient or a minimum number of variables have to be considered in a multi-variable measurement. If more than one variable is needed, also for precision in measurement, then how are these variables related so that different cases of gradient can be accounted for? Different RSMs involving various critical features of the gradient concept or combinations of these features can be constructed. The canonical solution will be the correct or desired construction of the gradient concept, factoring in generalisability for all cases and precision of measurement (see Section 2 for an exemplar CLD task on “Gradient of Linear Graphs” with discussions of the critical features, RSMs, and canonical solution).

Next, we will elaborate the next cornerstone of CLD, its two sequential phases. The ***problem solving phase*** starts with students solving a problem with the embedded mathematical concept to be constructed. Here, students are encouraged to develop various RSMs to address the question. RSMs marks the preliminary constructions of the concept. RSMs can be intuitive constructions at first. More sophisticated, formal mathematical RSMs can be developed as students move from individual attempts to co-constructions during group discussions facilitated by the teacher. To prepare for the instruction phase which typically occurs in the next lesson, students’ RSMs are collected by the teacher for analysis. During analysis, the RSMs are classified and ranked according to their affordances and constraints in addressing the critical features of the concept. For example, in “Gradient of Linear Graphs”, a RSM which allows for multiple cases of slopes with varying magnitudes and directions to be represented will be ranked higher than one which only addresses slopes with “positive gradients”.

During the ***instruction phase***, the teacher consolidates the RSMs constructed in the previous phase by engaging students in a series of discussions (class or group). The discussions elicit the critical features of the concept articulated in selected RSMs and highlights affordances and constraints of these RSMs in representing the cases. Deliberate and directed comparisons amongst the RSMs is crucial in helping students notice the critical features of the concept. Subsequent revisions of highly ranked RSMs, collectively, in negotiation under the tutelage of the teacher, will result in a full, comprehensive construction of the desired concept. Deep learning about the concept would have also resulted from the process. The instruction phase of CLD may continue with the application of the concept, an extension of it for special cases (e.g., gradient of an horizontal line), and the development of related skills (e.g., computation of ratio of height and length for gradient).

Finally, during the two phases of CLD, students also engage in four main actions: ***explore-generate***, ***compare-connect***. In Figure 2, the arrows moving out of the phases show the likely occurrences of various student-actions. Explore-generate are actions students undertake when they experiment with scenarios, cases, and examples of events presented in the problem. A range of mathematical arguments, computations, and representations are thus generated or constructed as RSMs. Compare-connect are actions which take place naturally during discussions because selected RSMs are compared and analysed for their affordances and constraints in encompassing the critical features of the concept. Students may draw connections between cases and across sets of prior knowledge as they look for similarities and differences amongst the RSMs. The double arrow between explore-generate and compare-connect signals the likely cyclical nature of these processes in each phase as part of development and refinement of RSMs.

In summary, CLD addresses one of the beliefs articulated in the Singapore Curriculum Philosophy (MOE, 2018): “Learning takes place individually and collaboratively, as children construct and co-construct meaning from knowledge and experiences. We guide learners to activate prior knowledge, and assimilate and accommodate new knowledge through exploration, and interaction with others. This allows them to build a strong foundation of knowledge by connecting new ideas and experiences with what they already know, thus facilitating the understanding of concepts and the application of what they have learnt to different contexts."

**How does the teacher facilitate Constructivist Learning Design?**

Teacher facilitation is important during both CLD phases. The main focuses of teacher facilitation during the problem solving phase and the instruction phase are ***experimentation*** and ***reflection*** respectively (see Figure 3). Before students attempt the CLD problem, the teacher may like to unpack the context and the goal of the problem.

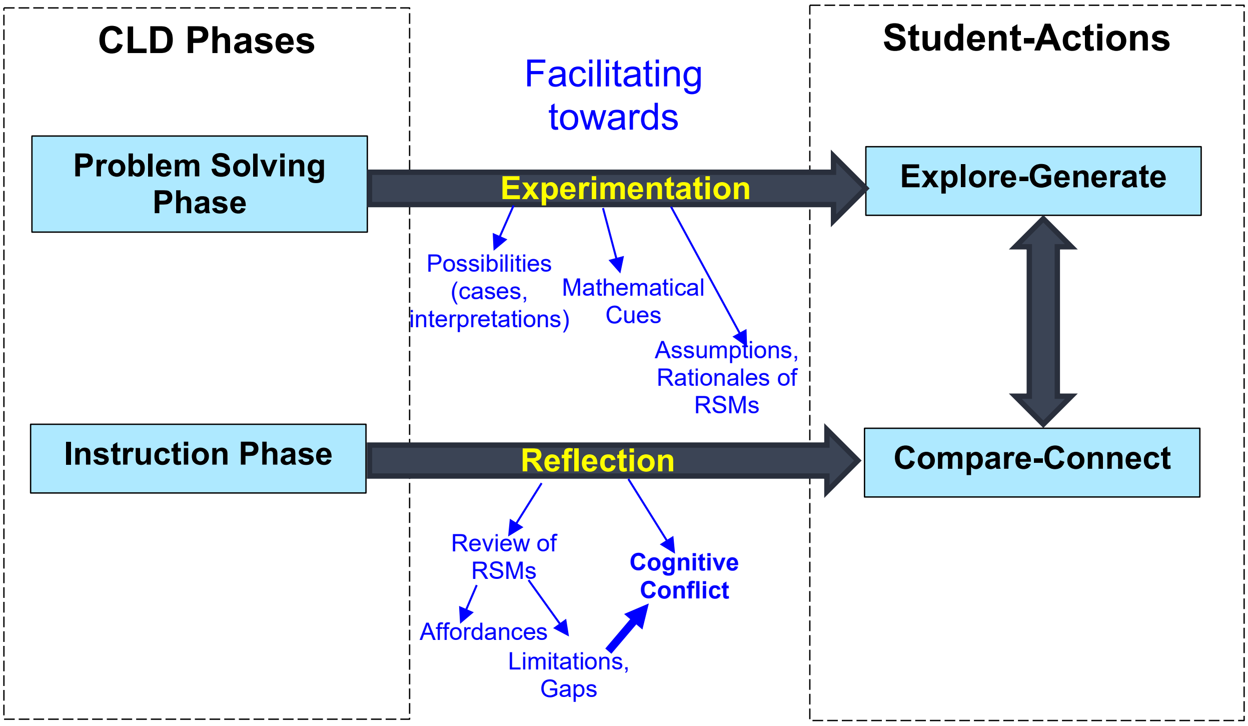



*Figure 3.* Teacher facilitation during CLD.

In facilitating towards experimentation, the teacher prompts students to explore different possibilities of interpreting the problem, its context, and cases. Other cases or scenarios related to the problem may surface from the discussions. Students can be encouraged to experiment with the scenarios, making mathematical calculations with different data so as to generate different RSMs. Key to teacher facilitation for experimentation is an open, non-judgemental way of seeking possibilities. Hence, prompting questions are typically targeted at eliciting cases for consideration, mathematical cues from the problem context (e.g., variables, dimensions for consideration), and any assumptions made. Once a RSM is constructed, teacher facilitation may be geared towards the development of more RSMs using other interpretations, data, and computations. Teacher questioning may also involve the rationale of development of RSMs and perhaps the critical features of the concept brought out by the RSMs.

In facilitating towards reflection, the teacher prompts for a review of the critical features in selected RSMs, and for comparisons between RSMs in terms of affordances and constraints in addressing all cases. Teacher can also prompt students to draw connections between mathematical ideas as they re-examine their thinking behind the RSMs. The main goal of reflection is to work towards the canonical solution or the desired construction of the concept with deep learning. Refinement of selected RSMs is necessary during teacher facilitation here. Key to teacher facilitation for reflection is provoke ***disequilibrium*** or ***cognitive conflict*** (see Cohen & Kim, 1999; Ojose, 2008; Olivier, 1989) as a RSM is examined particularly for gaps, limitations in representing cases, or contradiction in logic. For example, Solution B (one of the RSMs in the Gradient of Linear Graphs task; see Section 2) might have represented an intuitive way of gradient measurement using changes in height only. Here, teacher facilitation may highlight that the slopes with the "same change in height" look visibly different in terms of steepness on the given diagram. This brings to question the discrepancy between the beliefs behind Solution B and the observed data, prompting a refinement of the RSM because of a cognitive conflict.

In the next few sections, we will present four CLD units, complete with the critical features of the targeted concepts, analyses of RSMs ranges, identification of limitations or gaps in some of the RSMs, proposed teacher facilitation prompts for generic and RSM-specific discussions, and suggested facilitation moves during the lessons.

# SECTION 2:

## AN EXEMPLAR - GRADIENT OF LINEAR GRAPHS (SECONDARY ONE)

Section 2.1 below shows the problem task designed for targeted concept of "gradient of linear graphs" at Secondary One level. Students can work together as a group to explore-generate RSMs in response to the questions in this task.

## Section 2.1 Problem Task (Students' Version)

### The Mountain Trail

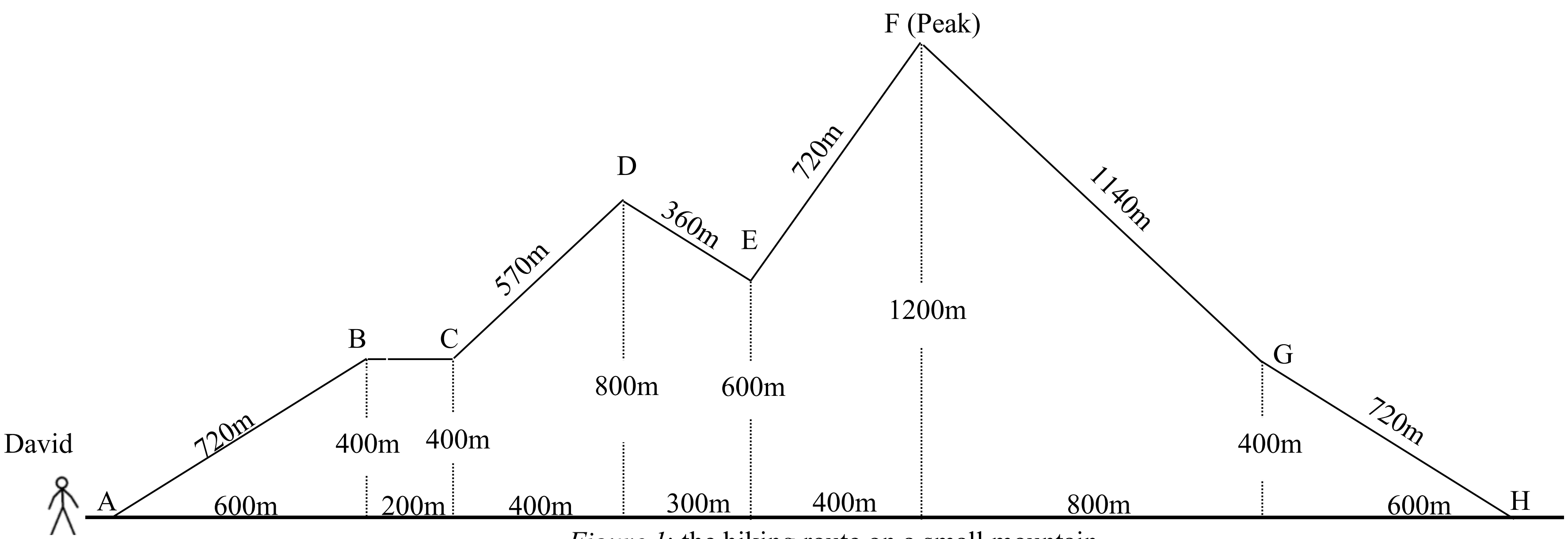



*Figure 1*: the hiking route on a small mountain

David enjoys hiking on a small mountain with a peak of 1200m. Figure 1 shows a sketch of David's trail, which has 7 sections. He starts at point A, hikes through points B to G, and ends at point H. The vertical heights, horizontal distances, and slope lengths (rounded to the nearest 10m) are also indicated in the figure.

Although he is a seasoned hiker, David notices that some sections are steeper compared to others**.** He seeks your group's help to describe **both the steepness and direction of the mountain's slopes mathematically**. Here is what you must do:

(1) Assuming that all other things are equal, please use the information provided in Figure 1 and come up with as many ways as possible to rank the various sections of the trail both in terms of their steepness and the direction.

(2) For each way of ranking in (1), justify your ranking **mathematically** to describe both the steepness and direction of different sections.

All the best, and remember, don't give up until you have come up with as many methods as possible!

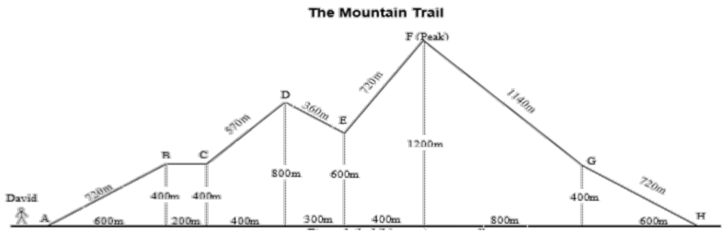
The design of the CLD problem encapsulates the critical features of the targeted concept. Section 2.2 below elucidates these features for the “gradient of linear graphs”.

## Section 2.2 Unpacking the Critical Features of Targeted Concept

| | |
|---|---|
| The **Concept** | **Gradient** is the measurement of steepness and direction of a line.<br>(a) In the secondary 1 syllabus, the concept is formulated as<br>$\frac{\text{change in magnitude and direction of variable 1}}{\text{change in magnitude and direction of variable 2}}$ or $\frac{\text{Vertical change}}{\text{Horizontal change}}$ or $\frac{\text{Rise}}{\text{Run}}$.<br>(b) In the secondary 3 syllabus, the gradient of a linear graph is formulated as $\frac{y_1 - y_2}{x_1 - x_2}$, where $(x_1, y_1)$ and $(x_1, y_2)$ are two points on the line. |
| **Critical Features** of the concept | There are four critical features to the concept of gradient:<br>(a) quantification/***magnitude*** of steepness;<br>(b) quantification of ***direction***;<br>(c) 2 dimensions/***variables*** used for measurement - the vertical height ($y$) and horizontal distance ($x$); and<br>(d) the 2 variables are in a ***ratio*** - change of vertical height ($y$) with respect to horizontal distance ($x$) is measured<br>These critical features could be used to classify students’ RSMs, and guide teacher facilitation moves during the problem solving phase. |
| **Big Ideas** underlying the concept | The concept of gradient is aligned to:<br>(a) ***Measures*.**<br>Gradient as a measure that quantifies the slope of the line and indicates both its steepness and direction.<br>(b) ***Invariance***.<br>Gradient of a line is invariant between any two points on the line, i.e., the ratio $\frac{y_1 - y_2}{x_1 - x_2}$ is constant regardless of the lengths and the position of the segments taken on slopes. For example, in the Mountain Trail problem above, DE and GH are two slopes with the same gradient, but they have different lengths and are at different positions of the trail. |

## Section 2.3 Organizing the RSMs

The RSMs produced for the Mountain Trail problem (on the right) during the problem solving phase can be classified according to whether and how much they address the critical features of the targeted concept. We present 11 RSMs (labelled Solutions A to K) classified into 5 solutions types according to the four critical features listed above. The Mountain Trail problem is reproduced on each page as we discuss each solution type to facilitate understanding of the RSMs.




