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‘Counting up’ Strategy as an Alternative to Solve Subtraction Problems with Regrouping

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Abstract. This paper intends to explore how the ‘counting up’ strategy on an empty number line could help children overcome their mistakes in performing column subtraction problems that require regrouping. Solving subtraction problems with regrouping in column strategy is often frustrating for most children because it involves regrouping numbers in place value. In this qualitative case study, how two second graders develop their thinking while engaging in a designed instructional activity were observed and analyzed. The data collection included observations, conversational interviews, and children’s written works. The results showed that the designed activity with a measurement context and an empty number line model is found to be helpful for children to develop the idea of ‘counting up’ in solving subtraction problems that require regrouping. The children gradually develop their counting up strategy from jumps of ones to jumps of tens to simplify the use of the empty number line model. The results help us understand that we must provide a meaningful support for children to explore various calculation strategies to improve their flexibility in solving subtraction problems.

INTRODUCTION

Inability in handling numbers flexibly in daily life situations has been a major concern for decades. When children start their primary school, they are introduced to the number sequence and counting of objects one by one, followed by addition and subtraction of single-digit numbers by counting on and counting back strategies using concrete materials or their fingers. Once they learn place value, teachers usually show them how to perform the column strategy or algorithm for solving addition and subtraction of two-digit or multi-digit numbers. Previous studies [1-3] agreed that the calculations become more problematic for children as they start regrouping numbers in solving addition and subtraction problems. Therefore, introducing the algorithm strategy prematurely without context problems and sufficient understanding of numbers can cause an overreliance on the use of algorithm and issues with addition and subtraction as a consequence [4, 5].

Based on literatures [6, 7], the operation of addition can be classified into two types, i.e. the aggregation (finding a total of two quantities) and the augmentation (finding an increased value), whereas the operation of subtraction has four different structures, i.e. partitioning (e.g. “how many are left?”), reduction (e.g. “what is the reduced price?”), comparison (e.g. “what is the difference?”), and inverse-of-addition (e.g. “how many more needed?”). The National Council of Teachers of Mathematics [8] proposed that students should have a meaningful understanding of numbers and operations as well as use efficient and accurate methods for computing, either using mental strategies or an algorithm. Apart from the strategy they use, students should be able to explain their strategies and be flexible in choosing the best strategy that fits for certain numbers. For example, the counting back strategy is considered more efficient to solve subtraction of “91 – 3”, but to find the difference of “91 – 89”, one might think that the counting up is a better method. While the algorithm is best to solve “56789 – 12345”, but to find the amount of money left after spending Rp29,850 and paying with a Rp50,000 cash (Rp = *Rupiah*) will suggest a different method than algorithm. Thus, we need to provide children with various situations to explore meaningful strategies in performing arithmetic operations.

Studies associated with counting strategies up to 20 [9-13] revealed children’s informal strategies and the utilised number facts in solving additions and subtractions. Other studies associated with counting multidigit numbers [1, 7, 14-17] indicated the use of models and contexts to help children acquire the place value concept as well as the number structure to lay the foundation for more formal strategies, including mental arithmetic strategies. Fuson *et al*

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[18], Treffers and Buys [19], Buys [20], and Thompson [21], defined three levels of calculations in solving addition and subtraction problems up to 100. At the bottom level, children solve addition and subtraction problems by counting one by one. For example, "13 – 5" is calculated as "13, 12, 11, 10, 9, 8" with the counting back strategy, starting from thirteen and counting back five times. The difficulty is that children must keep track accurately of how many they have just added or subtracted while counting. At the next level, children apply calculation by structuring and utilize methods of using tens, such as *decompose-tens-and-ones method* (splitting strategy) or *begin-with-one-number method* (stringing strategy), or even varying strategy, *mixed-method* and *change-both-numbers method*. At the formal level, children can carry out mental calculations or apply the algorithm method correctly.

