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A Study of Pre-Service Teachers' Performance on Two Calculus Tasks on Differentiation and Limit

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Abstract: The purpose of this paper is to report a part of a calculus research project, about the performance of a group of pre-service mathematics teachers on two tasks on limit and differentiation of the trigonometric sine function in which the unit of angle measurement was in degrees. Most of the pre-service teachers were not cognizant of the unit of angle measurement in the typical differentiation formula, and a number of participants recognized the condition on the unit of angle measurement but did not translate this to the correct procedure for performing differentiation. The result also shows that most of the participants were not able to associate the derivative formula with the process of deriving it from the first principle. Consequently, they did not associate it with finding $\lim_{x\to 0}\frac{\sin x}{x}$. In the process of evaluating this limit, the pre-service teachers exhibited further misconceptions about division of a number by zero.

Keywords: *Differentiation; limit; procedural knowledge; conceptual knowledge.*

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Introduction

Calculus is an important component for undergraduate Science, Technology, Engineering and Mathematics (STEM) education. Students' success in their STEM education depends very much on their conceptual knowledge of calculus (Sebsibe & Feza, 2020). It is also one of the key strands in the Singapore mathematics curriculum for upper secondary and the pre-university levels. However, relatively little is reported about calculus education in Singapore. In particular, students' attainment of calculus knowledge is rarely reported. It appears that much emphasis on research in Singapore mathematics education has been placed on elementary mathematics topics in the primary and lower secondary mathematics curriculum. This paper provides a glimpse into the knowledge of differentiation of the trigonometric sine function and the evaluation of a corresponding limit as part of the differentiation process of a group of pre-service teachers (PSTs) through analysing their performance of two calculus tasks.

This report is part of an ongoing research project on studying pre-university and beginning undergraduate students' calculus content knowledge. The participating PSTs were undergraduate student teachers at the start of the four-year degree program offered by the National Institute of Education, the sole teacher education institute in Singapore. They read mathematics as one of the academic subjects in their undergraduate studies. They had read O-Level Additional Mathematics in their secondary schools, and had recently graduated from the pre-university with A-Level Mathematics, in which calculus is one main component in the curriculum. The PSTs' performance in the calculus items could serve as a proxy for their school calculus content knowledge up to the pre-university level prior to their admission to the undergraduate studies.

Many studies have shown that many school students lack a good understanding of school calculus (e.g., Davis & Vinner, 1986; Judson & Nishimori, 2005). In particular, the concept of limit is difficult for most students. This difficulty could be attributed to its incongruity to the common daily use of the term "limit" (e.g., Tall & Vinner, 1981). This conflict of

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understanding of the term "limit" in the school calculus and the real world sense could pose an obstacle to their understanding of applications of calculus in the real world (Burn, 2005). Students' difficulty with calculus could also stem from their weak algebraic concepts or algebraic manipulative ability (e.g., Ng & Toh, 2008). In their study with a group of Singapore pre-university students, Ng and Toh (2008) showed that many of the students' errors in typical questions on integral calculus were direct consequences of their weak algebraic manipulation skills since their secondary school education.

A teacher's competency in subject knowledge is important as it likely affects how the subject is being taught in schools (Thompson, 1992; Toh, 2007a, 2007b). Through our collective experience in pre-service and in-service mathematics teacher education, we are concerned that many school mathematics teachers could be lacking in content mastery in school calculus. This concern is not unfounded as it is substantiated by existing international mathematics education literature. For example, the studies conducted by Masteroides and Zachariades (2004) and Huillet (2005) show that many teachers in their studies had difficulties with concepts related to limits and continuity of functions. Another small-scale study in Singapore conducted by Toh (2009) on a group of Singapore mathematics in-service teachers found that the participants turned to procedures rather than rely on their underlying calculus conceptual knowledge in answering a series of several calculus questions.

Literature Review

Procedural versus conceptual knowledge in calculus

In this section, we begin by reviewing the construct of procedural versus conceptual knowledge in mathematics education literature. Both procedural and conceptual knowledge are deemed to be essential components of mathematics content knowledge (Shulman, 1986). They are considered two important aspects of mathematical understanding (Wearne & Hiebert, 1988). According to Hiebert and Lefevre (1988), procedural knowledge consists of two main components: (1) knowledge about the formal language and the system of symbolic representation used in mathematics; and (2) knowledge of the algorithms required to complete mathematical tasks, which consists of "step-by-step instructions that prescribe how to complete tasks" (p. 6). On the other hand, conceptual knowledge is that which is characterized as "knowledge that is rich in relationships" (p. 5). Conceptual knowledge enables one to make connections across various conceptualizations of a particular concept, thereby resulting in one's ability to explain the algorithms used to complete mathematical tasks (Cho & Nagle, 2017). The ability to link conceptual and procedural knowledge results in an individual being able to develop a strong mathematical content knowledge (Serhan, 2015). Lin et al. (2013) cautioned that purely using a formula or procedure to simplify the entire mathematical problem solving process usually results in the students neglecting the underlying concepts for both conceptual and procedural understanding.