## TYPE 1 SOLUTIONS: Only *One Dimension/Variable* Considered

**Solution A**
***(uses horizontal distance)***

| | FG, | AB, | GH, | CD, | EF, | DE, | BC |
|---|---|---|---|---|---|---|---|
| | (800m), | (600m), | (600 m), | (400 m), | (400 m), | (300 m), | (200m) |
| Rank: | 1, | 2, | 2, | 4, | 4, | 6, | 7 |

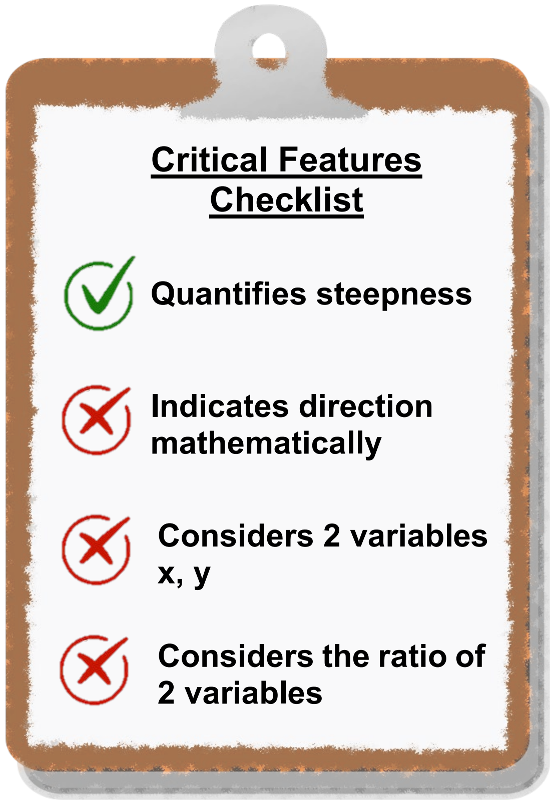
Explanation: FG has the longest horizontal distance so it is the steepest.

**Solution B**
***(uses change in vertical height)***

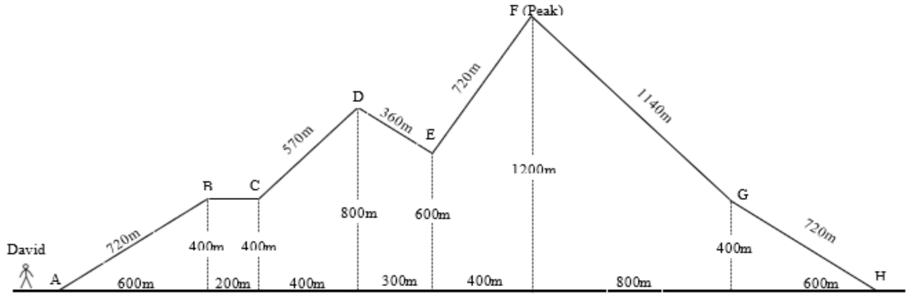
A-B -> 400 m upwards [NE] No. 3
B-C -> 0 m [E] No. 7
C-D -> 400 m upwards [NE] No. 3
D-E-> 200 m downwards [SE] No. 6
E-F-> 600 m upwards [NE] No. 2
F-G-> 800 m downwards [NE] No. 1
G-H-> 400 m downwards [NE] No. 3

**Solution C**
***(uses slope length)***

1st: F to G (1140m)
2nd: E to F (720m)
2nd: G to H (720m)
2nd: A to B (720m)
5th: C to D (570m)
6th: D to E (360m)
7th: B to C (200m)



**The Mountain Trail**




## TYPE 2 SOLUTIONS: Combination of 2 Dimensions/Variables Considered

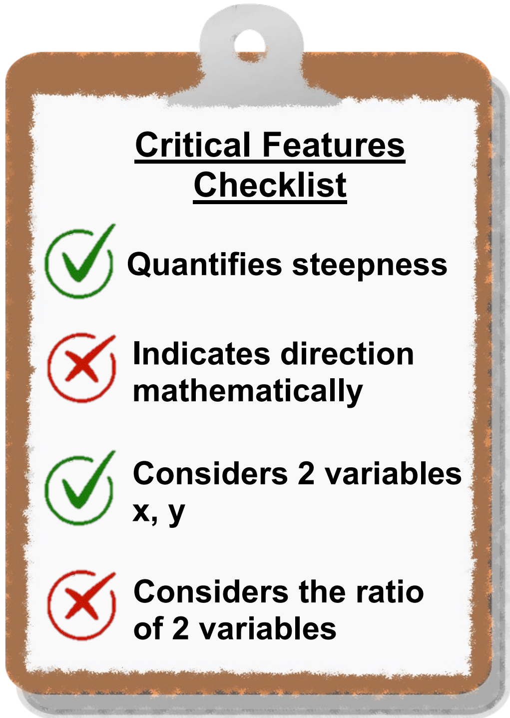
### Solution D
***(area under each slope)***

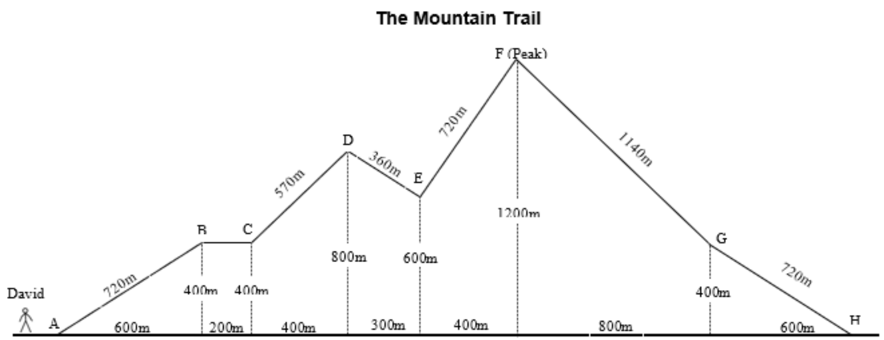
| | |
|---|---|
| AB | ½ x 600 x 400 = 120,000 |
| BC | 200 x 400 = 80,000 |
| CD | ½ x 400 x 400 = 80,000<br>400x 400 = 160,000<br>80,000 + 160,000 = 240,000 |
| DE | ½ x 300 x 200 = 30,000<br>300x 600 =180,000<br>30,000 + 180,000 = 210,000 |
| EF | ½ x 400 x 600 = 120,000<br>400x 600 = 240,000<br>240,000+ 120,000 = 360,000 |
| FG | ½ x 800 x 800 = 320,000<br>800x 400 = 320,000<br>320,000 + 320,000 =640,000 |
| GH | ½ x 600 x 400 = 120,000 |

Most steep = largest value
FG, EF, CD, DE, AB, GH, Bc

### Solution E
***(slope length minus horizontal distance)***

| Distance | | Steepness | |
|---|---|---|---|
| A to B (N-E)<br>720m | 720-600<br>=120m | 1st | F to G<br>340m |
| B to C (E)<br>200m | 200-200<br>=0m | 2nd | E to F<br>320m |
| C to D (N-E)<br>570m | 570-400<br>=170m | 3rd | C to D<br>170m |
| D to E (E-S)<br>360m | 360-300<br>=60m | 4th | A to B<br>120m |
| E to F (N-E)<br>720m | 720-400<br>=320m | 5th | G to H<br>120m |
| F to G (E-S)<br>1140m | 1140-800<br>=340m | 6th | D to E<br>60m |
| G to H (E-S)<br>720m | 720-600<br>=120m | 7th | B to C<br>0m |





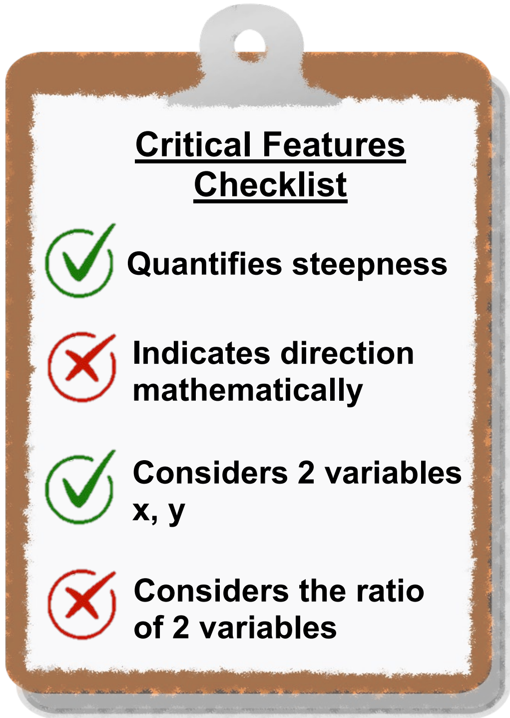

**TYPE 3 SOLUTIONS:** Ratio of 2 Dimensions/Variables Considered

## Solution F
***(slope length/ slope length minus horizontal distance)***

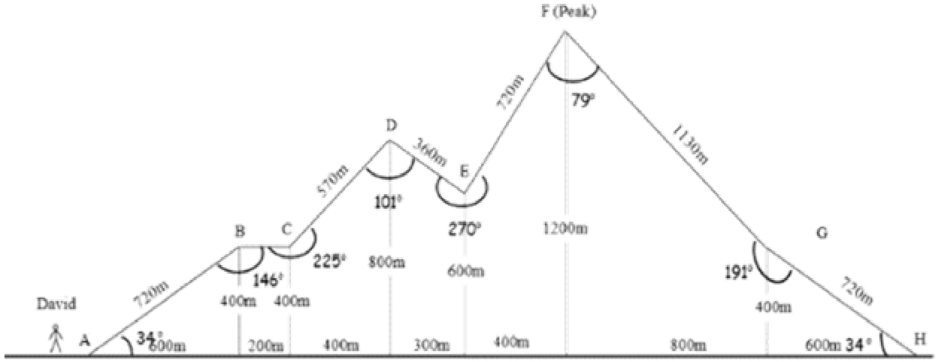
| | | Rank |
|---|---|---|
| AB | $\frac{720}{720-600} = 6.00$ | 3rd |
| BC | $\frac{200}{200-200}$ = err | ? |
| CD | $\frac{570}{570-400} = 3.35$ | 2nd |
| DE | $\frac{360}{360-300} = 6.00$ | 3rd |
| EF | $\frac{720}{720-400} = 2.25$ | 1st |
| FG | $\frac{1140}{1140-800} = 3.35$ | 2nd |
| GH | $\frac{720}{720-600} = 6.00$ | 3rd |

## Solution G
***(slope length/ horizontal distance)***

| | | |
|---|---|---|
| A-B | → | $\frac{720}{600}$ = 1.20 (3rd) |
| B-C | → | $\frac{200}{200}$ = 1.00 (4th) |
| C-D | → | $\frac{570}{400}$ = 1.425 (2nd) |
| D-E | → | $\frac{360}{300}$ = 1.20 (3rd) |
| E-F | → | $\frac{720}{400}$ = 1.80 (1st) |
| F-G | → | $\frac{1140}{800}$ = 1.425 (2nd) |
| G-H | → | $\frac{720}{600}$ = 1.20 (3rd) |



## Solution H
***(angles measured with changing reference points)***

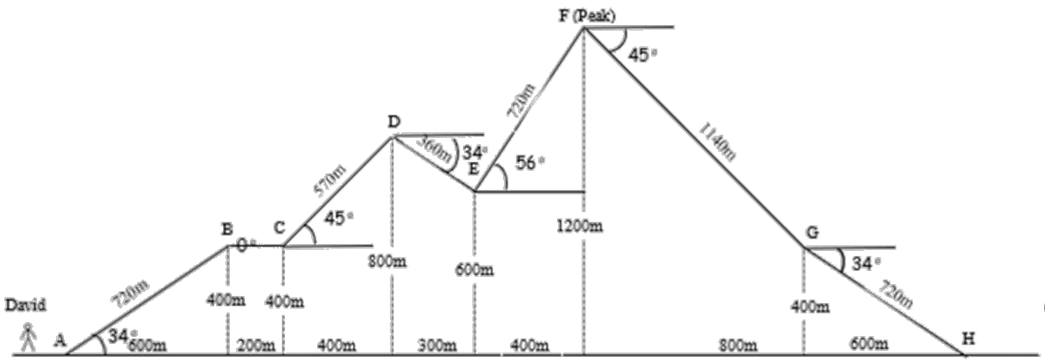



| Trail Point | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| Angles | 34° | 146° | 225° | 101° | 270° | 79° | 191° |

The smaller the angle, the steeper the slope.

