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MATHEMATICALLY-RICH GAMES

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ABSTRACT

In this paper, I will discuss the use of mathematically-rich games to develop in students certain skills and processes that are important in their daily and future workplace life. For example, students will learn through these games how to pose relevant and important questions when faced with a problem, how to formulate conjectures to solve the problem, what strategies or heuristics to use, and how to monitor their progress and their own thinking. The context is very real for these students because the outcome, whether they win or lose, matters to them.

1. INTRODUCTION

In 1997, the Ministry of Education in Singapore formulated a grand vision, *Thinking Schools, Learning Nations* (TSLN), which described a nation of thinking citizens capable of meeting the challenges of the future (Ministry of Education, Singapore, 1997). Since 2003, MOE has focused more on one aspect of TSLN: nurturing a spirit of *Innovation and Enterprise* (I&E) which will help to build up a core set of life skills and attitudes that we want in our students (Ministry of Education, Singapore, 2007). Then in 2004, Prime Minister Lee Hsien Loong

called on our teachers to "teach less to our students so that they will learn more" (Singapore Government, 2004). *Teach Less, Learn More* (TLLM) "builds on the groundwork laid in place by the systemic and structural improvements under TSLN, and the mindset changes encouraged in our schools under I&E" (Ministry of Education, Singapore, 2007). All these initiatives serve to emphasise the intention of the Singapore government and its Ministry of Education to nurture thinking and innovative students. In fact, as far back as 1988, in a paper 'Agenda for Action: Goals and Challenges' presented to the Parliament by then First Deputy Prime Minister Goh Chok Tong, he described the role of education was "to nurture inquiring minds, and to create a lively intellectual environment which will ultimately spread throughout Singapore society" (Yip & Sim, 1990, p. 3).

The central theme of the Pentagon Model, which is the framework of the Singapore mathematics curriculum, is mathematical problem solving (Wong, 1991). This also illustrates the emphasis placed on developing thinking and creative students through solving mathematical problems. Unfortunately, standard textbook exercises are not exactly real problems that stimulate critical thinking (Skovsmose, 2002). So there has always been emphasis on using some non-routine problem solving in mathematics teaching and assessment (Schoenfeld, 1985).

In this paper, I will look at the use of mathematically-rich games to engage students in their minds and hearts: their minds because these games promote mathematical, logical, analytical and creative thinking that are useful in daily and working lives; their hearts because most people like to play games (Ainley, 1988, 1990). I will begin with a review of some literature on the purpose of mathematically-rich games and then I will describe how two particular games can develop in students certain important thinking processes.

2. MATHEMATICALLY-RICH GAMES

Many mathematics educators (e.g. Frobisher, 1994; Skovsmose, 2002) believe in making mathematics real to the students. An example is to let students play mathematically-rich games such as the game of Nim and the Tower of Hanoi, which I will describe in Sections 3 and 4 in more details. The context is very real for these students because the outcome, whether they win or lose, matters to them (Ainley, 1988, 1990). So they may become more interested to find a strategy to win a game (Civil, 2002). When this happens, the students will be engaging in mathematical investigations and problem solving (Skovsmose, 2002). These types of activities parallel what academic mathematicians do in their daily working lives (Arcavi, 2002). Mathematics educators (e.g. Moschkovich, 2002) who believe in bringing academic mathematics to the level of the students will find that they can combine mathematicians' mathematics in an activity that makes sense to the students by letting them play a variety of mathematically-rich games (van Oers, 1996).

When students try to find a winning strategy, they may learn heuristics such as working backwards, looking for patterns and considering all possible scenarios (Ainley, 1988, 1990; Pólya, 1957); and processes such as specialising, generalising, conjecturing, justifying (Mason, Burton & Stacey, 1982) and problem posing (Silver, 1994), which I will describe in more details in Sections 2 and 3. The ability to synthesise a plan to solve the problem of finding a winning strategy is a creative process (Pope, 2005). All these heuristics, processes and thinking skills are important in daily and working lives (Carraher & Schliemann, 2002; Masingila, 2002).

3. THE GAME OF NIM

There are many versions of the game of Nim (Civil, 2002). In one version, two players take turn to take one, two or three pieces from a single pile of 12 pieces. The player who takes the last piece is the winner. Students are first exposed to the game and when the teacher keeps winning, some students may start wondering whether there is a winning strategy. Then the students can be led to investigate what the winning strategy is.

