First-countability, ω-Rudin spaces and well-filtered determined spaces☆

Xiaoquan Xu[a,*], Chong Shen[b], Xiaoyong Xi[c], Dongsheng Zhao[d]

[a]*Fujian Key Laboratory of Granular Computing and Applications, Minnan Normal University, Zhangzhou 363000, China*
[b]*School of Mathematical Sciences, Nanjing Normal University, Nanjing 210046, China*
[c]*School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, China*
[d]*Mathematics and Mathematics Education, National Institute of Education Singapore, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616*

Abstract

In this paper, we investigate some versions of d-space, well-filtered space and Rudin space concerning various countability properties. It is proved that every T_0 space with a first-countable sobrification is an ω-Rudin space and every first-countable T_0 space is well-filtered determined. Therefore, every ω-well-filtered space with a first-countable sobrification is sober. It is also shown that every irreducible closed subset in a first-countable ω-well-filtered space is countably directed, hence every first-countable ω^*-well-filtered d-space is sober.

Keywords: First-countability; Sober space; Well-filtered space; ω-Rudin space; ω-Well-filtered space; Countably directed set
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1. Introduction

In non-Hausdorff topology and domain theory, the d-spaces, sober spaces and well-filtered spaces form three of the most important classes (see [1,3-5,7-17, 19-25]). In [19, 22, 23], we introduced and studied the Rudin spaces, well-filtered determined spaces and ω-well-filtered spaces. Some relationships and links among these new non-Hausdorff topological properties, the well studied sobriety and well-filteredness were uncovered. In [23], it was proved that in a first-countable ω-well-filtered space X, every irreducible closed subset of X is directed under the specialization order of X. It follows immediately that every first-countable ω-well-filtered d-space is sober.

In the current paper, we continue studying some aspects of d-space, well-filtered space and Rudin spaces concerning countability. Employing countably directed sets, we define two new types of topological spaces — ω^*-d-spaces and ω^*-well-filtered spaces. It is proved that every T_0 space with a first-countable sobrification is an ω-Rudin space and every first-countable T_0 space is well-filtered determined. Therefore, every ω-well-filtered space with a first-countable sobrification is sober. From these, we obtain that if a T_0 space X is second-countable or first-countable with a countable underlying set, then X is an ω-Rudin space, and X is sober if it is additionally ω-well-filtered. We also show that in each first-countable ω-well-filtered space, every irreducible closed subset is countably directed, hence every first-countable ω-well-filtered ω^*-d-space is sober. Using the topological Rudin Lemma, we prove that a T_0 space X is ω^*-well-filtered iff its Smyth power space is ω^*-well-filtered iff its Smyth power space is an ω^*-d-space.

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*Corresponding author.
Email addresses: xiqxu2002@163.com (Xiaoquan Xu), shenchong0520@163.com (Chong Shen), littlebrook@jsnu.edu.cn (Xiaoyong Xi), dongsheng.zhao@nie.edu.sg (Dongsheng Zhao)

2. Preliminaries

In this section, we briefly recall some fundamental concepts and notations that will be used in the paper. Some basic properties of irreducible sets and compact saturated sets are presented. For further details, we refer the reader to [5,8,17].

For a poset P and $A \subseteq P$, let $\downarrow A = \{x \in P : x \leq a \text{ for some } a \in A\}$ and $\uparrow A = \{x \in P : x \geq a \text{ for some } a \in A\}$. For $x \in P$, we write $\downarrow x$ for $\downarrow\{x\}$ and $\uparrow x$ for $\uparrow\{x\}$. A subset A is called a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$).

For a set X, $|X|$ will denote the cardinality of X. Let $\mathbb{N}$ denote the set of all natural numbers with the usual order and $\omega = |\mathbb{N}|$, and let ω_1 denote the smallest cardinal number of all uncountable cardinalities. For a cardinal number κ and a poset P, let $P^{(<\kappa)} = \{F \subseteq P : F \text{ is nonempty and } |F| < \kappa\}$. The set $\{\uparrow F : F \in P^{(<\omega)}\}$ is denoted by $\mathbf{Fin}P$.