So from steepest to the least steep: A=H, F, D, B, G,

## Solution I
***(angles measured against a fixed horizontal reference)***




| Trail Section | AB | BC | CD | DE | EF | FG | GH |
|---|---|---|---|---|---|---|---|
| Angles | 34° up | Nil | 45° up | 34° down | 56° up | 45° down | 34° down |

Based on angles and direction, steepest to the least steep

(the bigger the angle, the steeper the slope)

Falling – DE= GH, FG;

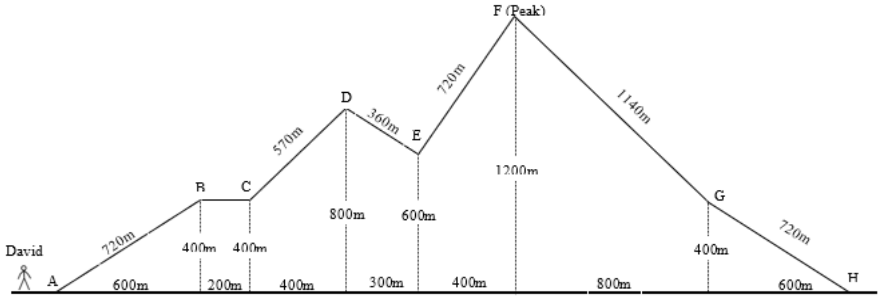
Rising – EF, CD, AB ;

Constant - BC

### Critical Features Checklist

- ✓ **Quantifies steepness**
- ✗ **Indicates direction mathematically**
- ✓ **Considers 2 variables x, y**
- ✓ **Considers the ratio of 2 variables**

The Mountain Trail




## **TYPE 5 SOLUTIONS:** Canonical RSMs – Targeted Concept Constructed

### Solution J

| Trail Section | $\frac{\text{Change in vertical height}}{\text{Slope length}}$ | Ranking |
|---|---|---|
| AB | $\frac{400-0}{720} \approx 0.56$ | 4 |
| B C | $\frac{400-400}{200} = 0.00$ | 7 |
| C D | $\frac{800-400}{570} = 0.70$ | 2 |
| D E | $\frac{600-800}{360} \approx -0.56$ | 4 |
| E F | $\frac{1200-600}{720} = 0.83$ | 1 |
| F G | $\frac{400-1200}{1140} = 0.70$ | 2 |
| G H | $\frac{0-400}{720} \approx -0.56$ | 4 |

### Solution K

| Trail Section | $\frac{\text{Horizontal distance}}{\text{change in vertical height}}$ | Ranking |
|---|---|---|
| AB | $\frac{600}{400-0} \approx 1.50$ | 4 |
| B C | $\frac{200}{400-400} = \infty$ | 7 |
| C D | $\frac{400}{800-400} = 1.00$ | 2 |
| D E | $\frac{300}{600-800} \approx -1.50$ | 4 |
| E F | $\frac{400}{1200-600} = 0.67$ | 1 |
| F G | $\frac{800}{400-1200} = -1.00$ | 2 |
| G H | $\frac{600}{0-400} \approx -1.50$ | 4 |

### Critical Features Checklist

- ✓ **Quantifies steepness**
- ✓ **Indicates direction mathematically**
- ✓ **Considers 2 variables x, y**
- ✓ **Considers the ratio of 2 variables**

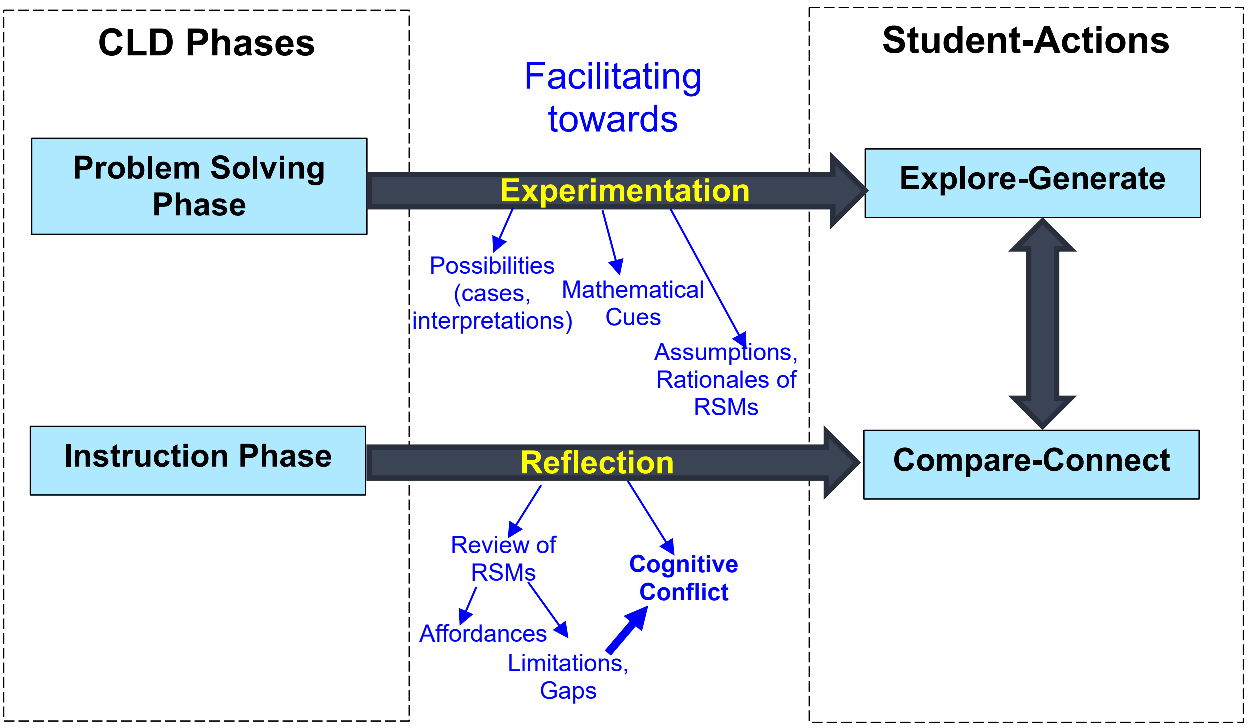
## Section 2.4 Suggested Teacher Facilitation

This section summarises and exemplifies the proposed teacher facilitation moves during the problem solving and instruction phases. Section 2.4.1 revisits generic but important facilitation moves for CLD problems that were explained in detail in Section 1. Section 2.4.2 suggests possible facilitation moves for specific solution types 1 to 5. Section 2.4.3 proposes a sample lesson outline for *consolidating* student learning during the instruction phase.

### Section 2.4.1 Generic Teacher Facilitation Moves

During the problem solving phase, teacher facilitation is mainly focused on ***experimentation*** where students explore possibilities of problem interpretation (activation of prior knowledge encouraged) and generate different RSMs. Here, teacher questioning may involve discussions of possible cases represented by each RSM, use of mathematical cues from the problem context to generate RSMs, and understanding the assumption and rationale behind an RSM.

Facilitation during the instruction phase targets at ***reflection***. Here, teacher prompts students to critically analyse each solution type (classified and deliberately sequenced ahead of time by the teacher) for its affordances in terms of addressing the critical features of the concept. Students are encouraged to compare different RSMs and draw connections between ideas. Next, the teacher activates student's thinking towards the construction of the targeted concept by highlighting instances of cognitive conflicts created by gaps or limitations in some RSMs. We include Figure 3 from Section 1 here again for a summary of facilitation focuses.



## Section 2.4.2 Solution-Specific Facilitation Moves

We now move on to solution-specific teacher facilitation moves. Such moves can be adopted whenever RSMs are developed/reviewed in either the problem solving or instruction phase. Suggested teacher facilitation questions are provided at times to provoke ***cognitive conflict*** for desired learning to occur. Here, students' attention is drawn to affordances and limitations of the RSM, moving them closer to the targeted concept.

## FACILITATING AND CONSOLIDATING TYPE 1 SOLUTIONS

### Example Solution(s)

**Solution A**
***(uses horizontal distance)***

| | FG, | AB, | GH, | CD, | EF, | DE, | BC |
|---|---|---|---|---|---|---|---|
| | (800m), | (600m), | (600 m), | (400 m), | (400 m), | (300 m), | (200m) |
| Rank: | 1, | 2, | 2, | 4, | 4, | 6, | 7 |

Explanation: FG has the longest horizontal distance so it is the steepest.

**Solution B**
***(uses change in vertical height)***

A-B -> 400 m upwards [NE] No. 3
B-C -> 0 m [E] No. 7
C-D -> 400 m upwards [NE] No. 3
D-E-> 200 m downwards [SE] No. 6
E-F-> 600 m upwards [NE] No. 2
F-G-> 800 m downwards [NE] No. 1
G-H-> 400 m downwards [NE] No. 3

**Solution C**
***(uses slope length)***

1st: F to G (1140m)
2nd: E to F (720m)
2nd: G to H (720m)
2nd: A to B (720m)
5th: C to D (570m)
6th: D to E (360m)
7th: B to C (200m)

### Facilitation Questions

**For Solution A:** The two slopes CD and EF have the same horizontal distance (400m). Can we say the two slopes are equally steep? Use a ruler and set square to check the diagram. The two slopes are not parallel in the diagram. In other words, the two slopes are not equally steep.

**For Solution B**: AB and CD have the same changes in vertical heights (400 m). Are they equally steep? However, visually on the graph, the two slopes are not equally steep.

**For Solution C**: Can *slope lengths* be used to measure steepness of slopes? Since AB and EF have the same slope lengths, are they equally steep? However, visually on the diagram, the two slopes are not equally steep.

| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | • Whilst students are generating RSMs, teachers addressing groups who are stuck or those who need to move beyond one solution type can consider using the facilitation questions here.<br>• The aim is to get students to examine if the proposed gradient measure concurs with visual differences in steepness and direction. |
| **"Instruction" phase** | Type 1 solutions consider **pertinent variables** to **quantify** steepness and direction. However, all 3 solutions here have only considered one variable. This is not sufficient to measure and compare the steepness of different slopes. Teachers can use of the facilitation questions here to show the gaps/limitations of each solution to distil the remaining critical features. |

## Example Solution(s)

**Solution D**
***(area under each slope)***

| | |
|---|---|
| AB | ½ x 600 x 400 = 120,000 |
| BC | 200 x 400 = 80,000 |
| CD | ½ x 400 x 400 = 80,000<br>400x 400 = 160,000<br>80,000 + 160,000 = 240,000 |
| DE | ½ x 300 x 200 = 30,000<br>300x 600 =180,000<br>30,000 + 180,000 = 210,000 |
| EF | ½ x 400 x 600 = 120,000<br>400x 600 = 240,000<br>240,000+ 120,000 = 360,000 |
| FG | ½ x 800 x 800 = 320,000<br>800x 400 = 320,000<br>320,000 + 320,000 =640,000 |
| GH | ½ x 600 x 400 = 120,000 |

Most steep = largest value
FG, EF, CD, DE, AB, GH, Bc

**Solution E**
***(slope length minus horizontal distance)***

| Distance | | Steepness | |
|---|---|---|---|
| A to B (N-E) 720m | 720-600 =120m | 1st | F to G 340m |
| B to C (E) 200m | 200-200 =0m | 2nd | E to F 320m |
| C to D (N-E) 570m | 570-400 =170m | 3rd | C to D 170m |
| D to E (E-S) 360m | 360-300 =60m | 4th | A to B 120m |
| E to F (N-E) 720m | 720-400 =320m | 5th | G to H 120m |
| F to G (E-S) 1140m | 1140-800 =340m | 6th | D to E 60m |
| G to H (E-S) 720m | 720-600 =120m | 7th | B to C 0m |

## Facilitation Questions

**For Solution D:** Is it always true that the bigger the area of the triangular space beneath the slope, the steeper the slope? Consider the areas below slopes **DE** and **GH**. They have the same steepness (they are parallel), but DE's area (210,000 $m^2$) is more than that of GH (120,000 $m^2$). Is DE steeper than GH?

**For Solution E**: This solution takes the difference between slope lengths and horizontal distances for each slope. If this difference is the same for two slopes, would the slopes have the same steepness then?
Consider slopes DE and GH. The difference in slope lengths and horizontal distance are 60 m and 120m respectively. However, is slope DE less steep than GH? (check the diagram with your rulers!)

| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | • Whilst students are generating RSMs, teachers addressing groups who are stuck or those who need to move beyond one solution type can consider using the facilitation questions here.<br>• The aim is to get students to examine if the proposed gradient measure concurs with visual differences in steepness and direction. |
| **"Instruction" phase** | Type 2 solutions consider a combination of **two dimensions/variables** when addressing steepness. However, the solutions are unable to show how one dimension *varies* or *changes* with another for slopes that have different steepness. Most also do not take into account the direction of the slope. Teachers can use of the facilitation questions here to show the gaps/limitations of each solution to distil the remaining critical features. |

## Example Solution(s)

**Solution G**
***(slope length/ horizontal distance)***

| | | |
|---|---|---|
| A-B | → | $\frac{720}{600}$ = 1.20 (3rd) |
| B-C | → | $\frac{200}{200}$ = 1.00 (4th) |
| C-D | → | $\frac{570}{400}$ = 1.425 (2nd) |
| D-E | → | $\frac{360}{300}$ = 1.20 (3rd) |
| E-F | → | $\frac{720}{400}$ = 1.80 (1st) |
| F-G | → | $\frac{1140}{800}$ =1.425 (2nd) |
| G-H | → | $\frac{720}{600}$ = 1.20 (3rd) |

**Solution F**
***(slope length/ slope length minus horizontal distance)***

| | | Rank |
|---|---|---|
| AB | $\frac{720}{720-600}$ = 6.00 | 3rd |
| BC | $\frac{200}{200-200}$ = err | ? |
| CD | $\frac{570}{570-400}$ = 3.35 | 2nd |
| DE | $\frac{360}{360-300}$ = 6.00 | 3rd |
| EF | $\frac{720}{720-400}$ = 2.25 | 1st |
| FG | $\frac{1140}{1140-800}$ =3.35 | 2nd |
| GH | $\frac{720}{720-600}$ = 6.00 | 3rd |

## Facilitation Questions

**For Solution F:** If the vertical height is the denominator, how can the solution be improved, since it results in error when horizontal slopes (e.g., BC) are used?