Calculation by structuring can be introduced using the empty number line model and researchers suggested that this should precede the use of algorithm [15, 22]. Clements *et al* [23] pointed out that there is a positive correlation between children's strategy diversity in arithmetic and their achievement. However, in fact, calculation by structuring that leads to the mental arithmetic strategies, is not highlighted in the students' mathematics textbook in Indonesia or in the curriculum for primary mathematics. Thus, primary teachers usually skip the level of calculation by structuring in their teaching practice in the classroom and move on to the algorithm straight away. The present study limits its scope to the counting up strategy as an alternative to solve subtraction problems with regrouping. Subtraction problems with regrouping often cause problems for children and have been identified as subtraction difficulties, thus the counting up strategy was proposed as an alternative [5, 9, 24]. Furthermore, the empty number line model introduced by the measurement context has been suggested as a meaningful model to represent children's strategies and to develop the counting up strategy in solving subtraction problems [14, 16, 22]. Therefore, the present study attempts to answer: How do children develop the counting up strategy on an empty number line to solve subtraction problems with regrouping using the measurement context?

METHOD

Participants

In this study, two second graders, namely Karen and Nina (pseudonyms) were observed during a two-hour activity. They were identified by their teacher as having difficulties in solving subtraction problems that require regrouping. Karen and Nina were at the age of eight and they were classmates in an Islamic private school in East Jakarta, Indonesia. Like many other schools, their school implemented the national curriculum from the Ministry of Education and Culture of Indonesia. This study was conducted in the middle of their second semester at the second grade, and they had been taught the algorithm strategy since they were at the first grade. Based on the curriculum, children at the second grade should be able to perform calculation up to 500. During the observation, Karen and Nina were constantly writing the minuend on top of the subtrahend whenever a subtraction problem was given. It gave us a hint that the algorithm could be the only means they know in explaining and writing solutions. Additionally, as average students, they basically knew how to perform the algorithm in solving subtraction problems of multidigit numbers that required regrouping, but several mistakes were identified during the study whenever a specific type of problems was given.

Procedures

In this qualitative case study [25], the data were collected by conducting careful observations, conversational interviews, and collecting children's written works. The whole activity was videotaped and the critical moments during the activity were documented and reported. Partial interviews were transcribed, specifically regarding on children's mistakes and evidences on how they constructed the idea of the counting up strategy as an effort to manage their mistakes. The qualitative analysis focused on the children's thinking development in constructing the counting up strategy on an empty number line introduced by the measurement context.

This study was conducted in a two-hour instructional activity after school hours. The sequence of activities and instructions were designed carefully to introduce children to the notion of the number line model in relation to the measuring context. By exploring the position of numbers and number relations, children were expected to develop the idea of 'counting up' strategy via tens or hundreds in solving subtraction problems. Following were the subtraction problems given during the study: "32 – 18", "61 – 59", "44 – 29", "93 – 27", "111 – 88", "102 – 78", "123 – 57", "123 – 117", "101 – 93", "100 – 67". Most of the problems were contextual, while some were not. The following section will elaborate further how the children solved the problems, what mistakes and difficulties occurred, and how the children constructed the ‘counting up’ strategy.

RESULT AND DISCUSSION

In the beginning of our study, we did not see any mistakes the children performed in solving two digit-subtraction problems with regrouping. They carefully undertook the algorithm procedures and solved the problems correctly. We can see from figure 1(a) and 1(b) that they performed the same procedures, the only difference is that Nina did not write the tens digit after regrouping while Karen wrote the tens digit after regrouping and signified the movement of borrowing from the tens to ones using a small curve on top of the minuend. By observing how they lifted their fingers while counting “13 – 7”, it was found that both Karen and Nina did the same counting back strategy, that is, lifting seven fingers then counting back from thirteen until all the seven fingers lifted were counted. This fact suggests to us that they did not know the counting up strategy before, and they did not apply the subtraction facts for numbers up to 20 in determining the difference.

(a) (b)

FIGURE 1. (a) Nina’s solution for 2-digit subtractions, (b) Karen’s solution for 2-digit subtractions

Nina’s explanation for Figure 1(a) “93 – 27”:

> *“I can not take away seven from three, so I borrow one to make thirteen, then seven is subtracted from thirteen equals six. Since nine has been taken away one, so now it is eight, eight take away two is six. So the answer is sixtysix.”*

However, it was found later that whenever three digits minuend and two digits subtrahend of a subtraction problem were given to Nina, the mistake associated with misalignment was apparent. Unlike Nina, Karen’s mistake was obvious when solving subtraction problems with zero digit in the minuend. The children apparently performed the algorithm and treated numbers as concatenated single digits without values, and thus solved the subtraction problems without reflection or understanding [1, 26]. Hence, the ‘counting up’ strategy was promoted after they experienced working with the number line in exploring number relations. The measuring activity and the emergence of number line model is described in the following section.