Teaching for procedural knowledge means to present to the students ready made definitions, notations and procedures without first giving deeper meanings to the concepts involved (Engelbrecht et al., 2009). On the other hand, teaching for conceptual knowledge is characterized by engaging students with mathematical reasoning in order to facilitate them to make connections to their prior knowledge (Brown et al., 2002). It has been widely reported that calculus is taught with much emphasis on the procedures and formulae rather than on the understanding of the underlying concepts (e.g., Lasut, 2015; Mendezabal & Tindowen, 2018). Some researchers believed that teaching for procedural knowledge does not help students understand the basic calculus concepts (e.g., Axtell, 2006). As we shall discuss towards the end of this paper, calculus in the Singapore mathematics curriculum is likely taught with much emphasis on procedural knowledge, as suggested by the evidence of the PSTs' performance in the calculus instrument and a review of the calculus component in the school curriculum.

According to the upper secondary level Additional Mathematics and pre-university H2 mathematics calculus curriculum as documented in the syllabuses prepared by the Singapore Ministry of Education (Ministry of Education [MoE], 2018a, 2018b), substantial differentiation formulae and techniques are taught at the secondary and pre-university levels. Students are first taught elementary techniques of differentiation (chain, product, quotient and composite rules) and various differentiation formulae (polynomial, rational, trigonometric, exponential and logarithmic functions) at the upper secondary level Additional Mathematics course. These techniques are extended to include more delicate techniques (implicit and parametric differentiation) at the pre-university level H2 Mathematics course. The curricula emphasize hierarchical and progressive learning on the formulae and techniques of differentiation. A summary of the coverage of the differentiation techniques is presented in Table 1.

Table 1: Differentiation techniques in the school curriculum at the upper secondary and pre-university levels

Upper Secondary (Additional Mathematics)	Pre-University level (H2 Mathematics)
Derivatives of x^n, for any rational n, sin x, cos x, tan x, e^x and ln x, together with constant multiples, sums and differences. Derivatives of products and quotients of functions. Derivatives of composite functions.	Differentiation of simple functions defined implicitly or parametrically.

The development of calculus in the Singapore mathematics curriculum builds on the notion of graphs in both Additional Mathematics and H2 Mathematics. The introduction of differentiation through covariational reasoning, namely, "the cognitive activity involved in coordinating two quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354), is not explicitly stated in the curriculum document. With regard to differentiation, obtaining the formulae of derivatives of functions using the first principle is not emphasized in both upper secondary level Additional Mathematics and the pre-university level H2 Mathematics. As evident from Table 1 showing the gist of the syllabus documents, the procedural knowledge of performing differentiation is the main focus in the school calculus curriculum in Singapore.

Some researchers have warned against excessive emphasis on procedural knowledge at the expense of developing conceptual understanding (e.g., Aspinwell & Miller, 1997; Mahir, 2009; Tatar & Zengin, 2016). However, other researchers have also deliberated on the importance of substantial exposure to procedural knowledge, which is required to build up an individual's procedural fluency. Substantial exposure to procedural knowledge forms the foundation of undergraduate calculus (e.g., Maciejewski & Star, 2016). Rittle-Johnson et al. (2001) believed that procedural and conceptual knowledge need not be mutually exclusive; they can be developed together. In fact, they asserted that an increase in one type of knowledge will lead to a positive gain in the other type of knowledge, and this will lead to further increase in the first.

The implications of the findings reported in this paper on the teaching of school calculus are discussed. In this paper, our report is guided by the following research question:

What is the pre-service teachers' knowledge of the formula for the derivative of sin x?

Knowledge related to differentiation formula

Some educators have recommended that conceptual understanding of derivatives and the formulae of the derivatives of various functions must be taught. In particular, Lim (2008) recommended that the algebraic derivation of the gradient function is important for students, as the derivation enables students to obtain "an analytic global gradient functions" (p. 42). However, Lim (2008) also cautioned that the algebraic manipulations involved in differentiation using the first principle usually result in obscuring the processes and principles of differentiation.

As an illustration, consider the knowledge required for finding the gradient function of f(x) = sin x. Let us assume the standard differentiation formula $\frac{d}{dx}(\sin x) = \cos x$ presented in most calculus textbooks (It is common knowledge that calculus students are usually presented with a list of formulae that must be rote-learned). One first needs to know that the unit of measurement of the angle must be in radian (in order for the usual formula $\frac{d}{dx}(\sin x) = \cos x$ to be valid). For the trigonometric function f(x) = sin x, in performing manipulation to obtain the first derivative, one needs to be concerned with various trigonometric identities such as the factor formula or the addition formula for trigonometric sine function. In addition, one will also be required to perform the algebraic manipulation leading towards obtaining the expression $\lim_{x\to 0}\frac{\sin x}{x}$. The entire process of obtaining the derivative from the first principle is more likely able to allow one to appreciate that the differentiation formula of trigonometric sine function is dependent on the value of this limit $\lim_{x\to 0}\frac{\sin x}{x}$. Next, the value of this limit can either be presented as a "standard value" to be memorized (in the case that the unit of measurement is radian), or its value could be derived conceptually by performing a basic comparison of the area of a sector with a triangle formed by the vertex angle and the two radii of the sector, as the vertex angle x becomes indefinitely small (Lee, 1993).