**For Solution G**: Both slopes AB and GH have the same ratio in each of the measures, but they have the different directions. Can you improve the measure such that it accounts for direction?

| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | • Whilst students are generating RSMs, teachers addressing groups who are stuck or those who need to move beyond one solution type can consider using the facilitation questions here.<br>• The aim is to get students to examine if the proposed gradient measure concurs with visual differences in steepness and direction. |
| **"Instruction" phase** | Type 3 solutions consider **two dimensions**, and how one dimension ***varies*** or ***changes*** with another to quantify the steepness of the slopes. They are conceptually close to the canonical concept in the measure of steepness. However, they do not take into account the direction of the slope. Teachers can use of the facilitation questions here to show the gaps/limitations of each solution to distil the remaining critical features. |

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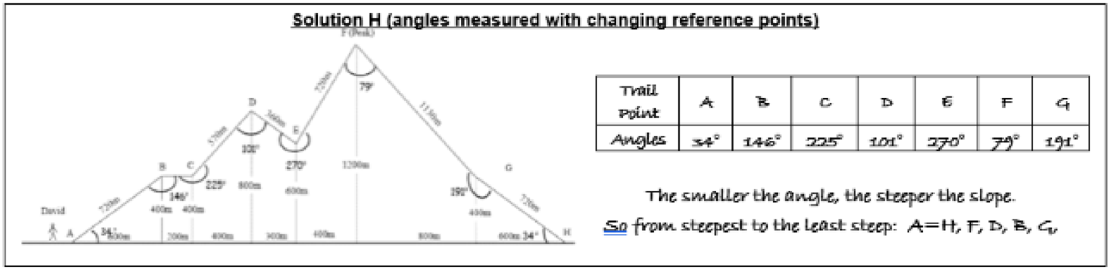
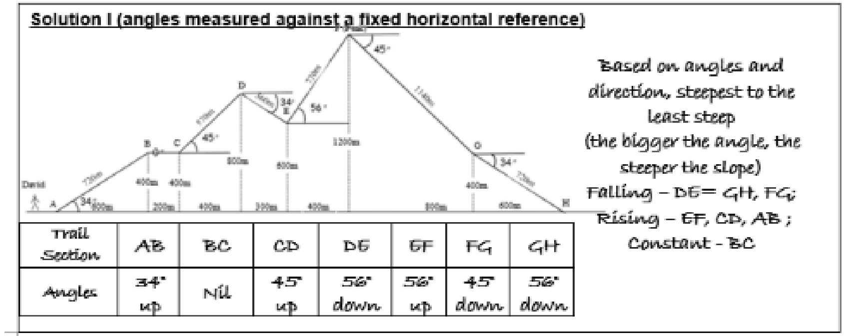
## Example Solution(s)




## Facilitation Questions

**<u>For Solution H:</u>** How would you improve the measure, since it measures elevation of points, rather than section of the trail?

**<u>For Solution I</u>**: Can you improve the measure such that it accounts for direction?

| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | • Whilst students are generating RSMs, teachers addressing groups who are stuck or those who need to move beyond one solution type can consider using the facilitation questions here.<br>• The aim is to get students to examine if the proposed gradient measure concurs with visual differences in steepness and direction. |
| **"Instruction" phase** | Type 3 solutions consider **two dimensions**, and how one dimension ***varies*** or ***changes*** with another to quantify the steepness of the slopes with the use of angles. However, they are not explicit in showing the direction of the slope (e.g., AB and GH) numerically. Teachers can use of the facilitation questions here to show the gaps/limitations of each solution to distil the remaining critical features. |

## Example Solution(s)

**Solution J**

| Trail Section | $\frac{\text{Change in vertical height}}{\text{Slope length}}$ | Ranking |
|---|---|---|
| AB | $\frac{400-0}{720} \approx 0.56$ | 4 |
| B C | $\frac{400-400}{200} = 0.00$ | 7 |
| C D | $\frac{800-400}{570} = 0.70$ | 2 |
| D E | $\frac{600-800}{360} \approx -0.56$ | 4 |
| E F | $\frac{1200-600}{720} = 0.83$ | 1 |
| F G | $\frac{400-1200}{1140} =0.70$ | 2 |
| G H | $\frac{0-400}{720} \approx -0.56$ | 4 |

**Solution K**

| Trail Section | $\frac{\text{Horizontal distance}}{\text{change in vertical height}}$ | Ranking |
|---|---|---|
| AB | $\frac{600}{400-0} \approx 1.50$ | 4 |
| B C | $\frac{200}{400-400} = \infty$ | 7 |
| C D | $\frac{400}{800-400} = 1.00$ | 2 |
| D E | $\frac{300}{600-800} \approx -1.50$ | 4 |
| E F | $\frac{400}{1200-600} = 0.67$ | 1 |
| F G | $\frac{800}{400-1200} = -1.00$ | 2 |
| G H | $\frac{600}{0-400} \approx -1.50$ | 4 |

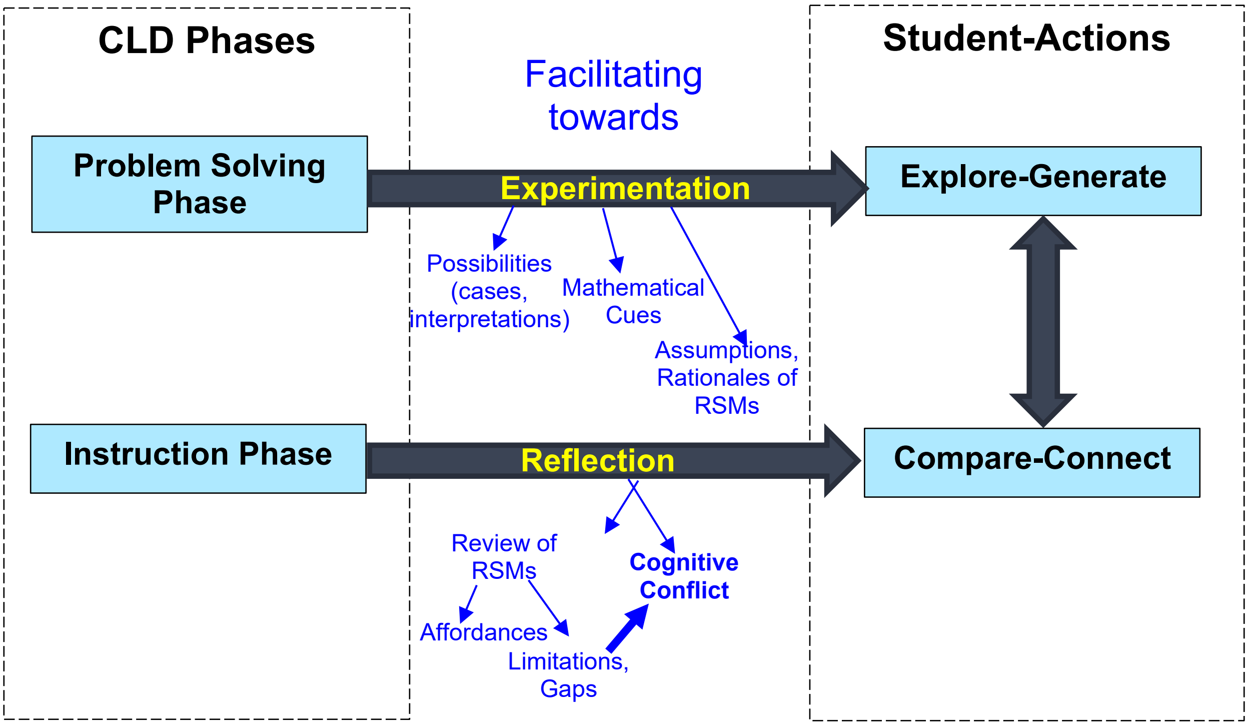
## Facilitation Questions

**For Solution J:** The values substituted into this formula will always fall within the range of -1 to 1. The ratio based on this indicates that the slope is the least steep, i.e., a horizontal line, when the ratio is 0. The slope is the steepest, i.e., a vertical line, when the ratio has an absolute value of 1. Although the interpretation of steepness based on this formula is intuitive compared to its inverse $\frac{\text{Slope length}}{\text{Change in vertical height}}$, one needs to be careful when that the measure takes a maximum absolute value of 1.

**For Solution K**: Although this solution makes sense, it is not convenient to interpret the gradient measure with it. This is because here, the ratio with a smaller absolute value indicates a steeper slope. This is counter-intuitive for many people.

| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | • Whilst students are generating RSMs, teachers addressing groups who are stuck or those who need to move beyond one solution type can consider using the facilitation questions here.<br>• The aim is to get students to examine if the proposed gradient measure concurs with visual differences in steepness and direction. |
| **"Instruction" phase** | All these methods are valid in terms of quantifying the steepness and direction of the slopes. They also included the necessary critical features of the targeted concept, especially showing how one variable varies with another. The solutions are improved versions of those in Type 3 – they take account of the direction as the vertical change can be either positive or negative. However, it seems that more care in interpretation may be needed when using these gradient measures. Teachers can use the facilitation questions here to prompt for different ways of presenting the proposed measures for easier interpretation. |

## Section 2.4.3 Instruction Phase: A sample lesson outline




We examined the solution specific facilitation moves earlier, examining how we can help students reflect on the affordances and constraints on their RSMs (i.e. solutions). In this section, we will move on to outline a lesson or the part of the lesson where the instruction phase occurs.

To prepare for the instruction phase, teachers can classify the different RSMs (or solutions) into various types depending on their similarities. Then, the solution types can be sequenced before drawing out explicit connections between the solution types and the targeted concept. The aim of teacher facilitation during the instruction phase is to encourage ***reflection***. Hence, students ***compare*** different approaches used to solve the problem, distil the critical features included in each solution type, and draw ***connections*** between solutions. With purposeful teacher prompts as scaffolding, students then move on to construct the targeted concept by refining existing RSMs. Table 2.1 provides additional details for suggested moves during the instruction phase.

*Table 2.1 Suggested teacher facilitation moves for the instruction phase (gradient of linear graph)*

<table>
<tr><th>When this Occurs</th><th>Suggested Facilitation Moves</th><th>Intentions Behind the Moves</th></tr>
<tr><td>Start of lesson</td><td><ul><li>Present the various solution types in gallery walk /display all solutions on the board</li></ul></td><td><ul><li>Highlight to students the rich array of solutions produced.</li></ul></td></tr>
<tr><td rowspan="2">During the lesson</td><td><u>Activate students’ solutions</u><ul><li>Teacher selectively describes representative solutions within each solution type in sequence.</li><li>Teacher helps students to progress towards the targeted concept:<br>For example, we discuss Type 1 to Type 5 solutions in sequence.<ul><li>Type 1 solutions consider only one variable and quantify the steepness to some extent.</li><li>From here, the solutions progress to those that consider a combination of two variables (Type 2), and a step further to those that consider the ratio of two variables (Type 3), and reach those that not only quantify the steepness but also indicate the direction of the slopes (Type 5).</li><li>Type 4 solutions are different from the others as they attempt to quantify the steepness and direction of the slopes using angles.</li></ul></li></ul></td><td><ul><li>The systematic review of the solutions should allow students time to distil the concept’s critical feature.</li></ul></td></tr>
<tr><td><u>Building upon the different solutions and analysing their affordances and constraints</u><ul><li>For each solution type, the teacher can show why the solution works and how it can be further improved.<ul><li>The analysis of the affordances and constraints of each solution can be found above.</li></ul></li></ul></td><td><ul><li>The sequence to considered here is the one proposed in this book, but teachers can try other sequences, as long as the critical features are brought up, compared and contrasted.</li></ul></td></tr>
</table>

| When this Occurs | Suggested Facilitation Moves | Intentions Behind the Moves |
|---|---|---|
| | - Highlight the critical features of the concept through compare and contrast.<br>For example,<br>➢ The teacher can point out that all the solutions attempt to quantify the steepness of the slopes to some extent and allows for ranking.<br>➢ However, Type 1 solutions consider only one variable, while Types 2, 3 and 5 solutions consider two or more variables in the calculation.<br>➢ Types 1, 2 and 3 solutions do not indicate the direction of the slopes, but Type 5 solutions have that advantage. | |
| | <u>Assemble the targeted concept</u><br>- Once the various solutions are discussed, the teacher will now show how the targeted concept looks like, and how it includes all the critical features.<br>- Teacher can get the students to solve the problem again. | • To show how the concept is being formulated and to show how it includes the critical features. |
| Practice and Homework | Teachers to give some practice questions to reinforce the concept, and higher order questions to further conceptual understanding. | • To reinforce the targeted concept. |

# SECTION 3:

## GRADIENT OF CURVES (SECONDARY THREE)

## Section 3.1 Problem Task (Students' version)

**Did the motorist exceed the speed limit?**

During peak hours, traffic accidents are very common along a road that is near a shopping mall, and a speed limit of **15 m/s** is set to regulate traffic.

A motorist is suspected to have exceeded the speed limit on this road on a particular morning. Data captured of the motorist's distance from a shopping mall during a 100 second time frame are shown in Table 1 below. The distance-time graph is also shown in Figure 1.

*Table 1*. Distance-time information of a motorist suspected of exceeding speed limit.

| Time (s) | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Distance from the shopping mall (m) | 0 | 40 | 80 | 120 | 166 | 233 | 346 | 569 | 1044 | 1170 | 1200 |

However, the motorist denied exceeding the speed limit, and sought the help of your group to defend his claim mathematically. Here is what your group must do:

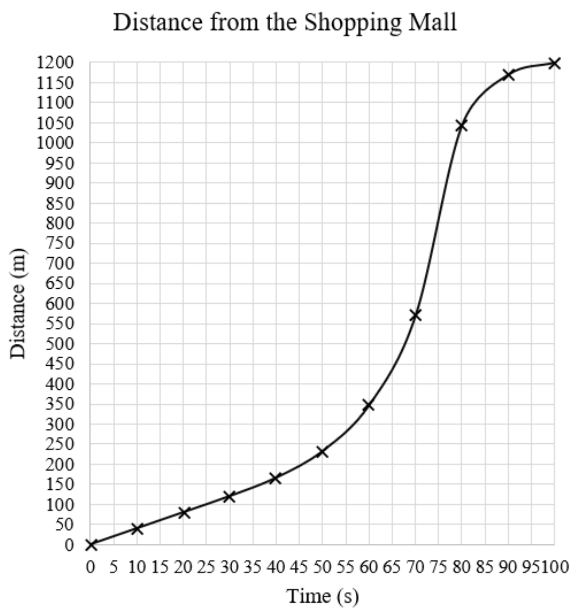



*Figure 1*. Distance-time graph of a motorist suspected of exceeding speed limit.

(1) Using the data and the graph provided, please come up with as <u>many different methods as possible</u> to estimate the motorist's speed at any given instance during the time frame. You can make use of the graph papers provided.

(2) For each method you use, decide whether the motorist exceeded the speed limit. If he did, please determine the exact time when the motorist first exceeded the speed limit.

All the best, and remember, don't give up until you have developed as many methods as possible to determine whether the motorist exceeded the speed limit!