The Measuring Activity and the Number Line Model

The sequence of activities started with measuring ribbon using the paper strip with tens structure, followed by positioning the numbers on the almost empty number line, finding the position of any given numbers on the number line to explore number relations, and then solving subtraction problems that required regrouping with or without the context of measurement. In exploring number relations, the children were asked to show where is the position of number 48 on the number line. By discussing how children got to the number 48, they could explore that 48 is closer to 50 than 40, that is 48 = 50 – 2 or 48 + 2 = 50. This kind of discussion will eventually scaffold children’s thinking so that they can use 50 as a reference point which will be useful for the later discussion.

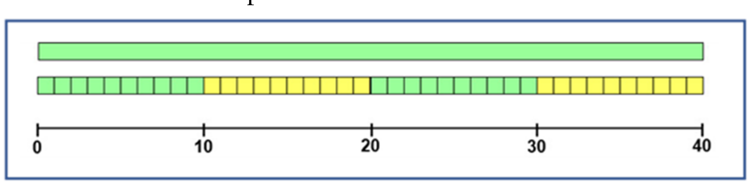
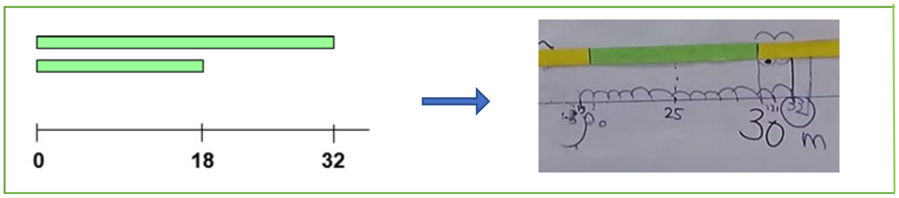


FIGURE 2. The emergence of the number line from a paper strip to measure the length of ribbon

Next, a subtraction problem that required regrouping was posed using the measurement context, i.e., "*If Karen has a 32-unit length of ribbon, and Nina has an 18 square length of ribbon, find the difference*." Nina was confident in solving the problem using the column strategy, and she was able to solve it correctly. However, although Karen knew how to solve it using the algorithm, she did not perform the algorithm. It seemed that she was inspired by the presentation of the ribbons and the number line (figure 3(a)) on the whiteboard, so her finger indicated that she performed counting one by one from a distance. After clarifying what Karen was doing, the instructor used Karen's strategy as a starting point to introduce jumps of ones on the number line from 18 to 32 (figure 3(b)). The children were asked to count up out loud from 18 to 32, while at the same time, the instructor was illustrating the jumps on the number line. This was one of the critical moments when the counting up strategy first emerged.



(a) (b)

FIGURE 3. The representation of Karen's counting strategy

Resolving Problems using the Counting Up strategy on the Empty Number Line Model

Nina and Karen were at the same class and were taught by the same teacher at school. They both were considered as average students, but they had different difficulties in solving certain kinds of subtraction problems. Nina was found to have difficulties in solving subtraction problems with three-digit minuend and two-digit subtrahend (figure 4). She constantly made the same mistake associated with the place value concept, i.e., misalignment. If we look at figure 4(a), she simply brought down the last digit of the minuend "1" and perform the algorithm for the subtraction between the first two digit of the minuend and the subtrahend. Because 8 cannot be subtracted from 1, so she indicated 11 as the result of borrowing from the hundreds place and gave 3 as the result of "11 – 8". Since there was no more left at the hundreds place of the minuend, she brought down the "8" of the subtrahend. The subtraction problem on figure 4(b) was supposed to be "123 – 57", and the mistake was obviously the same. Only the problem on figure 4(c) that seems uncomplete because she did not have any idea if the zero was in the middle of the number. The reason why she did not complete the answer shown in 4(c), because later she found out an easier way to solve it, that is the counting up strategy on the number line.