A closely related task to the above is to find the derivative $\frac{d}{dx}(\sin x^{\circ})$. The knowledge and the web of tasks required in finding this derivative is shown in Figure 1. Procedurally, the knowledge required to perform the differentiation of trigonometric functions is one's recognition of the unit of angle measurement to be in radian, distinct from the case where the unit of angle measurement is in degree. In the event that one recognizes this difference and appreciates that the typical differentiation formula holds only when the angle is measured in radian, one will convert the unit of measurement from degree to radian. Recognizing that when converted to radians, $\sin x^{\circ} = \sin\left(\frac{\pi x}{180}\right)$, which is indeed a composite of two functions, the individual will next apply the chain rule to perform the differentiation task.

The conceptual knowledge underlying the derivative of the sine function includes the knowledge of the value of $\lim_{x\to 0}\frac{\sin x}{x}$. Conceptual knowledge underlies the procedural knowledge as it enables the solver to understand the reasons for the procedural knowledge. It also enables the solver to retrieve and derive the procedure should the latter be forgotten. Alternatively, one can obtain the formula $\frac{d}{dx}(\sin x^{\circ})$ without converting the unit of measurement to radian, but by obtaining the derivative using the first principle. This will result in $\lim_{x\to 0}\frac{\sin x^{\circ}}{x}$, out of which this limit needs to be evaluated. These two tasks have resulted in many fallacious arguments and the various fallacies are discussed in great details in Toh et al. (in-press).

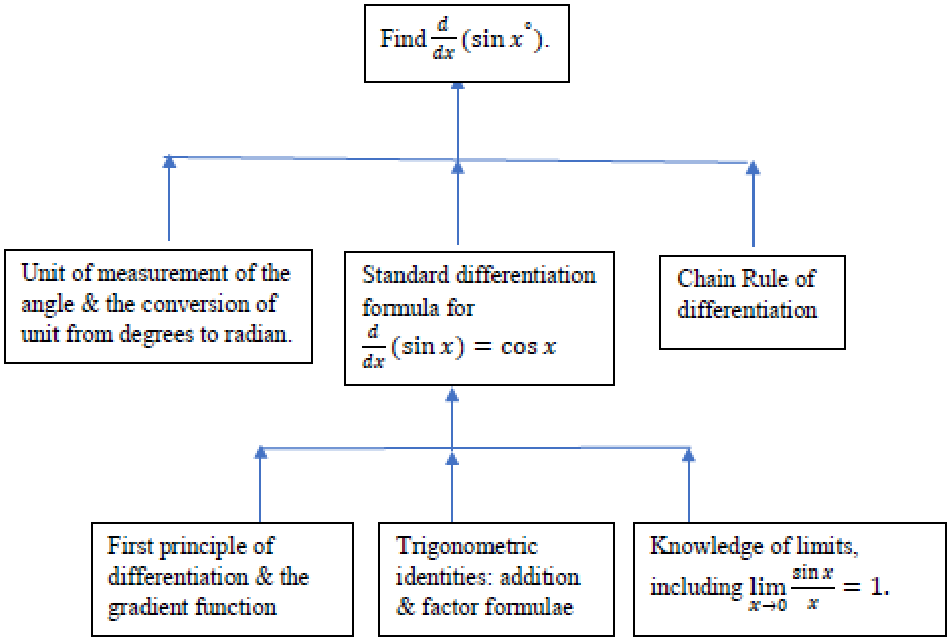

Figure 1. The hierarchy of tasks in obtaining the derivative formula

A learning hierarchy task analysis as illustrated by Figure 1 was introduced by Gagné (1968). Such an analysis provides an understanding of the task by identifying the sub-goals and sub-tasks required to complete the task (Gagné, 1968). In our task of finding the derivative $\frac{d}{dx}(\sin x^{\circ})$, the immediate subordinate sub-skills are: (1) the conversion of unit of angle measurement between degree and radian; (2) the application of the standard formula for $\frac{d}{dx}(\sin x)$, where the angle is measured in radians; and (3) the application of Chain Rule of differentiation. In sub-skill (2), three subordinate sub-skills can also be identified: the use of (a) the first principle of differentiation; (b) basic trigonometric identities, in particular, addition and factor formulae; and (c) $\lim_{x\to 0}\frac{\sin x}{x} = 1$ (where x is measured in radians).

Methodology

Developing the instrument

An instrument which consists of calculus tasks was designed for the study. The tasks were presented as multiple choice questions for the participants in an online environment. The instrument explored various dimensions of PSTs' school calculus content knowledge. Although it may be common practice for researchers to use open-ended tasks to assess various aspects of participants' reasoning, open-ended tasks lack the potential for extensive domain coverage (Parke et al., 2006). Thus, the research team decided to use multiple choice questions in the design of the tasks in the instrument. Each question in the instrument had five choices for the participants to select, and had only one correct answer and four distractors. To reduce the chance of the PSTs from getting the correct answer through mere guessing, each distractor was deliberated based on the potential students' underlying misconception for each of the concepts or plausible computation mistakes presented in each question. Together with a sufficient number of distractors, the chance of the participants giving the correct answer by guessing was further reduced (Haladyna, 2004). Conducting interviews or focus group discussion was not generally feasible in this study due to constraints of reality. In order to get more information about the participants' choice of answers, an open section accompanying each item was provided. The participants were encouraged to comment on how they had answered or to give any comment about each question.