A nonempty subset D of a poset P is *directed* if every two elements in D have an upper bound in D. The set of all directed sets of P is denoted by $\mathcal{D}(P)$. A subset $I \subseteq P$ is called an *ideal* of P if I is a directed lower subset of P. Let $\mathrm{Id}(P)$ be the poset (with the order of set inclusion) of all ideals of P. Dually, we define the notion of *filters* and denote the poset of all filters of P by $\mathrm{Filt}(P)$. A poset P is called a *directed complete poset*, or *dcpo* for short, if for any $D \in \mathcal{D}(P)$, $\vee D$ exists in P.

As in [5], the *upper topology* on a poset Q, generated by the complements of the principal ideals of Q, is denoted by $\upsilon(Q)$. A subset U of Q is *Scott open* if (i) $U = \uparrow U$, and (ii) for any directed subset $D \subseteq Q$ with $\vee D$ existing, $\vee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of Q form a topology, called the *Scott topology* on Q and denoted by $\sigma(Q)$. The space $\Sigma Q = (Q, \sigma(Q))$ is called the *Scott space* of Q. The upper sets of Q form the (*upper*) *Alexandroff topology* $\alpha(Q)$.

For a T_0 space X and $A \subseteq X$, the closure of A in X is denoted by $\mathrm{cl}_X A$ or simply by $\overline{A}$ if there is no confusion. We use $\leq_X$ to denote the *specialization order* on X: $x \leq_X y$ iff $x \in \overline{\{y\}}$. In the following, when a T_0 space X is considered as a poset, the partial order always means the specialization order provided otherwise indicated. Let $\mathcal{O}(X)$ (resp., $\mathcal{C}(X)$) be the set of all open subsets (resp., closed subsets) of X, and let $\mathcal{S}^u(X) = \{\uparrow x : x \in X\}$. Define $\mathcal{S}_c(X) = \{\overline{\{x\}} : x \in X\}$ and $\mathcal{D}_c(X) = \{\overline{D} : D \in \mathcal{D}(X)\}$.

A nonempty subset A of X is *irreducible* if for any $F_1, F_2 \in \mathcal{C}(X)$, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. Denote by $\mathsf{Irr}(X)$ (resp., $\mathsf{Irr}_c(X)$) the set of all irreducible (resp., irreducible closed) subsets of X. Clearly, every directed subset of X is irreducible. The space X is called *sober*, if for any $F \in \mathsf{Irr}_c(X)$, there is a unique point $a \in X$ such that $F = \overline{\{a\}}$.

Remark 2.1. In a T_0 space X, if $x \in X$ and $A \subseteq X$ such that $\overline{A} = \overline{\{x\}}$, then $\vee A$ exists in X and $x = \vee A = \bigvee \overline{A}$.

The following two lemmas on irreducible sets are well-known.

Lemma 2.2. *Let X be a space and Y a subspace of X. Then the following conditions are equivalent for a subset $A \subseteq Y$:*

(1) *A is an irreducible subset of Y.*
(2) *A is an irreducible subset of X.*
(3) *$\mathrm{cl}_X A$ is an irreducible subset of X.*

Lemma 2.3. *If $f : X \longrightarrow Y$ is continuous and $A \in \mathsf{Irr}(X)$, then $f(A) \in \mathsf{Irr}(Y)$.*

A T_0 space X is called a *d-space* (or *monotone convergence space*) if X (with the specialization order) is a dcpo and $\mathcal{O}(X) \subseteq \sigma(X)$ (cf. [5, 20]).

Definition 2.4. ([23, Definition 3.7]) A T_0 space X is called a *directed closure space*, DC *space* for short, if $\mathsf{Irr}_c(X) = \mathcal{D}_c(X)$, that is, for each $A \in \mathsf{Irr}_c(X)$, there exists a directed subset of X such that $A = \overline{D}$.