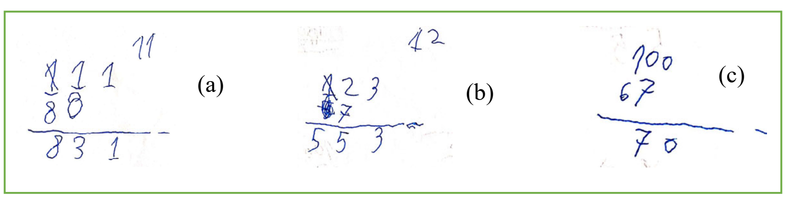
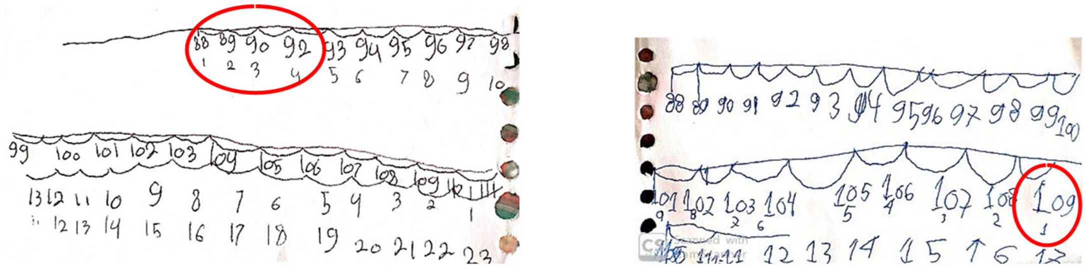


FIGURE 4. Nina's mistakes dealing with subtractions with different number of digits

"111 – 88"

The first mistake Nina encountered was when a three-digit minuend and a two-digit subtrahend problem "111 – 88" given (see figure 4(a)), she misaligned the position of the minuend and the subtrahend according to their place value. Thus, an idea of using the counting up strategy was proposed to the children, based on their prior discussion in solving "32 – 18". After thinking a while, the children started to draw a line and put the number 88 as a starting point. They drew jumps of ones and count from 88. Unfortunately, the space on the worksheet was not enough, so they continued drawing the second line under the first line. By having a closer observation on the children's drawing, we can interpret that they were struggling in aligning the jumps with the counting up process. Moreover, when the line was disconnected, the jump between the two consecutive numbers was not counted. Instead of counting the jumps, they counted the numbers that they have written out. Thus, during the discussion, Nina came up with 24 as the answer, instead of 23 (figure 5(b)). Unlike Nina, Karen gave 23 as the answer. But it was found out later that she did miscounting, because the number 91 was absent on her number line.



(a) (b)

FIGURE 5. Karen's (a) and Nina's (b) first attempts in performing the counting up strategy on an empty number line.

Their first attempts in drawing the number line looks cumbersome and exhausting. Moreover, if they performed jumps of one, the possibility of making mistake such as skip counting is bigger. Therefore, the instructor tried to persuade them to use jumps of tens and jumps via tens in their counting up strategy. Several problems were given until Nina was quite confident in performing jumps of tens and jumps via tens. However, Karen was still skeptical in performing jumps of tens on the empty number line.

"102 – 78"

After discussing the problem "111 – 88", several problems were posed. Instead of using the counting up strategy, they kept doing the algorithm in solving the subtraction problems of three-digit minuend and two-digit subtrahend. Thus, the instructor kept asking them to try using the counting up strategy on the number line to compare the solution. Unlike Nina, Karen did not have any issue with the misalignment. She was able to perform the algorithm correctly in almost all subtraction problems given, until the following mistake was found.

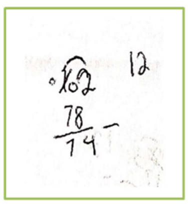


FIGURE 6. Karen's mistake in solving "102-78"

Karen's explanation for Figure 6:

> *"Because I can not subtract eigth from two, so I borrow from one* (the digit one in hundreds place), *so it becomes zero, and it becomes twelve* (the number two at ones place). *Eight is subtracted from twelve equals four. Because there is nothing left here* (in the minuend)*, so I bring down seven directly to get the result of seventy four.*

The excerpt above shows Karen's mistake in solving subtraction problems with regrouping when there is a zero digit in the minuend as in figure 6. She thought that when she could not borrow from the tens place, she could borrow from the hundreds place right away without understanding that exchanging one from the hundreds place gives 100 in the ones place. However, a discussion took place between the instructor, Karen, and Nina. The instructor tried to confirm Nina's answer (figure 7) and asked Karen to think once again. Although Nina was unsuccessful in solving the problem with the algorithm, but she decided to use the 'counting up' strategy on the number line and succeed to find the correct solution.