One of the heuristics used is working backwards. The students will need to ask themselves, "What is the end game like?" They may discover that the end game is to leave four pieces. If your opponent takes one piece, then you take all the remaining three pieces; if he or she takes two pieces, then you take all the remaining two pieces; if he or she takes the maximum three pieces allowed in this game, then you take the remaining one piece. So, in all the three possible scenarios, you win. This is another heuristic: consider all possibilities. Then the students need to ask themselves, "What should I do to ensure that there will be four pieces left in the end game?" This is yet another heuristic: break up the main problem into smaller problems to solve the smaller ones first. The students may then discover that they must leave eight pieces for the opponent to take. So, by working backwards and looking for patterns, the students may discover that the number of pieces they should leave for their opponent to take is a multiple of four because they should leave groups of four for their opponent to take at every turn of the game. In trying to find a winning strategy, the students not only engage in problem-solving heuristics but also in problem posing: how to ask relevant questions and pose smaller problems to solve first. Moreover, the students also learn some mathematical content: multiples. It is amazing how a simple game like this allows students to learn so many problem-solving heuristics, problem-posing techniques and even mathematical content.

But the game does not end here. What if the starting pile has 100 pieces? Should you start first or should you let your opponent starts first? It is impossible to examine all the possibilities for so many pieces but if the students have understood the winning strategy properly, then it is just a matter of checking whether 100 is a multiple of four or not. In this case, it is and so you should let your opponent starts first. But what if your opponent wants you to start first? This is a real-life problem. We need not just stop at the technicality of the winning strategy. We may need to address this real issue. One solution is to go along with your opponent's wish and wait for him or her to make a mistake. If your opponent does not know the winning strategy, then it is very likely for him to make many mistakes along the way, especially if there are so many pieces. If your opponent knows the winning strategy, then you just hope he makes a minor slip because with so many pieces, it may not be possible for him to ensure that the remaining pieces will always be a multiple of four. If he or she is careful, too bad, you lose.

This brings up another problem: if there are 100 pieces, how do *you* ensure that the remaining pieces will always be a multiple of four if you start first? It is not easy to always count the remaining pieces. But you can count the pieces being taken away and this must always be a multiple of four. In fact, at each stage, the total number of pieces taken away by your opponent and you must always be four. So if your opponent takes one piece, you take three pieces; if he or she takes two pieces, you take two pieces; if he or she takes three pieces, you take one piece. In this way, you will always ensure that the number of remaining pieces is a multiple of four.

But the game does not end here. What if the starting pile has 100 pieces and each player can take one, two, three or *four* pieces? Do students know how to find the winning strategy? Do they take 100 pieces and play a few games to find a winning strategy? Or do they know how to extend their previous finding and generalise? I have tried out this game with some pre-service and in-service teachers and some of them did not know how to generalise. In fact, a few of them thought that the remaining pile should still be a multiple of four! This suggests that these teachers might not have fully understood the winning strategy of the previous game: they thought that the pattern of multiples of four was a must, even when the rule of the game was changed.

"Generalisations are the life-blood of mathematics" (Mason, Burton & Stacey, 1982). But do students try to generalise on their own or will they wait for their teacher to pose them the question? When they do try to generalise, do they examine their conjecture properly through some deductive reasoning or do they rely on just inductive inference by looking at patterns which may turn out to be wrong? Students need to monitor their own thinking carefully or else they may go down the wrong track. These are important cognitive and metacognitive processes that can be developed through trying to find a basis for the winning strategy and these thinking processes are important in working lives (Carraher & Schliemann, 2002; Masingila, 2002).

Sometimes, we can even generalise further. For example, in this game, we can extend to the case where we have two piles of 12 pieces and two players will take turn taking one, two or three pieces from any one of these two piles. At each turn, the player cannot take the pieces from both piles. How does that change the winning strategy, if any? What if you have *m* piles of *n* pieces each? What if the number of pieces in each pile is different? Will this change the

winning strategy? In all these cases, there is not much change to the winning strategy although in the latter scenario, it may take a few moves to achieve a winning position.

But what if you can take any number of pieces from each pile containing different number of pieces? This will change the winning strategy quite a bit. An even more interesting variation is when you take out some pieces from a pile, you may break that pile into two separate piles. How can this be done? Figure 1 below shows a common game that some students in Singapore play but it is actually a more complicated version of the game of Nim. This game consists of a few rows of sticks. You can have as many rows as you want. Usually, there is one more stick as you go down each successive row, but there is actually no restriction: you can have one stick in the first row, three sticks in the second row, four sticks in the third row, etc., if you want.