## Section 3.2 Unpacking the Critical Features of the Targeted Concept

| | |
|---|---|
| The **Concept** | **Gradient of curves** is a measure that quantifies the slope of curves, defined as the gradient of the tangent to the curve at a specified point.<br>(a) It builds on students' understanding of the gradients of linear graphs introduced in Sec 1. However, unlike the gradient of a straight line which remains constant at different points on the graph, the gradient of a curve may vary at different points on the curve.<br>(b) The idea of deriving the tangent line through making two points on a curve infinitely close to each other is central in the construction of the measure of the slope of curves. |
| **Critical Features** of the concept | There are three critical features in the concept of gradient of curves:<br>(a) **Estimation** of the gradient of a curve at a point P (x, y) using the **gradient of a line PQ** (calculated as $\frac{y'-y}{x'-x}$), where Q (x', y') is another point on the curve.<br>(b) Finding the tangent to the curve at point P involves the idea of **narrowing the interval** between points P and Q, therefore making Q infinitely close to P.<br>(c) The gradient of a curve at point P is equal to the **gradient of the *tangent*** to the curve at P. |
| **Big Ideas** underlying the concept | The concept of gradient of curves is aligned to the ideas of measures, function, and diagrams:<br>(a) ***Measures***<br>The gradient of curves is used to measure or quantify the slope (i.e., the rate of change) of the curve at a point.<br>(b) ***Function***<br>The gradient function expresses the relationship between the points on the curve of the given function and the gradients of the curve at the points.<br>(c) ***Diagrams***<br>The gradient of a curve at a point can be represented in diagrams as the gradient of the tangent to the curve at the point. |

## Section 3.3 Facilitation Moves for the CLD

Solutions (i.e., RSMs) produced for the problem can be classified according to the critical features of the concept being addressed. There are three critical features underlying the concept of the gradient of curves: (1) estimating the gradient of a curve using gradient of the tangent at a point on the curve, (2) narrowing the interval between two points on the curve to obtain the tangent of a point on the curve, and (3) using a tangent to the curve to estimate the gradient of the curve. The following pages show the classification of solution types, an example solution representing each solution type, and solution-specific teacher facilitation moves for each solution type. Suggested facilitation questions are provided to provoke cognitive conflict for desired learning to occur.

### TYPE 1 SOLUTIONS: Intuitive Reasoning

**Critical Features Checklist**

- ✗ **Estimate the gradient of a curve using gradient of a line**
- ✗ **Narrowing the interval**
- ✗ **Obtaining the tangent and finding the gradient of the tangent**

### Example Solution

**Solution A**

***Identify a region of the graph where it starts to transit from a line to a curve***


*Figure 1.* Distance-time graph of a motorist suspected of exceeding the speed limit.

He exceeds at 40s-50s, because the graph starts to fluctuate from a straight line

### Facilitation Questions

The question requires you to estimate the motorist's speed at any given instance during the time frame. What are the possible ways to measure the speed of the motorist at your proposed time point/period?

How is the speed of the vehicle represented in the distance-time graph? How can you make use of the graph to estimate the speed of the vehicle at a given point in time?

| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | Teachers can consider using the facilitation questions above to encourage students to use the graph or data provided to quantify the speed of vehicle (i.e., gradient of distance-time graph) during the identified region of the graph to ascertain if the vehicle had exceeded 15m/s. |
| **"Instruction" phase** | ***Affordances*** – Type 1 solutions show that students recognise *shifts in the speed (or gradient)* from the point where the graph transits from a straight line to curve.<br><br>***Constraints*** – Type 1 solutions are based purely on visual observations without any attempt to *quantify* or *estimate* the speed (or gradient) at any interval or on any point. |

# TYPE 2 SOLUTIONS: Calculating Average Speed using Distance-Time Data at every 10 seconds

**Critical Features Checklist**

- ✓ Estimate the gradient of a curve using gradient of a line
- ✗ Narrowing the interval
- ✗ Obtaining the tangent and finding the gradient of the tangent

## Example Solution

**Solution B**

***Examine the average speed using distance-time data at every 10s***

| Time (s) | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Distance from the shopping mall (m) | 0 | 40 | 80 | 120 | 166 | 233 | 346 | 569 | 1044 | 1170 | 1200 |
| Speed (m/s) | 0 | 4 | 4 | 4 | 4.15 | 4.66 | 5.77 | 8.13 | 13.05 | 13 | 12 |

The motorist did not exceed the speed limit, since all of the average speeds within the 100 s timeframe are below 15 m/s.

## Facilitation Questions

Does your calculation show the motorist's speed at a particular time point or the average speed over a time interval?

Is the average speed you calculated represented on the graph? Why? What does average speed mean? How can you make use of the graph to find or estimate the speed at a point?

| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | Teachers can consider using the facilitation questions above to help students reflect how their solutions (average speed) are represented graphically, and to avoid calculating average speed with start point at the origin (0,0). |
| **"Consolidation" phase** | ***Affordances*** – In the graphical terms, Type 2 solutions use the concept of average speed to estimate the gradient of the curve at P. It is unclear if the students know how this is related to rate of change (distance over time) and what this means in the interpretation of the gradient of the curve.<br><br>***Constraints*** – In this solution type, it is assumed that instantaneous speed and average speed (over a duration of time) are similar. There is no attempt to find the speed of the vehicle at an instaneous point in time (interpreted from the context). In other words, the idea of narrowing the interval between two points on the graph to find the gradient of the curve at a point is not evident. |

# TYPE 3 SOLUTIONS: Average Speed at a Selected Time-Interval

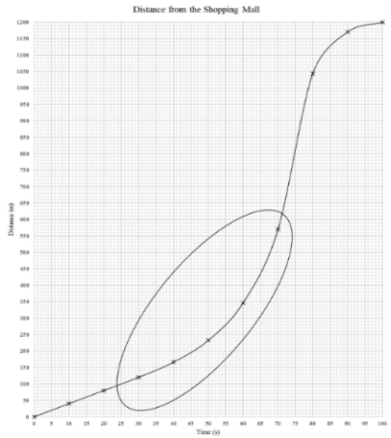
### Critical Features Checklist

- ✓ Estimate the gradient of a curve using gradient of a line
- ✓ Narrowing the interval
- ✗ Obtaining the tangent and finding the gradient of the tangent

## Example Solution

**Solution C**

***Identify region on the graph where the motorist is suspected to have exceeded the speed limit and calculate the average speed within the region***




Gradient between
$t$ = 30s and $t$ = 70s

$\frac{569-120}{70-30}$ = 11.23 m/s

The motorist did not exceed the speed limit.

## Facilitation Questions

What is so special about the region that you have identified compared to other part of the graph?

Does your calculation show the motorist's speed at a particular time point or the average speed over a time interval?

Can there be instances where the average speed *over a time interval* is below 15m/s, whereas the speed *at a point in time* during the interval is above

| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | Teachers can consider using the facilitation questions above to help students reflect if average speed within the selected region of the graph (e.g., the curved portion of the graph marked out in Solution C) gives an appropriate response to whether the motorist has exceeded the speed limit. Students can be encouraged to think about whether an instantaneous point in time is needed to conclude that the motorist has "begun" to exceed the speed limit. Teachers can discuss the possibility of further narrowing the interval between 2 points on the curve within the identified time-interval marked out on the graph so that the idea of tangent *at a point* on the curve where the speed limit is breached makes sense. |
| **"Instruction" phase** | ***Affordances*** – In the graphical terms, Type 3 solutions use the concept of *__gradient of a straight line__* to estimate the gradient of the curve (e.g., gradient is calculated using 2 points $t_1$ = 30s and $t_2$ = 70s). The solutions also demonstrate students' *__graphical interpretation skills__* and possible *__intuitive understanding__* of the slopes of distance-time curves as rates of change. The identification of a time-interval on the graph to focus on, by itself, may represent the beginning of the idea of using a tangent to calculate the gradient. This presents a possiblity of further narrowing of the interval between the 2 points used earlier.<br><br>***Constraints*** – Type 3 solutions, which examines the average speed within a region, do not specify the instantaneous speed at an exact point within the identified interval. |

### Critical Features Checklist

- ✓ Estimate the gradient of a curve using gradient of a line
- ✓ Narrowing the interval
- ✗ Obtaining the tangent and finding the gradient of the tangent

## Example Solution

**Solution D**

*Examine the average speed within each of the 10s interval*

| Time Interval | Average Speed |
|---|---|
| 0 - 10s | $\frac{40}{10}$ = 4 m/s |
| 10 - 20s | $\frac{40}{10}$ = 4 m/s |
| 20 - 30s | $\frac{40}{10}$ = 4 m/s |
| 30 - 40s | $\frac{46}{10}$ = 4.6 m/s |
| 40 - 50s | $\frac{67}{10}$ = 6.7 m/s |
| 50 - 60s | $\frac{113}{10}$ = 11.3 m/s |
| 60 - 70s | $\frac{223}{10}$ = 22.3 m/s |
| 70 - 80s | $\frac{475}{10}$ = 47.5 m/s |
| 80 - 90s | $\frac{126}{10}$ = 12.6 m/s |
| 90 - 100s | $\frac{30}{10}$ = 3 m/s |

The motorist first exceeded the speed limit of 15 m/s at the timeframe of 60-70s.

## Facilitation Questions

How small do you think the interval should be in order to give the most accurate estimate of the speed at a point in time?

What will the interval become if you make the interval infinitely small?

On the graph, we can draw lines joining 2 points within an interval of time. Think about what will happen to the line when the interval gets smaller and smaller? What happens to the line when the interval between the two points becomes infinitely small?

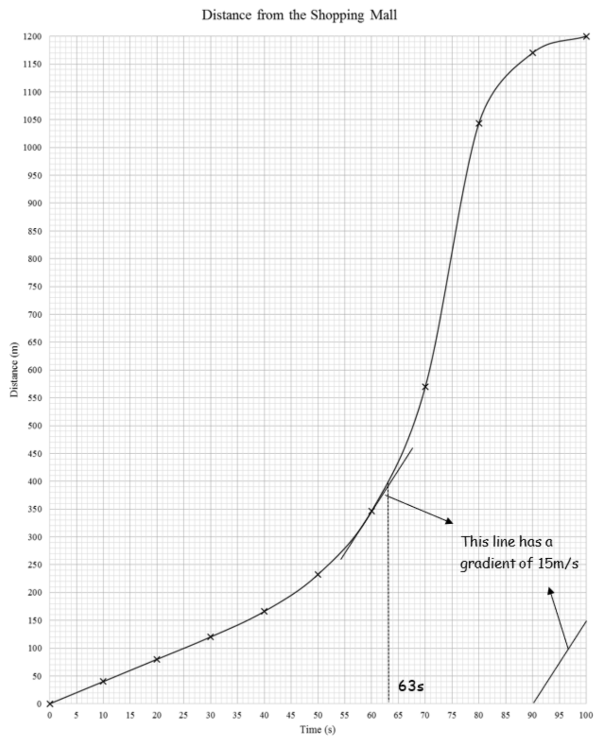
| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | Teachers can consider using the facilitation questions above to help students consider the possibility of narrowing the interval so that it becomes infinitely small, and how the line between the two points could eventually become the tangent of a point on the curve. |
| **"Instruction" phase** | ***Affordances*** – In the graphical terms, Type 4 solutions use the concept of the *gradient of a straight line* to estimate the gradient of the curve, and the attempt to *narrow down the time interval* between 2 points on the curve for gradient of a curve at a point is salient in these solutions.<br>***Constraints*** – Although students made attempts to use smaller intervals (e.g., intervals of 10s), these intervals may still not be small enough to provide an average speed that accurately reflect the instantaneous speed that is really needed to answer the question. |

## TYPE 5 SOLUTIONS: Instantaneous Speed at a Point

The solutions here indicate the desired construction of the gradient of curve concept.

### Solution E

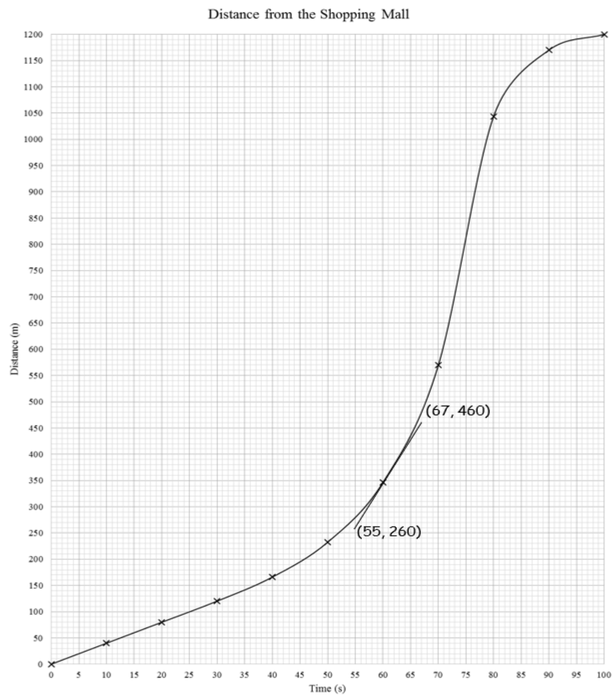
***Draw a line with gradient of 15m/s, and identify the point where the tangent of the curve is parallel to the line***




The motorist exceeded the speed limit of 15 m/s at 63 s.

### Solution F

***Draw a tangent line to the curve and calculate the gradient of the tangent line***




Gradient of tangent at 60s

$$= \frac{460 - 260}{67 - 55} = 16.67 \text{ m/s}$$

### Critical Features Checklist

- Estimate the gradient of a curve using gradient of a line
- Narrowing the interval
- Obtaining the tangent and finding the gradient of the tangent

# SECTION 4:

## QUADRATIC INEQUALITIES (SECONDARY THREE)

## Section 4.1 Problem Task (Students' version)

### A Quadratic Function Problem

A *quadratic function*, ***p*** of ***x***, is given below with an unknown ***a***

$$p = x^2 + (3a+1)\,x + (a^2+3a+1)$$

Together with your teammates, please solve the following two tasks.

**Task 1**: Determine **all possible values of *a*** to ensure that ***p* > 0** for **all real values of *x***.

Use as many methods as possible to find out all possible values for $a$. Once you are done with coming up with as many methods for Task 1, proceed to Task 2.