For any topological space X, $\mathcal{G} \subseteq 2^X$ and $A \subseteq X$, let $\Diamond_{\mathcal{G}} A = \{G \in \mathcal{G} : G \bigcap A \neq \emptyset\}$ and $\Box_{\mathcal{G}} A = \{G \in \mathcal{G} : G \subseteq A\}$. The symbols $\Diamond_{\mathcal{G}} A$ and $\Box_{\mathcal{G}} A$ will be simply written as $\Diamond A$ and $\Box A$ respectively if no ambiguity occurs. The *lower Vietoris topology* on $\mathcal{G}$ is the topology that has $\{\Diamond U : U \in \mathcal{O}(X)\}$ as a subbase, and the resulting space is denoted by $P_H(\mathcal{G})$. If $\mathcal{G} \subseteq \mathsf{Irr}(X)$, then $\{\Diamond_{\mathcal{G}} U : U \in \mathcal{O}(X)\}$ is a topology on $\mathcal{G}$. The *upper Vietoris topology* on $\mathcal{G}$ is the topology that has $\{\Box_{\mathcal{G}} U : U \in \mathcal{O}(X)\}$ as a base, and the resulting space is denoted by $P_S(\mathcal{G})$.

Remark 2.5. Let X be a T_0 space.

(1) If $\mathcal{S}_c(X) \subseteq \mathcal{G}$, then the specialization order on $P_H(\mathcal{G})$ is the set inclusion order, and the *canonical mapping* $\eta_X : X \longrightarrow P_H(\mathcal{G})$, given by $\eta_X(x) = \overline{\{x\}}$, is an order and topological embedding (cf. [5, 8, 17]).
(2) The space $X^s = P_H(\mathsf{Irr}_c(X))$ with the canonical mapping $\eta_X : X \longrightarrow X^s$ is the *sobrification* of X (cf. [5, 8]).

A subset A of a space X is called *saturated* if A equals the intersection of all open sets containing it (equivalently, A is an upper set in the specialization order). We shall use $\mathsf{K}(X)$ to denote the set of all nonempty compact saturated subsets of X and endow it with the *Smyth preorder*, that is, for $K_1, K_2 \in \mathsf{K}(X)$, $K_1 \sqsubseteq K_2$ iff $K_2 \subseteq K_1$. The space $P_S(\mathsf{K}(X))$, denoted shortly by $P_S(X)$, is called the *Smyth power space* or *upper space* of X (cf. [9, 17]). It is easy to verify that the specialization order on $P_S(X)$ is the Smyth order (that is, $\leq_{P_S(X)}=\sqsubseteq$). The *canonical mapping* $\xi_X : X \longrightarrow P_S(X)$, $x \mapsto \uparrow x$, is an order and topological embedding (cf. [9, 10, 17]). Clearly, X is homeomorphic to the subspace $\mathcal{S}^u(X)$ of $P_S(X)$ by means of ξ_X.

Lemma 2.6. *Let X be a T_0 space. For any nonempty family $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$, $\bigvee_{i\in I} K_i$ exists in $\mathsf{K}(X)$ iff $\bigcap_{i\in I} K_i \in \mathsf{K}(X)$. In this case $\bigvee_{i\in I} K_i = \bigcap_{i\in I} K_i$.*

Proof. Suppose that $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$ is a nonempty family and $\bigvee_{i\in I} K_i$ exists in $\mathsf{K}(X)$. Let $K = \bigvee_{i\in I} K_i$. Then $K \subseteq K_i$ for all $i \in I$, and hence $K \subseteq \bigcap_{i\in I} K_i$. For any $x \in \bigcap_{i\in I} K_i$, $\uparrow x$ is a upper bound of $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$, whence $K \sqsubseteq \uparrow x$ or, equivalently, $\uparrow x \subseteq K$. Therefore, $\bigcap_{i\in I} K_i \subseteq K$. Thus $\bigcap_{i\in I} K_i = K \in \mathsf{K}(X)$.