To achieve construct validity, the items in the instrument were scrutinized by all the team members of the calculus research project. The team members are university mathematics lecturers in three local universities in Singapore. In designing the tasks, the research team was guided by the report of Engelbrecht et al. (2009), who demonstrated several examples of items that were deemed to be conceptual by the researchers but turned out to be solved procedurally by the participants in the research. Furthermore, any disagreement among the researchers about the items to be included in the instrument were reviewed for their content and design, so that the tasks in the final version of the instrument had a consensus among the team members.

Researchers have asserted that assessing conceptual knowledge is difficult (e.g., Scheibling-Seve et al., 2020). Procedural knowledge may be assessed by the performance in a mathematical task (in our case, a multiple choice question). Conceptual knowledge is usually assessed from evaluating a procedure or judging its correctness. A calculation that is computed correctly might not tell us much about both an individual's conceptual or procedural knowledge. Thus, the open section for students' comments supplied us with information about their understanding or interpretation of the tasks. Moreover, a mistake in the answers provided by the students in performing the tasks could be an indication about an individual's lack of understanding (Barmby et al., 2007).

This paper reports the PSTs' knowledge of two of the tasks in the instrument, that on the derivative formula of the trigonometric sine function and the evaluation of the corresponding limit $\lim_{x\to 0}\frac{\sin x^{\circ}}{x}$. These two items are labelled as items 1 and 2 in Figure 2 below. The PSTs' performance in the two tasks was analysed and is reported below.

Item 1 What is $\frac{d}{dx}(\sin x^{\circ})$? (Here the angle x is measured in degrees.)

(A) $\cos x^{o}$ (B) $-\cos x^{o}$ (C) $\cos\frac{\pi x}{180}$ (D) $-\cos\frac{\pi x}{180}$

(E) $\frac{\pi}{180}\cos\frac{\pi x}{180}$

Please comment on the above item. You may explain your reason for the choice of your answer, or the level of difficulty of the item etc.

Item 2 As the angle x approaches 0, what does the value of $\frac{\sin x^{\circ}}{x}$ tend to? (Here x is in degrees.)

(A) 0 (B) 1 (C) $\frac{\pi}{90}$ (D) $\frac{\pi}{180}$

(E) $\frac{\pi}{360}$

Please comment on the above item. You may explain your reason for the choice of your answer, or the level of difficulty of the item etc.

Figure 2. The prototype version of the two items that are reported in this paper (labelled as Item 1 and Item 2)

Design of item 1 and the distractors

Item 1 was modified from a classical calculus question that has been found in the various editions of the popular undergraduate calculus textbook by Stewart. This question was also included in the latest edition of the textbook (Stewart, 2016, p. 207). In this item, the unit of angle measurement is in degrees. The original problem presented in Stewart (2016) gets the students to prove that $\frac{d}{dx}(\sin x^{o}) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right)$. In the design of our instrument, this question was converted into a multiple choice item with four distractors together with the correct answer. The procedural and conceptual knowledge associated with item 1 has been discussed and presented in the web of tasks in Figure 1. In particular, one crucial calculus knowledge required in solving item 1 is the chain rule for differentiation. Existing education literature on school calculus clearly shows that the chain rule is difficult for most students. The difficulty is attributed to the students' lack of conceptual understanding of composite functions (e.g., Gordon, 2005; Tall, 1992).

The distractors of item 1 were designed in consideration of the stages in the APOS Theory (Weller et al., 2009). The hypothesis of the APOS Theory deals with a learner's inclination to handle a mathematical problem situation by constructing *actions*, *processes* and *objects* and organizing them in *schemas* to enable them to make sense of and solve the problem. In other words, the APOS theory describes the learner's stages of learning. We shall describe the APOS stages with the potential responses to item 1.

At the initial stage of learning calculus, an individual may have difficulty with recalling the correct formula for differentiating the sine function. From our collective classroom experience, the incorrect statement $\frac{d}{dx}(\sin x) = -\cos x$ is very common among students, as this formula resembles another correct statement $\frac{d}{dx}(\cos x) = -\sin x$. Hence, the choices (B) and (D) were included for this problem. This stage is referred to as the "pre-action" stage of the APOS stages, in which the learners are unable to even recall a correct formula (in this case, the formula for the derivative of sine function) (Siyepu, 2015). An individual is considered to be at the *action* stage when he or she perceives an external stimulus. At this stage, the individual can perform little more than stating the correct formula $\frac{d}{dx}(\sin x) = \cos x$ without being able to derive the formula, and possibly he or she is not even cognizant of the conditions for which the formula is true (in particular, the unit of angle measurement in this case). Hence choice (A) was included as a distractor for this group of individuals. In the language of APOS theory, they have not reached the *process* or *object* stages. At the *process* stage, the individual can imagine performing transformation without executing each step explicitly. In particular, in relation to the task in item 1, the individual can recognize the need to convert the unit of measurement of angles to radian when given the expression $\frac{d}{dx}(\sin x°)$. However, they may not be able to execute the formula correctly. The option (C) reflects the choice of this group of individuals, who have not figured out how the conversion of unit leads to a change of procedure in performing differentiation. At the *object* stage, an individual is aware of a process stage in totality. In this case, the individual is not only able to express $\sin x°$ as $\sin\left(\frac{\pi x}{180}\right)$ but is also able to treat this as a composite function and apply chain rule of differentiation successfully. The individual is able to use a combination of rules to perform the process of differentiation correctly. The option (E), which is the correct answer for this item, identifies this group of individuals.