**Task 2**: Determine **all possible values of *a*** to ensure that ***p* > 0 whenever *x* > 0**.

Review the methods that you have obtained in Task 1, then come up with as many methods as possible to find out all possible values for $a$ for Task 2.

For both tasks, you may the blank papers and graph papers provided for your task.

All the best and don't give up!

## Section 4.2 Unpacking Quadratic Inequalities

| | |
|---|---|
| **The Problem Solving Process** | There are two major steps for both Tasks 1 and 2 in the problem:<br><br>Step 1. Identification of the necessary *conditions* involving the unknown "$a$" under which $p$ > 0 for all real values of $x$ (Task 1) or when $x$ > 0 (Task 2).<br><br>Step 2. Solving the quadratic inequalities obtained from Step 1 to find the range of values for $a$ such that $p$ > 0 for all real values of $x$ (Task 1) or when $x$ > 0 (Task 2).<br><br>The process of solving quadratic inequalities requires the incorporation of all relevant prerequisite knowledge:<br>(a) solving (simultaneous) linear inequalities;<br>(b) understanding the rule of signs in arithmetic (negative and positive rules);<br>(c) solving for roots for quadratic equations using the factorization, root formula, or completing the square methods;<br>(d) understanding the graphical representations and graphical characteristics of quadratic functions; and<br>(e) understanding the roots and discriminants of quadratic functions |
| **Critical Ideas/ Processes** | There are five critical ideas/processes in solving quadratic inequalities: an understanding of<br>(a) how to solve linear inequalities;<br>(b) algebraic methods to solve quadratic inequalities;<br>(c) the relationship between discriminants and the roots of the quadratic equations;<br>(d) the connections between the algebraic and graphical representations of quadratic functions (the function *p* and the discriminant function *D*) and the graphical characteristics (e.g., the *x*-intercept and *y*-intercept, the symmetrical line); and<br>(e) the connections between the roots of quadratic equations and the solutions of the related quadratic inequalities |
| **Big Ideas** underlying the process of solving quadratic inequalities | Quadratic inequalities are aligned with functions and diagrams:<br>(a) ***Functions***<br>Quadratic inequalities, like the quadratic functions and equations, represent rules that can be represented algebraically or graphically.<br>(b) ***Diagrams***<br>When solving quadratic inequalities, graphs are used to summarize the characteristics of the quadratic functions. They are constructed to help solve the quadratic inequalities. |

## Section 4.3 Facilitation Moves (Problem Task 1)

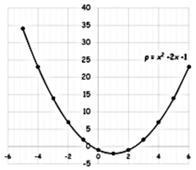
Solutions produced for **Problem Task 1** can be classified into five types. The following pages show the classification of solution types, as well as the example solution and solution-specific teacher facilitation moves for each of the solution type. Suggested facilitation questions are provided to provoke cognitive conflict for desired learning to occur.

## TYPE 1 SOLUTIONS: Examination of the given quadratic function *p*

### An Example Solution

**Solution A**

Substituting numbers for a to check whether it is true that p > 0 for all real values of x:

| Values of a | $p = x^2 + (3a+1)x + (a^2+3a+1)$ | Is it true that p > 0 for all real values of x? |
|---|---|---|
| $a = -1$ | $p = x^2 - 2x - 1$ | No. p < 0 when 0 < x < 2 (See the graph of $p = x^2 - 2x - 1$ on the right)  |
| $a = 0$ | $P = x^2 + x + 1$ | Yes (by graphing the function $p = x^2 + x + 1$) |
| $a = 1$ | $p = x^2 + 4x + 5$ | Yes (by graphing the function $p = x^2 + 4x + 5$) |
| $a = 2$ | $p = x^2 + 7x + 11$ | No (by graphing the function $p = x^2 + 7x + 11$) |
| … | … | … |

Based on the above, *a* is between 0 and 1 for *p* > 0 for all real values of *x*.

### Possible Critical Ideas/Processes Exhibited

- ✗ Solving linear inequalities
- ✗ Algebraic methods to solve quadratic inequalities
- ✗ The relationship between discriminants and roots of quadratic equations
- ? Connections between algebraic and graphical representations; show understanding of the graphical characteristics
- ✗ Connections between the roots of quadratic equations and the solutions of the related quadratic inequalities

### Facilitation Questions

1. Do the listed values give the exact range of the possible values for $\boldsymbol{a}$ that satisfy the requirement of the problem? Is there a more efficient, systematic way to find the range of values of $\boldsymbol{a}$?
2. What do you think the graph of the given quadratic function $p$ will look like? Can you use the graphical information to help you find conditions that the unknown $\boldsymbol{a}$ should satisfy?

### Pointers for teachers during the …

| | |
|---|---|
| **"Problem Solving" phase** | Teachers can consider using the facilitation questions above, to help students examine if the solution identifies the discriminant *D* of the quadratic function $p$, and whether both the graphical and the algebraic information of *D* is considered to solve the quadratic inequality *D* < 0. |
| **"Instruction" phase** | • Type 1 solution considers pertinent parts of information when finding the possible values of $\boldsymbol{a}$. Graphical information of quadratic fuctions has been considered when determining whether it's true that $p > 0$ for all real values of $x$ for the given value for $a$.<br>• However, it is unclear if the discriminant of the given quadratic function has been considered in order to find the values for $a$. |

## TYPE 2 SOLUTIONS: Consideration of discriminants and trial-and-error method to solve quadratic inequalities

### An Example Solution

**Solution B**

Since we want the function $p > 0$ for all real values of $x$, the discriminant D should be $< 0$.

So, $D = (3a+1)^2 - 4(a^2+3a+1) < 0$

$5a^2 - 6a - 3 < 0$

We then try different values of a to check which can result in the discriminant to be less than 0.

| Possible values of a | $y = 5a^2 - 6a - 3$ | Conclusion |
|---|---|---|
| $a = -1$ | $y = 8 > 0$ | The possible values of $a$ lie between 0 and 1. |
| $a = 0$ | $y = -3 < 0$ | |
| $a = 1$ | $y = -4 < 0$ | |
| $a = 2$ | $y = 5 > 0$ | |

### Critical ideas/processes

- Solving linear inequalities
- Algebraic methods to solve quadratic inequalities
- The relationship between discriminants and roots of quadratic equations
- Connections between algebraic and graphical representations of quadratic functions; shows understanding of the graphical characteristics
- Connections between the roots of quadratic equations and the solutions of the related quadratic inequalities

### Facilitation Questions

1. Are you able to check all possible values for $a$? Can you think of a more efficient, systematic way to find the range of values for $a$ from the quadratic inequality?
2. What algebraic manipulations can you do to the quadratic inequality that might help solve for $a$? What do you know about the quadratic function/ equation that will be relevant to solve for the values for $a$?
3. What do you think the graph of the discriminant will look like? Can you possibly use the graphical information to help you find the range of values for $a$?

### Pointers for teachers during the …

| | |
|---|---|
| **"Problem Solving" phase** | Teachers can consider using the facilitation questions above to get students to examine if the solution identifies the discriminant $D$ of the quadratic function $p$, and whether both the graphical and the algebraic information of $D$ is considered to solve the quadratic inequality $D < 0$. |
| **"Instruction" phase** | • Type 2 solutions seem to consider the relationship between the discriminant and the number of roots of a quadratic equation, and possibly show an understanding of $D < 0$ means no real roots. Solutions also may show students' initial attempts to solve quadratic inequalities and an understanding that there is a range of feasible values of $a$.<br>• However, the solutions do not seem to demonstrate connections between algebraic and graphical representations of quadratic function $D$. They also do not make use of either an algebraic or a graphical method to solve the quadratic inequality $D < 0$. |

## TYPE 3 SOLUTIONS: Consideration of discriminant and algebraic manipulations to solve quadratic inequalities

| An Example Solution | Critical ideas/processes | Facilitation Questions |
|---|---|---|
| **Solution C**<br><br>Since we want the function p >0 for all real values of x, the discriminant D should be < 0.<br>So, $D = (3a+1)^2 - 4(a^2+3a+1) < 0$<br>$5a^2 - 6a - 3 < 0$<br>$5a^2 - 6a < 3$<br>$a(5a - 6) < 3$<br>So we have $a < 1$, $(5a - 6) < 3$ | ✗ Solving linear inequalities<br><br>✓ Algebraic methods to solve quadratic inequalities<br><br>✓ The relationship between discriminants and roots of quadratic equations<br><br>✗ Connections between algebraic and graphical representations of quadratic functions; shows understanding of the graphical characteristics<br><br>✗ Connections between the roots of quadratic equations and the solutions of the related quadratic inequalities | 1. What algebraic manipulations to the quadratic inequality will help solve for $a$? How is finding the root(s) of the corresponding quadratic equation relevant to solving the quadratic inequality?<br><br>2. What do you think the graph of the discriminant will look like? Can you possibly make use the graphical information of the discriminant to help you find the range of values for $a$? |

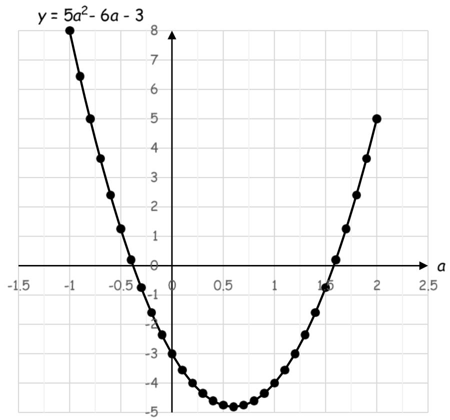
| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | Teachers can consider using the facilitation questions above, to get students to examine if the solution identifies the discriminant $D$ of the quadratic function $p$, and whether both the graphical and the algebraic information of $D$ is considered to solve the quadratic inequality $D < 0$. |
| **"Instruction" phase** | • Type 3 solutions consider the relationship between the discriminant and the number of roots of a quadratic equation, and understand that $D < 0$ means no real roots. Solutions make attempts to solve quadratic inqualities systematically by carrying out algebraic manipulations.<br>• However, there is little attempt to continue using algebraic methods to solve for the roots of the quadratic equation to find the range of values for $a$. The solutions do not demonstrate an understanding of connections between algebraic and graphical representations of quadratic functions. No graphical information is used to solve the quadratic inequality. Teachers can make use of the facilitation questions to show the constraints of the solution to distil the critical features. |

## TYPE 4 SOLUTIONS: Finding the roots of quadratic functions to solve quadratic inequalities algebraically

| An Example Solution | Critical ideas/processes | Facilitation Questions |
|---|---|---|
| **Solution D**<br><br>Since we want the function p >0 for all real values of x, the discriminant D = $5a^2 - 6a - 3$ should be < 0.<br><br>The two roots of $5a^2 - 6a - 3 = 0$ are<br>$\frac{6 \pm \sqrt{36+4*5*3}}{10}$ or $\frac{3 \pm 2\sqrt{6}}{5}$<br>Hence $5a^2 - 6a - 3 = 5(a - \frac{3-2\sqrt{6}}{5})(a - \frac{3+2\sqrt{6}}{5}) < 0$<br>So $a < \frac{3-2\sqrt{6}}{5}$ or $a < \frac{3+2\sqrt{6}}{5}$ | ✓ Solving linear inequalities<br><br>✓ Algebraic methods to solve quadratic inequalities<br><br>✓ The relationship between discriminants and roots of quadratic equations<br><br>✗ Connections between algebraic and graphical representations of quadratic functions; shows understanding of the graphical characteristics<br><br>✗ Connections between the roots of quadratic equations and the solutions of the related quadratic inequalities | 1. Have you checked whether you solved the quadratic inequality correctly? Suppose $mn < 0$ ($m$ and $n$ are two real numbers), would $m$ and $n$ be both positive? Why?<br><br>2. What do you think the graph of the discriminant will look like? Can you use the graphical information to help you check whether the range of $a$ that you have found is correct? |

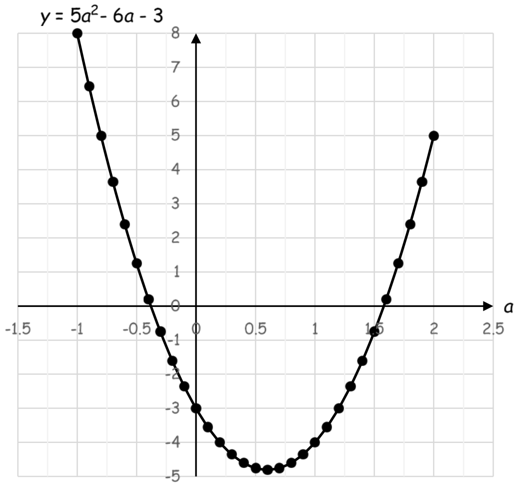
| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | Teachers can consider using the facilitation questions above to get students to examine if the solution identifies the discriminant $D$ of the quadratic function $p$, and whether both the graphical and the algebraic information of $D$ is considered to solve the quadratic inequality $D < 0$. |
| **"Instruction" phase** | • Type 4 solutions consider the relationship between the discriminant and the number of roots of a quadratic equation, and understand that $D < 0$ means no real roots. The solutions make use of algebraic methods to find the roots of the quadratic equations corresponding to the quadratic inequality, and subsequently the solutions to the quadratic inequality. The solutions also demonstrate the skills to solve (simultaneous) linear inequalities.<br>• However, since no graphs are used, the solutions do not demonstrate an understanding of the connections between algebraic and graphical representations of quadratic functions. |