Conversely, if $\bigcap_{i\in I} K_i \in \mathsf{K}(X)$, then $\bigcap_{i\in I} K_i$ is an upper bound of $\{K_i : i \in I\}$ in $\mathsf{K}(X)$. Let $G \in \mathsf{K}(X)$ be any other upper bound of $\{K_i : i \in I\}$, then $G \subseteq K_i$ for all $i \in I$, and hence $G \subseteq \bigcap_{i\in I} K_i$, that is, $\bigcap_{i\in I} K_i \sqsubseteq G$, proving that $\bigvee_{i\in I} K_i = \bigcap_{i\in I} K_i$. □

Lemma 2.7. ([17, Proposition 7.21]) *Let X be a T_0 space.*

(1) *If $\mathcal{K} \in \mathsf{K}(P_S(X))$, then $\bigcup \mathcal{K} \in \mathsf{K}(X)$.*
(2) *The mapping $\bigcup : P_S(P_S(X)) \longrightarrow P_S(X)$, $\mathcal{K} \mapsto \bigcup \mathcal{K}$, is continuous.*

A T_0 space X is called *well-filtered* if it is T_0, and for any open set U and filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$.

Remark 2.8. The following implications are well-known (cf. [5]):

$$\text{sobriety} \Rightarrow \text{well-filteredness} \Rightarrow d\text{-space}.$$

For a T_0 space X, let $\mathcal{D}^\omega(X) = \{D \subseteq X : D \text{ is countable and directed}\}$ and $\mathcal{D}^\omega_c(X) = \{\overline{D} : D \in \mathcal{D}^\omega(X)\}$.

Definition 2.9. ([22, Definition 3.1]) A poset P is called an *ω-dcpo*, if for any $D \in \mathcal{D}^\omega(P)$, $\vee D$ exists.

Definition 2.10. ([22, Definition 3.4]) Let P be a poset. A subset U of P is called *ω-Scott open* if (i) $U = \uparrow U$, and (ii) for any countable directed set D with $\vee D$ existing, $\vee D \in U$ implies that $D \cap U \neq \emptyset$. All ω-Scott open sets form a topology on P, denoted by $\sigma_\omega(P)$ and called the *ω-Scott topology.* The space $\Sigma_\omega P = (P, \sigma_\omega(P))$ is called the *ω-Scott space* of P.

Clearly, $\sigma(P) \subseteq \sigma_\omega(P)$. The converse need not be true, see Example 6.9 in Section 6.

Definition 2.11. ([22, Definition 3.6]) A T_0 space X is called an ω-d-space (or an *ω-monotone convergence space*) if for any $D \in \mathcal{D}^\omega(X)$, there exists a (unique) point $x \in X$ with $\overline{D} = \overline{\{x\}}$, or equivalently, $\mathcal{D}^\omega_c(X) = \mathcal{S}_c(X)$.

Some characterizations of ω-d-spaces were given in [22, Proposition 3.7].

Definition 2.12. ([22, Definition 3.9]) A T_0 space X is called *ω-well-filtered*, if X is and for any countable filtered family $\{K_n : n < \omega\} \subseteq \mathsf{K}(X)$ and $U \in \mathcal{O}(X)$, it holds that

$$\bigcap_{n<\omega} K_n \subseteq U \ \Rightarrow\ \exists n_0 < \omega, K_{n_0} \subseteq U.$$

Proposition 2.13. *For an ω-well-filtered space X and a countable filtered family $\{K_n : n < \omega\} \subseteq \mathsf{K}(X)$, $\bigcap_{n<\omega} K_n \in \mathsf{K}(X)$.*