Design of item 2 and its distractors

Item 2, which involves the limit of the function $\frac{\sin x°}{x}$ as x tends to 0, is modified from a critical part of deriving the derivative of sine function using the first principle (Toh, 2007a). The critical part of the derivation makes use of $\lim_{x\to 0}\frac{\sin x}{x} = 1$, where the angle x is measured in radians. This item was included in this instrument as an attempt to assess the participants' conceptual understanding related to item 1, that is, obtaining the derivative of the sine function is dependent on the value of this limit. The procedural knowledge of evaluating $\lim_{x\to 0}\frac{\sin x°}{x}$ begins with recognizing the unit of angle measurement given in degrees, and the need to convert from degrees to radian before applying the standard result $\lim_{x\to 0}\frac{\sin x}{x} = 1$. The conceptual understanding of this limit formula is to relate the areas of a sector with a triangle sharing the same vertex angle, and by considering the case when the vertex angle tends to zero. By comparison of the sector and the triangle using the appropriate formulae, it could be seen that the limit holds only when the angle x is measured in radian (Lee, 1993). One could use the above method to compare the area of a sector and a right-angled triangle with the common subtended angle x being measured in degrees to obtain $\lim_{x\to 0}\frac{\sin x°}{x} = \frac{\pi}{180}$. This item was included in the instrument in order to gauge the PSTs' ability to relate the derivative of the trigonometric sine function with the value of $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Much of the PSTs' reasoning in evaluating the limit in item 2 would have to be captured through the open comment section that they were encouraged to complete. For participants who attempt to explore using spreadsheets or graphing calculators, they will likely discover that the value of $\frac{\sin x°}{x}$ becomes very small as x decreases. Hence, a very likely incorrect generalization would be to erroneously deduce this limit to be zero (as the value of $\frac{\pi}{180}$ is very small and hence could be mistaken to be 0), hence the distractor (A) was included. For an individual who calculates $\frac{\sin x}{x}$ for small values of x without converting the unit of measurement from degrees to radian, he or she will likely obtain the answer 1. Furthermore, some participants could have memorized the result $\lim_{x\to 0}\frac{\sin x}{x} = 1$ without taking note of the unit of the angle measurement. Hence the distractor (B) was included. Distractors (C) and (E) reflect possible miscalculation that could be made by the participants, who might have recognized the need for angle unit measurement and are aware of $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Piloting the Instrument

The instrument was piloted on a group of 20 PSTs, who had newly graduated from local post-secondary education. Among them, 18 of the PSTs had graduated from local pre-university course and two from local polytechnic diploma courses. They had read upper secondary Additional Mathematics. The PSTs who had graduated from pre-university courses had read H2 Mathematics courses, and the two PSTs had read diploma mathematics courses. In all the mathematics courses, calculus is an important strand in their curriculum. They were admitted to the four-year degree programme in the Singapore National Institute of Education, Nanyang Technological University, and would obtain a Bachelor Degree with a Diploma in Education upon their graduation. They would be reading mathematics as one of their undergraduate academic subjects. The calculus instrument was administered at the time when they were about to begin their first modules of undergraduate mathematics, namely, Calculus and Linear Algebra.

The PSTs participated on a voluntary basis, and their participation was in accordance with the Research Ethics guideline of the University (the entire cohort consisted of 22 PSTs, but 2 PSTs opted out of this study). An electronic version of the instrument was administered to the PSTs. The participants were encouraged to complete all the items and to provide the rationale for their choice of answers, or to comment on each of the items in the open space provided below each item. To increase the validity of the results of their performance, the researchers rode on the affordance of technology to have the sequence of all the items, including the choices of answers for each item, randomized for every individual participant. This was implemented to prevent the mutual influence of the peers' answers on their responses. The instrument was administered within a common time slot in a computer laboratory. The PSTs were encouraged to complete all the items as quickly as possible, although they were given the maximum time of one hour. They were allowed to use electronic or graphing calculators, or spreadsheets. All their responses were captured by the system and tabulated on a spreadsheet, which was analyzed by the researchers. The study on the PSTs' performance in items 1 and 2 is presented in the following section.

Results

The PSTs' choices to items 1 and 2 are tabulated in Table 2. Note that as the sequence of all the items in the instrument and the sequence of the choices of each item were randomized for each participant, the online system tabulated the PSTs' choices according to the prototype version of the instrument (Figure 2). The multiple choices that were selected by the PSTs as the answers were captured and tabulated according to the labelling in the prototype version.