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Figure 1. Game of Nim

Two players take turn to cancel any number of *consecutive* sticks in any row. At each turn, the player cannot cancel the sticks in more than one row. If a player cancels any consecutive sticks in a row which do not involve the two end sticks, the row is broken up into two rows or piles (see Figure 2). Then the players cannot cancel any consecutive sticks across these

‘cancelled sticks’: they must treat the row as two new rows now and they can only cancel any number of *consecutive* sticks in any one row at a turn. The player who cancels the last stick wins. (In another version that is commonly played in Singapore, the player who cancels the last stick loses. This will involve a slight change of strategy at the end game but I do not want to complicate the issue here. I just use the version that is similar to the game of Nim above.)

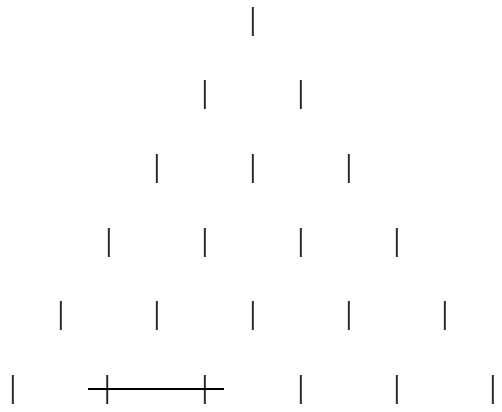

Figure 2. Last row became two separate rows

As you may have observed, this game is actually a complicated version of the game of Nim where there are more than one pile (or row) of pieces (or sticks) where the number of pieces in each pile is different. At each turn, the player can remove *any* number of pieces from any pile. Each pile can be broken into two separate piles if the pieces taken from the pile at each turn do not involve the two end pieces. Students can also investigate whether there is a winning strategy for this version of the game of Nim. In fact, there is a winning strategy which involves a complicated theoretical basis and a simpler practical approach, based on this theoretical basis, of maintaining a certain structure for the opponent, but it is beyond the scope of this paper to explain this.

4. TOWER OF HANOI

The Tower of Hanoi was invented by Edouard Lucas, a French mathematician, in 1883 (Lawrence Hall of Science, 2004). He was inspired by a Hindu legend which said the priests in a temple were given a stack of 64 gold discs, each one a little smaller than the one beneath it, and they were to transfer the 64 discs from one of the three poles to another, with two constraints: only one disc could be moved at a time and the smaller disc could not be placed under a bigger disc. When the priests finished this task, the world would end. So this puzzle was sometimes referred to as the Tower of Brahma (because the Hindu worshipped the god Brahma) or the End of the World Puzzle. A prototype of the Tower of Hanoi is shown in Figure 3 below.

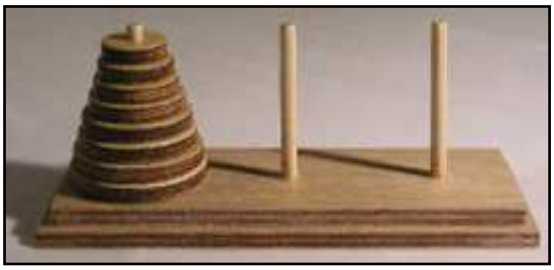

Figure 3. Tower of Hanoi

The first task is to find out how to move the discs from the first pole to the last pole. Heuristics involved include simplifying the problem by using a smaller number of discs first, looking for patterns, working backwards and using logical reasoning. For example, if we start with five discs on the first pole, to move the fifth disc to the last pole, we must somehow or rather move all the first four discs to the middle pole and then we can move the fifth disc from the first pole to the last pole (see Figure 4).

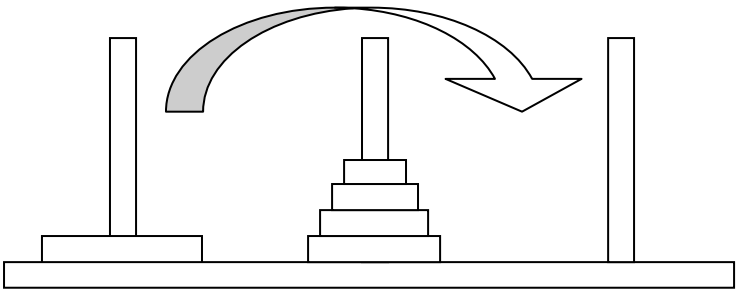

Figure 4. Fifth disc goes to last pole

But to move the first four discs to the middle pole, we must somehow or rather move the first three discs to the last pole and then we can move the fourth disc from the first pole to the middle pole (see Figure 5).