# TYPE 5 SOLUTIONS: Graphical method to solve quadratic inequalities

| An Example Solution | Critical ideas/processes | Facilitation Questions |
|---|---|---|
| **Solution E**<br><br>Since we want the function $p > 0$ for all real values of $x$, the discriminant D of $p$ should be $< 0$.<br>The range of values for $a$ is $0.38 < a < 1.58$, where the two numbers are **estimated** $x$-intersects on the graph of $D = 5a^2 - 6a - 3$. | ✗ Solving linear inequalities<br>✗ Algebraic methods to solve quadratic inequalities<br>✓ The relationship between discriminants and roots of quadratic equations<br>✓ Connections between algebraic and graphical representations of quadratic functions; shows understanding of the graphical characteristics<br>✓ Connections between the roots of quadratic equations and the solutions of the related quadratic inequalities | 1. Do the listed values give the exact range of the possible values for $a$ that satisfy the requirement of the problem? Is there a different way to find the range of values of $a$?<br>2. What do you think the graph of the given quadratic function $p$ will look like? Can you make use the graphical information to help you find conditions that the unknown $a$ should satisfy? |

| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | Teachers can consider using the facilitation questions above to get students to examine if the solution identifies the discriminant $D$ of the quadratic function $p$, and whether both the graphical and the algebraic information of $D$ is considered to solve the quadratic inequality $D < 0$. |
| **"Instruction" phase** | • Type 5 solutions use graphical methods to solve the quadratic inequalities and find the correct range of the values for $a$.This demonstrates an understanding of the connections between algebraic and graphical representations of quadratic functions, and an understanding of graphical characteristics, e.g., the $x$-intersect. Solutions clearly demonstrate the understanding of the connections between the roots of quadratic equations and the solutions of the related quadratic inequalities.<br>• However, solutions do not use the algebraic method to find the exact values of the two roots and the exact range of the values of $a$. |

# TYPE 6 SOLUTIONS: (Canonical Solution) Algebraic and Graphical Methods to Solve Quadratic Inequalities

| An Example Solution | Critical ideas/processes |
| --- | --- |
| **Solution E**<br><br>To ensure $p > 0$ for all real values of $x$, the discriminant D of $p$ should be < 0:<br>$$D = 5a^2 - 6a - 3 < 0$$<br>Solving $D = 5a^2 - 6a - 3 = 0$ gives the two roots $\frac{3 \pm 2\sqrt{6}}{5}$.<br>From the graph, $D < 0$ when $\frac{3-2\sqrt{6}}{5} < a < \frac{3+2\sqrt{6}}{5}$. | ✓ Solving linear inequalities<br>✓ Algebraic methods to solve quadratic inequalities<br>✓ The relationship between discriminants and roots of quadratic equations<br>✓ Connections between algebraic and graphical representations of quadratic functions; shows understanding the graphical characteristics<br>✓ Connections between the roots of quadratic equations and the solutions of the related quadratic inequalities |

## Section 4.4 Facilitation Moves (Problem Task 2)

Solutions for Problem Task 2 are built upon those from Problem Task 1. However, Problem Task 2 puts more emphasis on the graphical understanding of the quadratic functions and the related quadratic inequalities. Hence, the analysis of the solutions for Problem Task 2 will skip over the first three critical ideas/processes listed above (i.e., these relate to algebraic methods of solving quadratic inequalities that exist in the solutions for Problem Task 1). Instead, the focus will be on the remaining two critical ideas/processes that are predominant in Task 2:

(a) Understanding of the connections between algebraic and graphical representations of quadratic functions & the graphical characteristics.
(b) Understanding of the connections between the roots of quadratic equations and the solutions of the related quadratic inequalities.

The analysis will focus on the quadratic function $p$, and examine whether the solutions for Problem Task 2 demonstrate (a) and (b) listed here in relation to $p$. Specifically, for (a) in the above, we examine whether the solutions demonstrated an understanding of the connection between the algebraic and graphical representation of the quadratic function $p$, and whether there is consideration of graphical characteristics, e.g., its $x$-intercept(s), and $y$-intercept, and the symmetrical line. For (b) in the above, we examine whether the solutions show an understanding of graphical connections between the roots of the quadratic function $p = 0$ and the solutions of its related quadratic inequality $p > 0$.

The following pages show the classification of solution types for Problem Task 2, as well as an example solution, and solution-specific teacher facilitation moves for each solution type. Suggested facilitation questions are provided to provoke cognitive conflict for desired learning to occur.

## TYPE 1 SOLUTIONS: Consideration of the Discriminant *D* of the Quadratic Equation $p$

| An example Solution | Critical ideas/processes | Facilitation Questions |
|---|---|---|
| **Solution A**<br><br>To ensure $p > 0$ for all $x > 0$, the discriminant D of the given quadratic equation $p$ should be less than 0.<br><br>$D = 5a^2 - 6a - 3 < 0$<br><br>Solving the quadratic inequality gives $\frac{3-2\sqrt{6}}{5} < a < \frac{3+2\sqrt{6}}{5}$. | Connections between algebraic and graphical representations of quadratic functions; shows understanding of the graphical characteristics<br><br>Connections between the roots of quadratic equations and the solutions of the related quadratic inequalities | 1. Does the case that the discriminant $D < 0$ include all the values of $a$ that possibly result $p > 0$ for all $x > 0$? How about the value $a = 5$? How about value $a = 10$?<br><br>2. Is it possible that $p > 0$ for all $x > 0$ when $D < 0$? What does the graph of $p$ look like in this case?<br><br>3. What are the possible graphs of the quadratic function $p$, given different values of a? Can you show all the possible graphs of $p$ and identify what values of $a$ will ensure $p > 0$ for all $x > 0$? |

| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | Teachers can consider using the facilitation questions above to get students to think about the different graphs where $p > 0$ for all $x > 0$, and how to identify and construct the algebraic conditions for each case before solving for the values of $a$. |
| **"Instruction" phase** | • Type 1 solutions demostrate an understanding that the discriminant $D < 0$ ensures $p > 0$ (for all real values of $x$).<br>• However, the solutions overlook that Problem Task 2 specifies a restricted domain of $x$ (i.e., $x > 0$), rather than all values of $x$ as in Problem Task 1. Solutions do not demonstrate an understanding of the connections between graphs of quadratic functions and graphical characteristics of quadratic functions. |

## TYPE 2 SOLUTIONS: The range of the values of *a* in Task 2 is a subset of that in Task 1 and consists of the positive values of *a* obtained in Task 1

| An Example Solution | Critical ideas/processes | Facilitation Questions |
|---|---|---|
| **Solution B**<br><br>Consider finding the positive values of $a$ obtained in Task 1 such that $p > 0$ whenever $x > 0$.<br><br>The range of values of $a$ from Task 1 is $\frac{3-2\sqrt{6}}{5} < a < \frac{3+2\sqrt{6}}{5}$.<br>Since $x > 0$, if $a > 0$, we then have<br>$p = x^2 + (3a+1)x + (a^2+3a+1) > 0$.<br>So the range of values of $a$ is $0 < a < \frac{3+2\sqrt{6}}{5}$. | Connections between algebraic and graphical representations of quadratic functions; shows understanding of the graphical characteristics<br><br>Connections between the roots of quadratic equations and the solutions of the related quadratic inequalities | 1. Does the range of values that you have found include all the values of $a$ that possibly result $p > 0$ for all $x > 0$? How about the value $a = -0.1$? How about value $a = 5$?<br><br>2. Is it possible that the range of values of $a$ in Problem Task 2 is larger than that in Problem Task 1, since Task 2 requires $p > 0$ for all $x > 0$, rather than for any value of $x$?<br><br>3. What are the possible graphs of the quadratic function $p$, given different values of $a$? Can you show all the possible graphs of $p$ and identify what values of a will ensure $p > 0$ for all $x > 0$? |

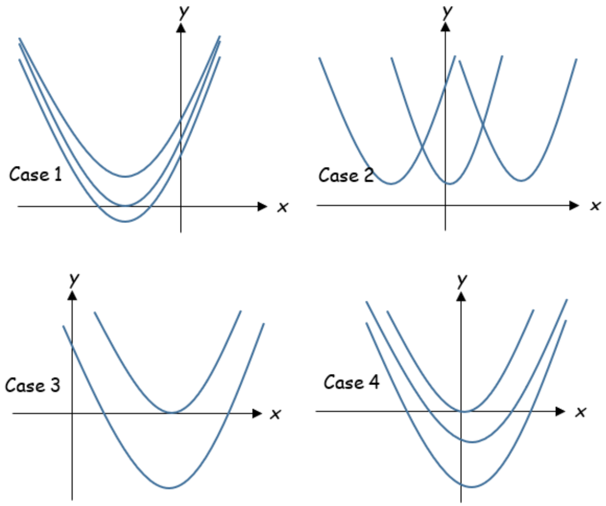
| Pointers for teachers during the … | |
|---|---|
| **“Problem Solving” phase** | Teachers can consider using the facilitation questions above to get students to think about the different graphs where $p > 0$ for all $x > 0$, and how to identify and construct the algebraic conditions for each case before solving for the values of $a$. |
| **“Instruction” phase** | • Type 2 solutions consider a restricted set of the values of $a$ to ensure $p > 0$ for all $x > 0$.<br>• However, the solutions fail to realise that the range of $a$ should be widened because of the restricted domain of $x$, (i.e., $x > 0$ in Problem Task 2). Solutions do not demonstrate the understanding of connections between graphs of the quadratic functions and graphical characteristics of quadratic functions. No graphical information is presented. |

## TYPE 3 SOLUTIONS: Consideration of the x-intercept(s) or y-intercept of the given quadratic function $p$

| An Example Solution | | Critical ideas/processes | Facilitation Questions |
|---|---|---|---|
| **Solution C** | **Solution D** | | |
| To ensure $p > 0$ for all $x > 0$, the y-intercept of the quadratic function $p$, should be greater than 0.<br>When $x = 0$, $p = a^2 + 3a + 1$. The coefficient of $x^2$ is 1.<br>When $x > 0$, $p > a^2 + 3a + 1$.<br>So we have $a^2 + 3a + 1 \geq 0$.<br>Solving the quadratic inequality gives $a \leq \frac{-3-\sqrt{5}}{2}$ or $a \geq \frac{-3+\sqrt{5}}{2}$. | To ensure $p > 0$ for all $x > 0$, the roots of the quadratic function $p$<br>$\frac{-3a-1 \pm \sqrt{5a^2-6a-3}}{2}$ should be less than 0.<br>Let $\frac{-3a-1+\sqrt{5a^2-6a-3}}{2} \leq 0$.<br>$\sqrt{5a^2-6a-3} \leq -3a-1$<br>$5a^2-6a-3 \leq 9a^2+6a+1$<br>$a^2 + 3a + 1 \geq 0$.<br>Solving the quadratic inequality gives $a \leq \frac{-3-\sqrt{5}}{2}$ or $a \geq \frac{-3+\sqrt{5}}{2}$. | Connections between algebraic and graphical representations of quadratic functions; shows understanding the graphical characteristics.<br>(though graphs are not used)<br><br>Connections between the roots of quadratic equations and the solutions of the related quadratic inequalities:<br><br>Solution C: No<br>Solution D: to some extent<br>(though graphs are not used) | 1. Does the range of values that you have found consists of all the values of $a$ that result $p > 0$ for all $x > 0$? Can you check whether it is true when $a = -3$? How about $a = -10$?<br>2. What are the possible graphs of the quadratic function $p$, given different values of $a$? Can you show all the possible graphs of $p$?<br>3. What can you observe about the x-intercept(s) or y-intercept of the graphs that satisfy $p > 0$ for all $x > 0$? Are there any other characteristics besides the x-intercept(s) or y-intercept? |

| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | Teachers can consider using the facilitation questions above to get students to think about the different graphs where $p > 0$ for all $x > 0$, and how to identify and construct the algebraic conditions for each case before solving for the values of $a$. |
| **"Instruction" phase** | • Type 3 solutions consider the pertinent information of the quadratic function $p$, the $x$- and $y$-intercepts.<br>• However, other important information including the discriminant is overlooked. No graphical information is presented. Solutions do not adequately demonstrate an understanding of connections between graphs of the quadratic functions and the graphical characteristics of quadratic functions. |

## TYPE 4 SOLUTIONS: Graphical illustration of the possible cases of the quadratic function *p*

| An Example Solution | Critical ideas/processes | Facilitation Questions |
|---|---|---|
| **Solution D**<br>Possible graphs of the quadratic function p are presented.<br>Both the graphs in Case 1 and Case 2 satisfy that $p > 0$ for all $x > 0$<br> | Connections between algebraic and graphical representations of quadratic functions; shows understanding the graphical characteristics<br>:<br><br>Connections between the roots of quadratic equations and the solutions of the related quadratic inequalities: to some extent | 1. How do you characterise the graphs in different cases? What common conditions do they share?<br><br>2. Can you construct the conditions for $p > 0$ for all $x > 0$ mathematically using the inequalities? What can you observe about the x-intercept(s), y-intercept, the discriminant, etc., of these graphs that satisfy $p > 0$ for all $x > 0$? |

| **Pointers for teachers during the …** | |
|---|---|
| **“Problem Solving” phase** | Teachers can consider using the facilitation questions above to get students to think about the different graphs where $p > 0$ for all $x > 0$, and how to identify and construct the algebraic conditions for each case before solving for the values of $a$. |
| **“Instruction” phase** | • Type 4 solutions consider all the possible graphs of the quadratic function $p$, and are able to identify those that satisfy $p > 0$ for all $x > 0$. The graphical information presented demonstrates the understanding of the graphical characteristics of a quadratic function.<br>• However, the graphical information is not translated to algebraic inequalities that help characterize the conditions. The solutions also do not seem to adequately demonstrate an understanding of connections between graphs of the quadratic functions and graphical characteristics of quadratic functions. |

## TYPE 5 SOLUTIONS: (Canonical solution) Graphical and algebraic representation of the possible cases of the quadratic function *p*

### An Example Solution

**Solution E**

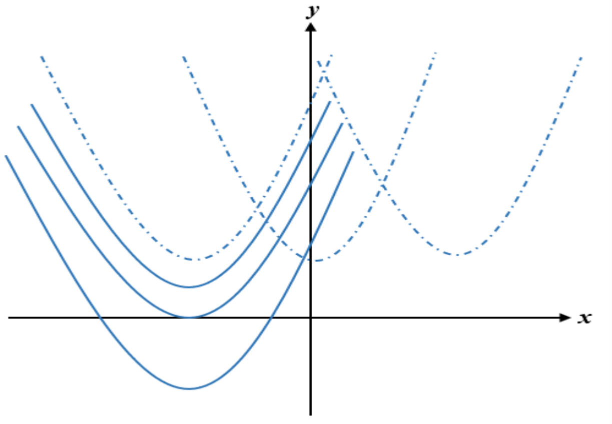
Consider the following two cases where $p > 0$ for all real values of $x$:

**Case 1**: The graphs are fully above the $x$-axis and $p > 0$ for all real values of $x$, illustrated by the three graphs in dashed lines. There are two conditions:

1. The coefficient of $x^2 > 0$ (fulfilled)
2. The discriminant of the $p$ function < 0, i.e.,

   $(3a+1)^2 - 4\,(a^2+3a+1) < 0$

Solving the quadratic inequality gives $\frac{3-2\sqrt{6}}{5} < a < \frac{3+2\sqrt{6}}{5}$ .