Table 2: A tabulation of the frequency of the choices selected by the PSTs for items 1 and 2

Items	A	B	C	D	E
1	12 (60%)	0	3 (15%)	1 (5%)	4* (20%)
2	11 (55%)	8 (40%)	0	1* (5%)	0

*correct answer

From the collated results in Table 2, four participants (20%) obtained the correct answer for $\frac{d}{dx}(\sin x°)$ in item 1. Only one (5%) of the participants gave the correct answer for $\lim_{x\to 0}\frac{\sin x°}{x}$ in item 2.

The PSTs' performance in item 1

For the clarity of our discussion and to maintain anonymity, all the participants were labelled from S1 to S20 in this paper. Four participants (S1, S9, S11 and S18) gave the correct answer (E) for item 1. It was evident that they were operating at least at the *object* stage of the APOS Theory. They recognized the necessity to convert the unit of measurement from degrees to radians, and managed to express the resultant expression as a composite function in order to be able to use the chain rule of differentiation to obtain the derivative successfully. According to two participants S1 and S11, who had included their comments about this item in the instrument,

> S1: Differentiation is normally done in radians. So *x* degrees equals pi times *x* divided by 180 [degrees]. Differentiating the resultant expression will give the option 1 [i.e. option (E) in the prototype version was randomized as option 1 in student S1's version] as the answer.

> S11: Vaguely remember from jc [junior college, which is a pre-university institution in Singapore], not even sure I am right.

Three participants (S2, S19 and S20) selected option (C) as the answer for this item. According to two of the three participants (S19 and S20) who provided their comment for this item, they were clearly cognizant that the unit of measurement in differentiation formula must be in radian. However, as shown in this instrument, they did not manage to translate this to the process of finding the derivative correctly, although they had recognized that cosine is the derivative of the sine function. They clearly had not transcended the *process* stage of the APOS Theory.

> S19: In [obtaining the] derivative [of trigonometric functions], the angle used is in radians not in degrees.

> S20: Differentiation for trigonometric functions have to be in radians, not in degrees. 180 degrees equals pi [radians], so that x degrees equals (x)(pi)/(180) and differentiating sin[e] gives cos[ine].

The participants S1, S9, S11, S18, S2, S19 and S20 (who chose (C) and (E) as their answers for item 1) could remember the condition on the unit of measurement of angle in differentiating trigonometric functions. Only the four participants S1, S9, S11 and S18 managed to translate their knowledge correctly to perform this task of differentiating using the chain rule. Only one student S3 (5%) gave the incorrect response as option (D). He did not manage to recall the correct formula for the derivative of sine function, though the selection of this option as the answer was an indication that he was able to recall the condition on the unit of angle measurement. However, he had memorized the formula for differentiating sine function incorrectly, and was not able to translate the function with the unit conversation as a composite function.

It was also evident that the remaining 60% of the participants (S4, S5, S6, S7, S8, S10, S12, S13, S14, S15, S16 and S17) were operating at most at the *action* stage of the APOS Theory. They had memorized the formula for the derivative of sine function without being aware of the attached condition for the unit of measurement of the angle.

The PSTs' performance in item 2

As indicated in Table 2, a total of 11 PSTs (S2, S5, S6, S8, S11, S12, S13, S16, S18, S19 and S20, forming 55% of the responses) chose the option (A) as the answer, that is, $\lim_{x\to 0}\frac{\sin x^{\circ}}{x}=0$. The responses of the participants who had chosen option (A) and had included their comment on this item are shown below:

S5: The issue here is that the denominator cannot be 0. But the numerator will also give 0 at the same time.
S6: Quite easy to do, since the numerator is zero.
S11: 0/0 = 0 [i.e. zero divided zero equals zero]
S12: sin 0 = 0
S16: If the graph for this is extrapolated, it will tend to 0.
S18: sin (0) = 0. So 0 divided by a number close to 0 is still 0.
S19: As derived from the graph of y = sin *x*, as *x* tends to 0, the graph of sin *x* tends to 0 as well.

For the above PSTs who had selected option (A) as the answer for item 2, it was clear that the students S5, S6, S12 and S18 had selected their choice of the answer by substituting 0 into the numerator of the expression $\frac{\sin x^{\circ}}{x}$. They had used the statement that 0 divided by any number (including zero) equals 0. The PST S11 had identified explicitly that zero divided by zero gives the value zero, a very common misconception among both students and teachers. Although the PSTs S16 and S19 appeared to have used a graphical argument, their argument is based on the graph of *y* = sin *x* and its behavior as "*x* tends to 0". It appeared that they had used a graphical approach to inspect the limit, but had misinterpreted the value of the *y*-intercept as being zero. Contrary to the researchers' expectation, none of the participants had responded by using a spreadsheet or a graphing calculator to tabulate the values of $\frac{\sin x^{\circ}}{x}$ for small values of *x*. The use of spreadsheet or graphing calculator for finding the limit was also not observed by the researchers when they walked around the computer laboratory as the PSTs were completing the instrument.

A total of 8 PSTs (S1, S3, S4, S7, S9, S10, S14 and S15, forming 40% of the responses) selected the option (B).