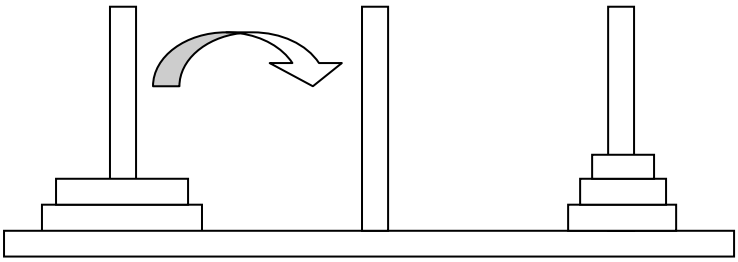

Figure 5. Fourth disc goes to middle pole

But to move the first three discs to the last pole, we must somehow or rather move the first two discs to the middle pole and then we can move the third disc from the first pole to the last pole (see Figure 6).

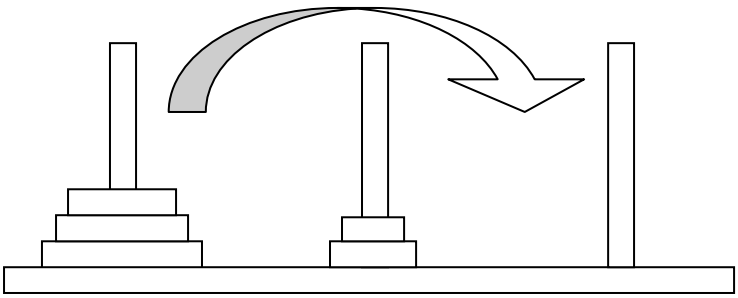

Figure 6. Third disc goes to last pole

But to move the first two discs to the middle pole, we must move the first disc to the last pole and then we can move the second disc from the first pole to the middle pole (see Figure 7).

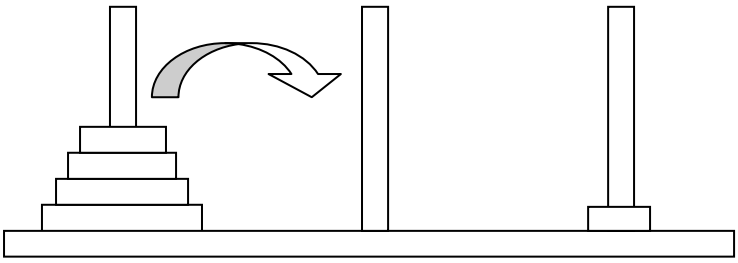

Figure 7. Second disc goes to middle pole and first disc goes to last pole

So we observe that there is a pattern. The fifth disc must go to the last pole, the fourth disc to the middle pole, the third disc to the last pole, the second disc to the middle pole and the first disc to the last pole (see Figure 8). And there is a logical reasoning for why this pattern occurs which I have already explained.

Figure 8. Pattern Searching in Tower of Hanoi

This argument is very powerful. At any time when you are stuck, you can ask what the next objective is and then use the same reasoning. For example, in moving five discs from the first pole to the last pole, and you have just moved the fifth disc to the last pole (see Figure 9), what should you do next?

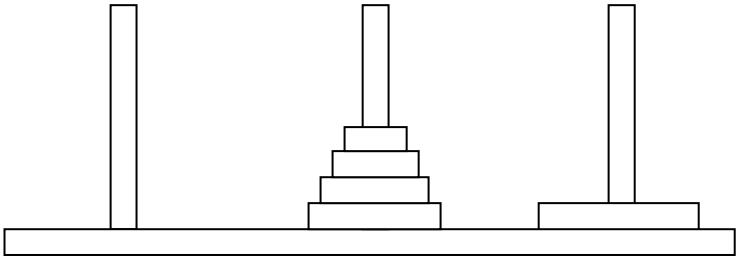

Figure 9. What is the next objective?

The next objective is of course to move all the four discs from the middle pole to the last pole. Using the same argument as above, the fourth disc will have to go to the last pole, the third disc to the first pole, the second disc to the last pole and so we have to move the first disc to the first pole (see Figure 10). If we are not careful and move the first disc to the last pole, then we will end up moving all the four discs from the middle to the first pole instead of to the last pole.