Proof. Clearly, $\bigcap_{n<\omega} K_n$ is saturated and $\bigcap_{n<\omega} K_n \neq \emptyset$ (otherwise, $\bigcap_{n<\omega} K_n = \emptyset$ implies $K_m = \emptyset$ for some $m < \omega$, a contradiction). Now we verify that $\bigcap_{n<\omega} K_n$ is compact. Let $\{U_i : i \in I\}$ be an open cover of $\bigcap_{n<\omega} K_n$. As X is ω-well-filtered, we have $K_{n_0} \subseteq \bigcup_{i\in I} U_i$ for some $n_0 < \omega$. By the compactness of K_{n_0}, there is $J \in I^{(<\omega)}$ such that $K_{n_0} \subseteq \bigcup_{i\in J} U_i$, and hence $\bigcap_{n<\omega} K_n \subseteq K_{n_0} \subseteq \bigcup_{i\in J} U_i$. Thus $\bigcap_{n<\omega} K_n \in \mathsf{K}(X)$. □

Proposition 2.14. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is ω-well-filtered.*
(2) *For every continuous mapping $f : X \longrightarrow Y$ from X to a T_0 space Y and a countable filtered family $\{K_n : n < \omega\} \subseteq \mathsf{K}(X)$, $\uparrow f\left(\bigcap_{n<\omega} K_n\right) = \bigcap_{n<\omega} \uparrow f(K_n)$.*
(3) *For every continuous mapping $f : X \longrightarrow Y$ from X to a ω-well-filtered Y and a countable filtered family $\{K_n : n < \omega\} \subseteq \mathsf{K}(X)$, $\uparrow f\left(\bigcap_{n<\omega} K_n\right) = \bigcap_{n<\omega} \uparrow f(K_n)$.*
(4) *For every continuous mapping $f : X \longrightarrow Y$ from X to a sober space Y and a countable filtered family $\{K_n : n < \omega\} \subseteq \mathsf{K}(X)$, $\uparrow f\left(\bigcap_{n<\omega} K_n\right) = \bigcap_{n<\omega} \uparrow f(K_n)$.*

Proof. (1) ⇒ (2): It is proved in [4, Lemma 8.1] for sober spaces and the proof is valid for ω-well-filtered (cf. [23, Theorem 5.1]). For the sake of completeness, we present the proof here. It needs only to check $\bigcap_{n<\omega} \uparrow f(K_n) \subseteq \uparrow f\left(\bigcap_{n<\omega} K_n\right)$. Let $y \in \bigcap_{n<\omega} \uparrow f(K_n)$. Then for each $n < \omega$, $\overline{\{y\}} \cap f(K_n) \neq \emptyset$, that is, $K_n \cap f^{-1}(\overline{\{y\}}) \neq \emptyset$. Since X is ω-well-filtered, $f^{-1}(\overline{\{y\}}) \cap \bigcap_{n<\omega} \uparrow f(K_n) \neq \emptyset$ (otherwise, $\bigcap_{n<\omega} \uparrow f(K_n) \subseteq X \setminus f^{-1}(\overline{\{y\}})$, which implies that $K_m \subseteq X \setminus f^{-1}(\overline{\{y\}})$ for some $m < \omega$, a contradiction). It follows that $\overline{\{y\}} \cap f\left(\bigcap_{n<\omega} \uparrow f(K_n)\right) \neq \emptyset$. This implies that $y \in \uparrow f\left(\bigcap_{n<\omega} \uparrow K_n\right)$. So $\bigcap_{n<\omega} \uparrow f(K_n) \subseteq \uparrow f\left(\bigcap_{n<\omega} \uparrow K_n\right)$.

(2) ⇒ (3)⇒ (4): Trivial.