S1: As *x* tends to 0, sin *x* tends to 0. So 0/0 = 1 [i.e. zero divided by zero equals one].
S4: L'Hospital's rule
S7: As $x \rightarrow 0$, $\sin x \rightarrow 0$. So 0/0 is undefined. In the context of this question, as undefined is not an option, the second best answer is 1.
S14: When sin *x* is small, sin *x* is tentatively (approximately) *x*.
S15: Have to differentiate sin *x* / *x* as limit of sin *x* / *x* as *x* tends to 0 is 1. So consider cos *x* / 1 as $x \rightarrow 0$, which gives the answer 1.

The arguments for the students who responded by selecting option (B) as the answer for this item could be classified under three categories: the PSTs who (1) mistook the value of 0/0 to be 1, as illustrated by the PSTs S1 and S7; (2) had used small angle approximation for sin *x* (ie., $\sin x \approx x$); and (3) applying L'Hospital's rule for differentiation. The participants whose reasoning were classified under categories (2) and (3) had failed to take note of the unit of angle measurement in performing differentiation of trigonometric functions. The PSTs S4 and S15 had obviously read beyond the pre-university mathematics content to have encountered the procedural knowledge in L'Hospital's Rule. For the two PSTs S4 and S15, they obviously had made use of the derivative formula for sine function in order to compute the limit in item 2. Only one participant S17 responded with the correct answer which is (D). He did not include any comment for his choice of answer for this item. However, we found that the participant S17 had not selected the correct answer for item 1.

Matching students' performance in items 1 and 2

The four participants who gave the correct response for the derivative of the function $\sin(x^{\circ})$ with respect to *x* in item 1 did not respond correctly for item 2. Similarly, the only student (S17) who gave the correct response for item 2 did not give the correct response for item 1. This is a clear indication that the students did not see the relation between the formula for the derivative of the trigonometric sine function and the value of $\lim_{x\to 0}\frac{\sin x}{x}$. The students who obtained the correct answer for the derivative of sin (x^{o}) responded to the two items independently of each other, evidently failed to articulate the connection between them.

The objective of the two items in the calculus instrument was to assess the PSTs' knowledge of the derivative of the sine function. Using the first principle of obtaining the derivative is built on the knowledge of the value of $\lim_{x\to 0}\frac{\sin x}{x}$. The two pairs of consistent responses between items 1 and 2 are: the pair of consistent correct responses consisting of option (E) for item 1 and option (D) for item 2 (recognizing that is $\frac{d}{dx}(\sin x^{\circ})=\frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right)$ and $\lim_{x\to 0}\frac{\sin x^{\circ}}{x}=\frac{\pi}{180}$) ;

and the pair of consistent incorrect responses with option (A) for item 1 versus option (B) for item 2 (recognizing that is $\frac{d}{dx}(\sin x^{\circ}) = \cos(x^{\circ})$ and $\lim_{x\to 0}\frac{\sin x^{\circ}}{x} = 1$) as illustrated in Table 3.

Table 3: *Consistent responses of items 1 and 2*

Item 1	**Item 2**	**No. of students**	**Students**
$\frac{d}{dx}(\sin x^{\circ}) = \frac{\pi}{180}\cos x^{\circ}$ Option (E)	$\lim\limits_{x\to 0}\frac{\sin x^{\circ}}{x} = \frac{\pi}{180}$ Option (A)	0	-
$\frac{d}{dx}(\sin x^{\circ}) = \cos x^{\circ}$ Option (A)	$\lim\limits_{x\to 0}\frac{\sin x^{\circ}}{x} = 1$ Option (B)	2	S4 and S15

As shown in Table 3, none of the students who obtained the correct response for item 1 had the consistent pair of correct response for item 2. Two PSTs (S4 and S15) had the pair of consistent incorrect responses for items 1 and 2 (i.e. option (A) for item 1 and option (B) for item 2). The researchers investigated their rationale for the choice of their responses for the two items. A short discussion with a duration of not more than 3 minutes was conducted with the PSTs S4 and S15 after the data was collected and analysed. A short excerpt of the two PSTs' comments for their matching was summarized below:

PSTs S4 and S15 had not been aware of the condition required on the unit of angle measurement to be in radian in performing differentiation of trigonometric functions in obtaining the typical result $\frac{d}{dx}(\sin x) = \cos x$. Hence, they selected their response (A) for item 1. PST S4 had read up on L'Hospital's Rule for finding limits in his polytechnic education (prior to the university admission), and S15 had learnt the rule in the pre-university Further Mathematics course (a more advanced course offered at the pre-university level taken by very few students. In this group of 20 PST, only S15 had read Further Mathematics course at the pre-university). Not being cognizant of the condition of unit of angle measurement for differentiation, they had performed L'Hospital's Rule erroneously.

$$\lim_{x\to 0}\frac{\sin x^{\circ}}{x} = \lim_{x\to 0}\frac{\cos x^{\circ}}{x} = \cos 0 = 1.\text{* (incorrect response)}$$

S4 and S15 explained that they had not made use of $\lim_{x\to 0}\frac{\sin x}{x}$ to obtain the derivative of sin *x*, which was memorized as a standard formula. On the other hand, they had used the derivative formula of sin *x* to evaluate $\lim_{x\to 0}\frac{\sin x}{x}$. However, not being cognizant of the condition of the unit of angle measurement, they gave this pair of consistent incorrect responses. The rules of differentiation and L'Hospital's Rule were learnt by the PSTs procedurally without being mindful of the condition of the formulae and the procedure.