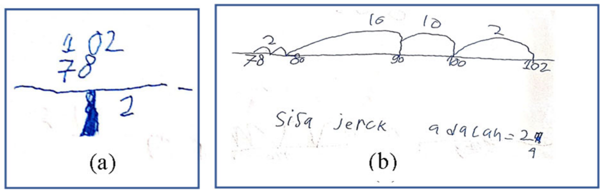


FIGURE 7. Nina's solution for "102 – 78"

Meanwhile, Karen was thinking and trying to make sense the problem by drawing her number line. This time, she counted the number of jumps instead of the numbers itself. She signified a span from 78 to 80 by drawing a bigger leap and wrote 2 above the number line. Then she continued to count up by ones, but she only wrote the tens numbers such as 90 and 100. She consciously drew ten small jumps between 90 and 100, and signified the position of 100, 101, and 102 by a small stroke in between the jumps. She suddenly shouted out "*wait, wait, I made a mistake*". She pointed to her number line and showed her final answer (figure 8). She succeeded to resolve the problem using the 'counting up' strategy on an empty number line.

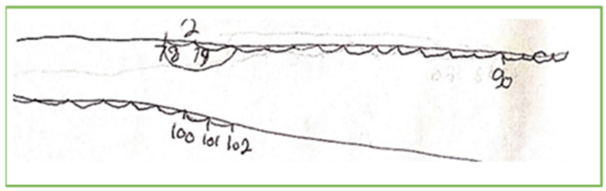


FIGURE 8. Karen's counting up strategy from 78 to 102

At that time, she was unsure whether one big jump of ten will cover the same distance as ten small jumps of one, thus she avoided performing a jump of ten. From figure 8, we can also see that Karen wrote the numbers in between two jumps with a stroke signifying the positions. This indicates that she already has a conceptual understanding of measuring a distance, that is accumulation of distance [14]. In the next discussion, Karen was encouraged to perform jumps of ten to simplify her counting up strategy. Below (figure 9) is one example of Karen's 'counting up' strategy by performing jumps via tens and jumps of tens on an empty number line.

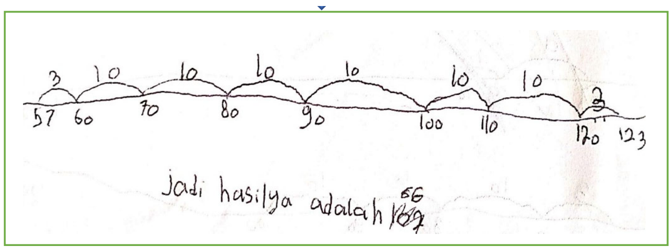


FIGURE 9. Karen's 'counting up' strategy on an empty number line to solve "123 – 57"

The findings of this study is in line with [27] and [28], i.e. although children have been exposed with classroom's instructions in developing mental arithmetic strategies, their overreliance on the algorithm has produced ineffective ways in solving multi-digit subtraction problems. They were constantly putting numbers in column order and trying to solve every subtraction problem by the algorithm. However, children's common mistakes in performing algorithm was obvious and thus teachers must be aware of the difficulties and provide a meaningful situations and model to help children develop the idea of calculation by structuring before they are taught the algorithm strategy.

CONCLUSION

This study helps us understand how a meaningful instruction can be used to enhance children's thinking to exploit an alternative strategy, i.e. the 'counting up' strategy to solve subtraction problems that require regrouping. From the measurement context, the number line emerges as a model of the situation to represent the length measurement, and to explore number relations. Later, the number line model is used as a model for representing children's 'counting up' strategy in solving subtraction problems. The number line model can be used as a representation to scaffold children's strategy from counting by ones to counting by tens or via tens in performing the 'counting up' strategy. This study found that when children solved subtraction problems with the algorithm, they handled numbers as digits and often ignored the value of numbers, thus the mistakes happened. In contrast, although it was cumbersome to work with the number line model in the beginning, the number line model allowed children to treat numbers as values. In addition, teachers should play an important role in initiating children's discussion and guiding the discussion whenever there is a room for improving children's thinking. Hence, teachers must have a sufficient understanding of pedagogical and content knowledge of mathematics concepts.

We acknowledge that this is a small-scale study with only two children as participants. Thus, a recommendation for further research would be conducting this study in a classroom situation by addressing the psychological development of children's thinking as individual and the development of classroom community in a social-constructivism perspective.

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