(4) ⇒ (1): Let $\eta_X : X \to X^s\ (= P_H(\mathsf{Irr}_c(X)))$ be the canonical topological embedding from X into its soberification. Suppose that $\{K_n : n < \omega\} \subseteq \mathsf{K}(X)$ is a countable filtered family, $U \in \mathcal{O}(X)$, and $\bigcap \mathcal{K} \subseteq U$. If $K_n \not\subseteq U$ for each $n < \omega$, then by Lemma 3.1, $X \setminus U$ contains a minimal irreducible closed subset A that still meets all K_n. By condition (4) we have $\bigcap_{n<\omega} \uparrow_{\mathsf{Irr}_c(X)} \eta_X(K_n) = \uparrow_{\mathsf{Irr}_c(X)} \eta_X\left(\bigcap_{n<\omega} \uparrow K_n\right) \subseteq \uparrow_{\mathsf{Irr}_c(X)} \eta_X(U) = \Diamond_{\mathsf{Irr}_c(X)} U$. Clearly, $A \in \bigcap_{n<\omega} \uparrow_{\mathsf{Irr}_c(X)} \eta_X(K_n)$, and whence $A \in \Diamond_{\mathsf{Irr}_c(X)} U$, that is, $A \cap U \neq \emptyset$, being in contradiction with $A \subseteq X \setminus U$. Thus X is ω-well-filtered. □

3. Rudin spaces and well-filtered determined spaces

In Section 3, we recall some results about the topological Rudin Lemma, Rudin spaces, ω-Rudin spaces, well-filtered determined spaces and ω-well-filtered determined spaces in [10, 15, 22, 23] that will be used in the next three sections.

Rudin's Lemma is a useful tool in non-Hausdorff topology and plays a crucial role in domain theory (see [3,5-9,11]). Heckmann and Keimel [10] presented the following topological variant of Rudin's Lemma.

Lemma 3.1. (Topological Rudin Lemma) ([10, Lemma 3.1]) *Let X be a topological space and $\mathcal{A}$ an irreducible subset of the Smyth power space $P_S(X)$. Then every closed set $C \subseteq X$ that meets all members of $\mathcal{A}$ contains a minimal irreducible closed subset A that still meets all members of $\mathcal{A}$.*

For a T_0 space X and $\mathcal{K} \subseteq \mathsf{K}(X)$, let $M(\mathcal{K}) = \{A \in \mathcal{C}(X) : K \cap A \neq \emptyset \text{ for all } K \in \mathcal{K}\}$ (that is, $\mathcal{K} \subseteq \Diamond A$) and $m(\mathcal{K}) = \{A \in \mathcal{C}(X) : A \text{ is a minimal member of } M(\mathcal{K})\}$.

Based on the topological Rudin Lemma, Rudin spaces and well-filtered determined spaces were introduced and studied in [23] (see also [19]).

Definition 3.2. ([23, Definition 4.6 and Definition 4.7]) Let X be a T_0 space and A a nonempty subset of X.

(a) The set A is said to be a *Rudin set*, if there exists a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ such that $\overline{A} \in m(\mathcal{K})$ (that is, $\overline{A}$ is a minimal closed set that intersects all members of $\mathcal{K}$). Let $\mathsf{RD}(X)$ denote the set of all closed Rudin sets of X.
(b) The space X is called a *Rudin space*, RD *space* for short, if $\mathsf{Irr}_c(X) = \mathsf{RD}(X)$, that is, every irreducible (closed) set of X is a Rudin set.

Definition 3.3. ([23, Definition 6.1]) Let X be a T_0 space and A a nonempty subset of X.

(a) The set A is called a *well-filtered determined set*, WD *set* for short, if for any continuous mapping $f : X \longrightarrow Y$ to a well-filtered space Y, there exists a (unique) $y_A \in Y$ such that $\overline{f(A)} = \overline{\{y_A\}}$. Denote by $\mathsf{WD}(X)$ the set of all closed well-filtered determined subsets of X.
(b) The space X is called a *well-filtered determined* space, WD *space* for short, if all irreducible (closed) subsets of X are well-filtered determined, that is, $\mathsf{Irr}_c(X) = \mathsf{WD}(X)$.