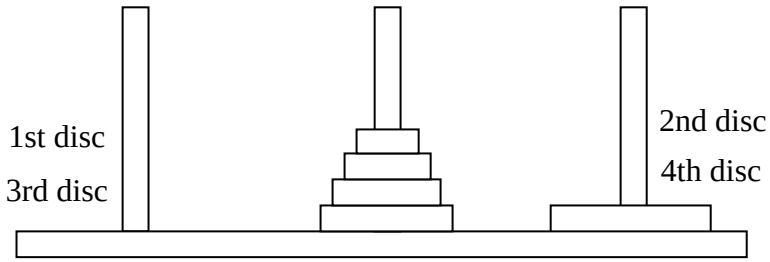

Figure 10. First disc should go to first pole

I had observed many pre-service and in-service teachers playing this game but very few of them were able to discover this method on their own. Some were able to discover this after some guidance but others had difficulty understanding this strategy even after I had explained it to them with the help of a concrete manipulative. It suggests that it is not easy to be analytical, even for mathematics teachers.

The next question is to ask what is the minimum number of steps required to move *n* number of discs from the first pole to the last pole. If you follow the steps described above without making any mistakes, then it will be the least number of steps. One way is to physically count the steps for $n = 1, 2, 3, 4, \ldots$ and try to observe if there is any pattern. The table in Figure 11 shows the minimum number of moves to move *n* discs from the first pole to the last pole.

No. of Discs, *n*	1	2	3	4	5	6
Minimum No. of Moves	1	3	7	15	31	63

Figure 11. Minimum Number of Moves

But is there a pattern in the minimum number of moves? This is yet another heuristic: try to link this sequence to an obvious sequence that you know. After some thoughts, some students may observe that each term of this sequence is one less than each corresponding term of the powers of two: 2, 4, 8, 16, 32, 64, … So the minimum number of moves to move *n* discs from the first pole to the last pole is $2^n - 1$.

But why this pattern? Is there a reason for it? If you think carefully, subtracting one in the formula $2^n - 1$ is a bit strange because we should add one instead. Some teachers were able to observe the following pattern. Suppose we start with five discs. First, we use the minimum number of steps, *m*, to move the first four discs from the first pole to the middle pole (see Figure 4 above). Then we move the fifth disc from the first pole to the last pole (this is one step). Finally, we use the minimum number of steps, *m*, to move the first four discs from the middle pole to the last pole. So the total number of steps should be $2m + 1$. But the formula involves subtraction of one instead of addition of one. Why? This is because $m = 2^4 - 1$ and

so $2m + 1 = 2(2^4 - 1) + 1 = 2^5 - 2 + 1 = 2^5 - 1$. Thus, although we add one in the above pattern, but because m is already in that form, we end up with a subtraction of two, and together with the addition of one, this gives a final subtraction of one.

Another interesting question is to ask, according to the Hindu legend described above, when the end of the world will come. Suppose the gold discs are small enough for the priests to move each disc from one pole to another pole in one second. Then it will take $2^{64} - 1$ seconds to move the 64 gold discs from the first pole to the last pole. The number $2^{64} - 1$ looks small but it is equal to 1.84×10^{19}, correct to three significant figures. So $2^{64} - 1$ seconds is approximately 5.12×10^{15} hours which is approximately 2.14×10^{14} days, and this is approximately 5.85×10^{11} years. Thus the end of the world will occur 585 billion years later, so you and I don't have to worry!

Therefore, this game not only provides opportunity for students to develop logical and analytical thinking but also to learn some mathematics which includes number patterns, powers of a number and how powerful the power of a number is: $2^{64} - 1$ may look small but the index 64 results in a very big number 1.84×10^{19}.

5. CONCLUSION

Many students like to play games. The use of mathematically-rich games can engage students in their hearts because the context is real and games are definitely more fun than working on a piece of academic mathematical work. But through playing these games, students also have the opportunity to develop critical and analytical thinking skills which are important not only in mathematics but also in their daily and future working lives. Best of all, students also learn

some mathematics through these games, and this type of investigative activities reflect what academic mathematicians do in their working lives. The use of mathematically-rich games has the potential to create a "microcosm of mathematical culture" (Schoenfeld, 1987, p. 213) in the classroom where students engage in activities that are central to academic mathematicians' practices.

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