Discussion

It is clearly noted that very few of the PSTs answered items 1 and 2 correctly, and most of their responses to the two items were not consistent. This could be suggestive that they had learnt the differentiation formula by rote, and had not seen the connection between obtaining the derivative and evaluating the corresponding limit. The PSTs had appeared to have answered the two questions independently without taking note of the relation between the two items. As the objective of this research project was about the PSTs' knowledge of school calculus and not about their ability to evaluate limit, item 2 was eventually removed from the instrument.

Most of the PSTs in this study had not been cognizant of the unit measurement in the typical formula for the differentiation of trigonometric functions. The PSTs who were aware of the angle unit measurement did not manage to successfully translate this knowledge into the procedure of performing differentiation with angle measurement in degrees. This is not surprising as the procedure involves converting the function into composite functions followed by using the chain rule of differentiation, both of which are difficult for many students and teachers. The same item 1 was administered by Toh (2009) on a group of practicing mathematics teachers' content knowledge of calculus. Similar to the result of the study reported in this paper, many of the participating in-service teachers in Toh (2009) were not aware of the condition on the unit of angle measurement in finding derivatives of trigonometric functions. Without a conceptual understanding in calculus, the solvers' first response to a novel problem (the stimulus) was to look for appropriate formulae to respond to the problem, and they attempted to apply the correct formula to this "stimulus" without taking note of the appropriate condition of application. This was termed as *tentative solution starts* by Selden et al. (1999). To us, this stage seems to correspond to the action stage of the APOS theory. Only relatively few respondents have reached the process stage (in being able to perform transformation using more complex rules of differentiation) or transcend the process stage of the APOS theory.

Researchers have asserted that differentiation using the first principle is difficult for most students (Muzangwa & Chifamba, 2012; Robert & Speer, 2001). It is thus not surprising that this is not taught in the secondary school calculus curriculum, with a brief mention in the pre-university calculus curriculum. The PSTs' inability to respond to the two items correctly could be the tip of an iceberg; it could possibly be the result of their almost exclusive focus on procedural knowledge (which begins with calculus at the secondary mathematics curriculum) without an in-depth understanding of calculus concepts (which are taught at the pre-university level). The spiral approach to the school curriculum has resulted in calculus first being taught at the secondary level with an emphasis on procedures, and a more complete calculus knowledge is taught iteratively at the pre-university level (Toh, 2021). With reference to calculus in particular, Orton (1983) warned that it could be difficult for students to replace their imperfect knowledge acquired at the earlier stage by a more complete knowledge which was taught at a later stage. The study in Toh (2021) shows that this difficulty is still true for students today after more than 35 years later.

Conclusion

Two decades ago, Lim-Teo et al. (2000), in their study on pre-university and tertiary students' attitude towards calculus, reported that a large number of students in their study learnt calculus by memorizing and applying formulae and procedures. Consequently, these students were likely to face more difficulty when they encountered more advanced calculus concepts at the higher level. This report also verified that the difficulty in solving a modified task without a conceptual understanding of the calculus concepts is also faced by pre-service teachers more than two decades later.

Limitations

Readers should be cautioned of the limitation of this study. Firstly, the sample size of this study only consisted of one cohort of pre-service teachers. We are mindful of not over-generalizing the results in this report. The lack of generalizability due to the small sample size was compensated by the fact that the group of PSTs in this study belong to the best candidates for teacher education programme in terms of their academic performance at the secondary and pre-university levels. Thus, we have the "best-case scenario": if the candidates of the "upper bound" have such conceptual difficulty with calculus, it is likely that the other candidates could have similar difficulty. Secondly, the design of the study did not cater for interview or focused group discussion with the participants. This was compensated by the inclusion of the open section for the candidates to include their feedback. Although we are cognizant of the limitation in the conclusion presented here, we hope this would spur the interest of more researchers in the area of calculus education research.

Recommendations

Törner et al. (2014), in the special issue on calculus education in the ZDM – Mathematics Education journal, lamented that calculus is not receiving much attention from the researchers in Europe as it is not perceived as a predominant subject. The same situation occurs in Singapore: a quick search of research database shows the scarcity of studies on calculus education. This is not surprising as calculus is not among the topics in most international comparative studies.

As calculus is an important pre-requisite for STEM education at the undergraduate level, a review of both school calculus content and pedagogy from secondary to pre-university levels is timely. Researchers (e.g., Thompson et al., 2013) have called for reconceptualizing a conceptual approach to calculus through the use of technology, instead of "a retention of traditional calculus ideas now supported by dynamic graphics for illustration and symbolic manipulation for computation" (Tall, 2010, p. 2).

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Authorship Contribution Statement

Tin Lam Toh: Concept and design, data analysis, completed the first draft of the manuscript, critical revision of manuscript, securing funding. Pee Choon Toh: Drafting manuscript, data analysis, revision of manuscript, securing funding. Kok Ming Teo: Drafting manuscript, data analysis, revision of manuscript, securing funding. Ying Zhu: Drafting manuscript, data analysis, revision of manuscript, securing funding.

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