Proposition 3.4. ([23, Proposition 6.2]) *Let X be a T_0 space. Then $\mathcal{D}_c(X) \subseteq \mathsf{RD}(X) \subseteq \mathsf{WD}(X) \subseteq \mathsf{Irr}_c(X)$.*

In the class of T_0 spaces, by Proposition 3.4, we have the following implications:

$$\text{Sober} \Rightarrow \mathsf{DC} \Rightarrow \mathsf{RD} \Rightarrow \mathsf{WD}.$$

A topological space X is *locally hypercompact* if for each $x \in X$ and each open neighborhood U of x, there is $\uparrow F \in \mathbf{Fin}X$ such that $x \in \mathrm{int}\uparrow F \subseteq \uparrow F \subseteq U$ (cf. [3]). A space X is called *core-compact* if $(\mathcal{O}(X), \subseteq)$ is a *continuous lattice* (cf. [5]).

Proposition 3.5. ([3, Proposition 3.2]) *Let X be a locally hypercompact T_0 space and $A \in \mathsf{Irr}(X)$. Then there exists a directed subset $D \subseteq \downarrow A$ such that $\overline{A} = \overline{D}$. Therefore, X is a DC space, and hence a Rudin space.*

Proposition 3.6. ([23, Theorem 6.10 and Theorem 6.15]) *Let X be a T_0 space.*

(1) *If X is locally compact, then X is a Rudin space.*
(2) *If X is core-compact, then X is a WD space.*

It is not known wether every core-compact T_0 space is a Rudin space. By Proposition 3.6, we immediately deduce the following result.

Corollary 3.7. ([15, Theorem 3.1], [23, Theorem 6.15]) *Every core-compact well-filtered space is sober.*

In [22], the following countable versions of Rudin spaces and WD spaces were introduced and studied.

Definition 3.8. ([22, Definition 5.1]) Let X be a T_0 space and A a nonempty subset of X.

(a) The set A is said to be an ω-*Rudin set*, if there exists a countable filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ such that $\overline{A} \in m(\mathcal{K})$. Let $\mathsf{RD}_\omega(X)$ denote the set of all closed ω-Rudin sets of X.
(b) The space X is called ω-*Rudin space*, if $\mathsf{Irr}_c(X) = \mathsf{RD}_\omega(X)$ or, equivalently, all irreducible (closed) subsets of X are ω-Rudin sets.

Definition 3.9. ([22, Definition 5.4]) Let X be a T_0 space and A a nonempty subset of X.

(a) The set A is called an ω-*well-filtered determined set*, WD_ω *set* for short, if for any continuous mapping $f : X \longrightarrow Y$ to an ω-well-filtered space Y, there exists a (unique) $y_A \in Y$ such that $\overline{f(A)} = \overline{\{y_A\}}$. Denote by $\mathsf{WD}_\omega(X)$ the set of all closed ω-well-filtered determined subsets of X.
(b) The space X is called ω-*well-filtered determined*, ω-WD *space* for short, if $\mathsf{Irr}_c(X) = \mathsf{WD}_\omega(X)$ or, equivalently, all irreducible (closed) subsets of X are ω-well-filtered determined.

For a T_0 space X, it was proved in [22, Proposition 5.5] that $\mathcal{S}_c(X) \subseteq \mathcal{D}_c^\omega(X) \subseteq \mathsf{RD}_\omega(X) \subseteq \mathsf{WD}_\omega(X) \subseteq \mathsf{Irr}_c(X)$. Therefore, every ω-Rudin space is ω-well-filtered determined.

Definition 3.10. A T_0 space X is called an ω-DC *space* if $\mathsf{Irr}_c(X) = \mathcal{D}_c^\omega(X)$, that is, for each $A \in \mathsf{Irr}_c(X)$, there exists a countable directed subset of X such that $A = \overline{D}$.