**Case 2:** The $y$-intercept is positive, and the symmetrical line of the graph is on the negative $x$-axis. These are illustrated by the three graphs in solid lines.

There are three conditions:

1. The coefficient of $x^2 > 0$ (fulfilled)
2. The value of $y$-intercept > 0, i.e., $a^2+3a+1 > 0$ …①
3. The symmetrical line ($x = -\frac{b}{2a}$) is on the negative $x$-axis side, i.e., $-\frac{(3a+1)}{2} < 0$ …②

Solving ① gives $a < \frac{-3-\sqrt{5}}{2}$ or $a > \frac{-3+\sqrt{5}}{2}$. Solving ② gives $a > -\frac{1}{3}$. Hence $a > -\frac{1}{3}$ (the intersection of the range for ① and ② above).

Conclusion: combining Cases 1 and 2, to ensure $p$ is always positive whenever $x > 0$, we should take the union of $\frac{3-2\sqrt{6}}{5} < a < \frac{3+2\sqrt{6}}{5}$ (Case 1) and $a > -\frac{1}{3}$ (Case 2), that is, $\boldsymbol{a > \frac{3-2\sqrt{6}}{5}}$.

### Critical ideas/processes

- Connections between algebraic and graphical representations of quadratic functions; shows understanding the graphical characteristics
- Connections between the roots of quadratic equations and the solutions of the related quadratic inequalities

# SECTION 5:

## STANDARD DEVIATION (SECONDARY FOUR)

# Section 5.1 Problem Task (Students' version)

## Football Fever

The organizers of the Premier League Federation have to decide which one of two players - Mike Arwen and Dave Backhand – should receive the "The Most Consistent Player" award. Table 1 shows the number of goals that each striker scored in a 16-year period.

*Table 1: Number of goals scored by two strikers in the Premier League between 2003 and 2018*

| Year | Mike | Dave |
|---|---|---|
| 2003 | 14 | 12 |
| 2004 | 10 | 10 |
| 2005 | 15 | 17 |
| 2006 | 10 | 14 |
| 2007 | 15 | 11 |
| 2008 | 11 | 14 |
| 2009 | 15 | 14 |
| 2010 | 12 | 15 |
| 2011 | 16 | 15 |
| 2012 | 13 | 16 |
| 2013 | 17 | 16 |
| 2014 | 13 | 13 |
| 2015 | 18 | 12 |
| 2016 | 13 | 14 |
| 2017 | 18 | 13 |
| 2018 | 14 | 18 |

The organizers agreed to approach this decision *mathematically* by designing a **measure of consistency.** They decided to get your group's help. Here is what you must do:

**(1) Design as many different measures of consistency as you can.**

**(2) Your measure of consistency should make use of all data points in the table.**

**(3) Show all working and calculations on the papers provided.**

All the best, and remember, don't give up until you have developed several ways of calculating consistency!

## Section 5.2 Unpacking the Critical Features of the Targeted Concept

| | |
|---|---|
| The **Concept** | **Standard deviation** (SD) is a measure that quantifies the *variability* or *spread* of a data set, i.e., the extent to which data points in the data set deviate from a fixed reference point.<br>(a) SD is formulated as $\sqrt{\frac{\sum(X-\bar{X})^2}{N}}$, and it concerns the degree of spread or dispersion relative to the *mean*.<br>(b) The notion of spread can be observed visually through graphical representations of the data set (e.g., histograms), and can also be informed from other related measures like range and interquartile range. |
| **Critical Features** of the concept | There are five critical features to the concept of SD:<br>(a) SD **quantifies** the degree of deviation from the mean. This provides an exact and precise indication of the variability of the data.<br>(b) SD takes the average deviation of **all data points** from the mean. The consideration of all data points gives the formula its power (i.e., more numerical information in a data set).<br>(c) SD has the **same unit** as the data set, since the final output takes the square root of the average squared deviations. This ensures that the measure has the same dimensionality as the data used.<br>(d) SD is **invariant** under the transformation of the data set (e.g., data rearrangement)<br>(e) SD is sensitive or **easily affected by extreme data values**, such as outliers. |
| **Big Ideas** underlying the concept | The concept of SD is aligned to the ideas of measures and invariance:<br>(a) ***Measures***<br>SD is a measure that quantifies the variability or spread.<br>(b) ***Invariance***<br>SD remains unchanged when the order of the data points is rearranged, or when the data are translated. |

## Section 5.3 Facilitation Moves for the CLD

Solutions produced for the problem can be classified according to the critical features of the concept. There are five critical features underlying the concept of SD: (1) Quantifying the degree of deviation from the mean [***precision***], (2) taking account of data size N and pertinent information of all data points [***power***], (3) same unit as the given data [***dimensionality***], (4) invariance under rearrangement and translation, and (5) easily affected by extreme data points [***sensitivity***]. The following pages show the classification of solution types, as well as the example solution and solution-specific teacher facilitation moves for each of the solution type. Suggested facilitation questions are provided to provoke cognitive conflict for desired learning to occur.

## TYPE 1 SOLUTIONS: Sums and Measures of Central Tendencies

**Critical Features Checklist**

| | Sum of Goals & Mean | Median & Mode |
|---|---|---|
| Quantifying the degree of deviation from the mean [precision] | ✗ | ✗ |
| Taking account of data size N and pertinent information of all data points [power] | ✓ | ✗ |
| Same unit as the given data [dimensionality] | ✓ | ✓ |
| Invariance under rearrangement and translation | ✗ | ✗ |
| Easily affected by extreme data points [sensitivity] | ✓ | ✗ |

### Example Solution

**Solution A**

***Calculate the mean, median, mode, and sum of goals for Mike and Dave.***

| | Mike | Dave |
|---|---|---|
| Sum of Goals | 224 | 224 |
| Mean | $\frac{224}{16} = 14$ | $\frac{224}{16} = 14$ |
| Median | 14 | 14 |
| Mode | 13 and 15 | 14 |

Answer: Mike and Dave are equally consistent. Their means, medians and modes are the same.

### Facilitation Questions

Consider the goals scored by the following players:

| Player 1 | Player 2 | Player 3 |
|---|---|---|
| 13, 14, 14, 14, 15 | 14, 15, 15, 15, 16 | 11, 14, 14, 14, 17 |

Players 1 and 3 have the same means, medians, and modes of 14, but the consistency is obviously different. Players 1 and 2 have different means but the same consistency.

| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | Teachers can consider using the facilitation questions above to encourage students to see that measures of central tendencies alone cannot provide an indication of consistency of data points. |
| **"Instruction" phase** | ***Affordances*** – Means, medians, and modes, which share the *same unit* with the data set given, are *possible fixed point(s)* that attempt to identify the central positions within a set of data. For the mean, it also takes into account the *sample size*.<br>***Constraints*** – Type 1 solutions *lack precision* as they do not convey the variability of the data set. Mean, median, and mode also vary under data translation (e.g., adding a constant to every value), and hence *lack invariance*. For mode and median, they only take part of the data into consideration (*lacking power*) and may not be affected by changes in individual data points (*lacking sensitivity*). |

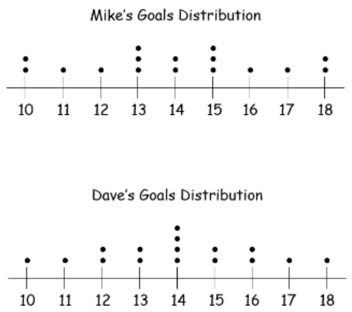
### Critical Features Checklist

- Quantifying the degree of deviation from the mean [precision]
- Taking account of data size N and pertinent information of all data points [power]
- Same unit as the given data [dimensionality]
- Invariance under rearrangement and translation
- Easily affected by extreme data points [sensitivity]

## Example Solution

**Solution B**

***Dot diagram***




Distribution: evenly spread out.

Distribution: Clustered around 14, with 14 as the highest.

Answer: Mike Arwen. His goals are spread out evenly.

## Facilitation Questions

From your tables/ diagrams, can you tell which player is the most "consistent" in his score? Why?

There are many data points here. Do you think your diagram is able to describe the "consistency" of data?

Can you describe/ summarise what you have found from your diagram/ table using a number/ formula? The number/ formula must be able to tell us which player is the most consistent in scoring.

| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | Teachers can consider using the facilitation questions above to help students evaluate their graphs and reflect on the essence of variability of data, and attempt to quantify this variability. |
| **"Instruction" phase** | ***Affordances*** – Type 2 solutions give a *visual idea* of the spread of the data. Graphical representations consider *all data points* when plotting the data. The visual presentation of the variability of the data is *invariant* if the data is rearranged or translated. However, Type 2 solutions have shown some of the critical features of the concept partially, as seen from the incomplete ticks on the checklist.<br>***Constraints*** – Graphical representations *do not quantify* the magnitude of the spread. While they make use of all data points, but they do not make salient other relevant information like distance between points. Depending on the type of representations, graphical representations *may be sensitive* to the appearance of outliers. Some misconceptions may arise when interpreting graphs, e.g., students might interpret dot diagrams with similar number of dots for all data values as a sign of consistency. |

### Critical Features Checklist

- ✓ Quantifying the degree of deviation from the mean [precision]
- ✓ Taking account of data size N and pertinent information of all data points [power]
- ✗ Same unit as the given data [dimensionality]
- ✓ Invariance under rearrangement and translation
- ✗ Easily affected by extreme data points [sensitivity]

## Example Solution

### Solution C

***Comparing the frequency of years or goals above, below, and at the mean***

| | Mike | Dave |
|---|---|---|
| No. of years player scored 14 goals | 2 | 4 |
| No. of years player scored above 14 goals | 7 | 6 |
| No. of years player scored below 14 goals | 7 | 6 |

Answer: Dave is the most consistent because he scored the most at 14 and least number of goals below and above 14 compared to the other two players.

## Facilitation Questions

Consider the following example:

| Player 1 | Player 2 |
|---|---|
| 12, 14, 14, 14, 16 | 10, 14, 14, 14, 18 |

Both the data sets above have the same number and sum of goals below and above the average (14) but they are obviously different in consistency.

| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | Teachers can consider using the facilitation questions to show how counting based on a specified criteria/ interval fails to account for relevant information like distance between data points. |
| **"Instruction" phase** | ***Affordances*** – Type 3 solutions attempt to *quantify* information relating to the frequency of data. The methods consider *all the data points*, and the frequencies obtained from counting are *invariant* if the data is rearranged or translated.<br><br>***Constraints*** – While Type 3 solutions make use all data points, they did not account for the *relative deviation from the mean*. |

## TYPE 4 SOLUTIONS: Deviations Considered

### Critical Features Checklist

- ✗ Quantifying the degree of deviation from the mean [precision]
- ✓ Taking account of data size N and pertinent information of all data points [power]
- ✓ Same unit as the given data [dimensionality]
- ✗ Invariance under rearrangement and translation
- ✓ Easily affected by extreme data points [sensitivity]

### Example Solution

**Solution D**

***Summing up/ averaging the year on year differences in performance with signs intact***

| Mike | | Dave | |
|---|---|---|---|
| 14 | -4 | 12 | -2 |
| 10 | 5 | 10 | 7 |
| 15 | -5 | 17 | -3 |
| 10 | 6 | 14 | -3 |
| 15 | -5 | 11 | 3 |
| 11 | 5 | 14 | 1 |
| 15 | -4 | 14 | -1 |
| 12 | 4 | 15 | 1 |
| 16 | -4 | 15 | -3 |
| 13 | 5 | 16 | 4 |
| 17 | -5 | 16 | -3 |
| 13 | 6 | 13 | 3 |
| 18 | -5 | 12 | -2 |
| 13 | 5 | 14 | -1 |
| 18 | -4 | 13 | 5 |
| 14 | | 18 | |

Most consistent = Least sum of differences
Mike = 0
Dave =6

Answer: Mike has least ups and downs, so he is the most consistent.

### Facilitation Questions

Consider the following example:

Suppose Player 1's goal distribution looks like this: 13, 14, 15. His sum of year on year differences = 2. However, if the order of his goals is changed to 13, 15, 14, his sum of year on year differences = 3. Although the number of goals is the same in both cases, the solution changes when the order changes.

| Pointers for teachers during the … | |
|---|---|
| **"Problem Solving" phase** | Teachers can consider using examples of different scenarios to help students see the limitations of the solution methods. |
| **"Instruction" phase** | ***Affordances*** – The solutions attempt to *quantify* consistency and capture information of *relative distance* from particular points of data. They have the *same units* with the given data and give measures that are *invariant* if the data is translated. It take *all information in the data points* into consideration.<br><br>***Constraints*** – The solution uses previous years' scores as reference points, which are not fixed. The values will therefore change when the data is rearranged (*lack invariance*). |

## Solution E

***Variance and standard deviation (SD)***

Mike $\dfrac{[(14-14)^2 + (10-14)^2 + (15-14)^2 + (10-14)^2 + (15-14)^2 \ldots (14-14)^2]}{16} = 6.00$ (Variance)

$\sqrt{6.00} \approx 2.45$ (SD)

Dave $\dfrac{[(12-14)^2 + (10-14)^2 + (17-14)^2 + (14-14)^2 + (11-14)^2 \ldots (18-14)^2]}{16} = 4.38$ (Variance)

$\sqrt{4.38} \approx 2.09$ (SD)

## Solution K

*Mean absolute difference (MAD)*

**Mike:**

$$\frac{|(10-14)|+|(15-10)|+|(10-15)|+|(15-10)|+|(11-16)|\ldots\ldots+|(14-18)|}{16-1} \approx 4.27$$

**Dave:**

$$\frac{|(10-12)|+|(17-10)|+|(14-17)|+|(11-14)|+|(14-11)|\ldots\ldots+|(18-13)|}{16-1} \approx 2.13$$

Answer: Dave is the most consistent as his average of fluctuations is the least.

### Critical Features Checklist

- ✓ Quantifying the degree of deviation from the mean [precision]
- ✓ Taking account of data size N and pertinent information of all data points [power]
- ✓ Same unit as the given data [dimensionality]
- ✓ Invariance under rearrangement and translation
- ✓ Easily affected by extreme data points [sensitivity]

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