By [22, Theorem 5.11], we have the following result.

Proposition 3.11. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is sober.*
(2) *X is an ω-DC and ω-d-space.*
(3) *X is an ω-DC and ω-well-filtered space.*
(4) *X is an ω-Rudin and ω-well-filtered space.*
(5) *X is an ω-well-filtered determined and ω-well-filtered space.*

4. ω^*-Scott topologies and ω^*-d-spaces

We now introduce and study two new types of spaces.

Definition 4.1. A nonempty subset D of a poset P is called *countably directed* if every nonempty countable subset of D has an upper bound in D. The set of all countably directed sets of P is denoted by $\mathcal{D}^{\omega^*}(P)$. The poset P is called a *countably directed complete poset*, or ω^*-*dcpo* for short, if for any $D \in \mathcal{D}^{\omega^*}(P)$, $\vee D$ exists in P.

Clearly, $\{\{x\} : x \in P\} \subseteq \mathcal{D}^{\omega^*}(P) \subseteq \mathcal{D}(P)$.

Example 4.2. For the countable chain $\mathbb{N}$ (with the usual order of natural numbers), $(\mathbb{N}^{(<\omega)}, \subseteq)$ is directed in $2^{\mathbb{N}}$, but not countably directed.

Definition 4.3. A subset U of a poset P is ω^*-*Scott open* if (i) $U = \uparrow U$, and (ii) for any countably directed subset D with $\vee D$ existing, $\vee D \in U$ implies $D \cap U \neq \emptyset$. All ω^*-Scott open subsets of P form a topology, called the ω^*-*Scott topology* on Q and denoted by $\sigma_{\omega^*}(Q)$. Let $\Sigma_{\omega^*} Q = (Q, \sigma_{\omega^*}(Q))$.

Clearly, $\upsilon(P) \subseteq \sigma(P) \subseteq \sigma_{\omega^*}(P) \subseteq \alpha(P)$. In general, $\sigma(P) \neq \sigma_{\omega^*}(P)$ as shown in Example 6.9 below.

Remark 4.4. For a poset P, the ω^*-Scott topology $\sigma_{\omega^*}(P)$ on P is at the same time a σ-lattice (of subsets). A σ-*lattice* of subsets is a collection of subsets that is closed under countable unions and countable intersections. Indeed, for any countable family $\{U_n : n < \omega\} \subseteq \sigma_{\omega^*}(P)$, $\bigcap_{n<\omega} U_n$ is clearly an upper subset of P. Let D be a countably directed subset of P for which $\vee D \in \bigcap_{n<\omega} U_n$. Then for each $n < \omega$, $\vee D \in U_n$, whence there is $d_n \in D \cap U_n$. As D is countably directed, there is $d^* \in D$ such that $d^* \geq d_n$ for all $n < \omega$, and hence $d^* \in \bigcap_{n<\omega} U_n$. Thus $\bigcap_{n<\omega} U_n \in \sigma_{\omega^*}(P)$.

Definition 4.5. A T_0 space X is said to be an ω^*-d-space (or an ω^*-*monotone convergence space*), if every subset D that is countably directed relative to the specialization order of X has a sup, and the relation $\sup D \in U$ for any open set U of X implies $D \cap U \neq \emptyset$.

For a T_0 space X (endowed with the specialization order), let $\mathcal{D}_c^{\omega^*}(X) = \{\overline{D} : D \in \mathcal{D}^{\omega^*}(X)\}$. Then $\mathcal{S}_c(X) \subseteq \mathcal{D}_c^{\omega^*}(X) \subseteq \mathcal{D}_c(X)$.

Proposition 4.6. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is an ω^*-d-space.*
(2) *$\mathcal{D}_c^{\omega^*}(X) = \mathcal{S}_c(X)$, that is, for any $D \in \mathcal{D}^{\omega^*}(X)$, the closure of D has a (unique) generic point.*
(3) *X is an ω^*-dcpo and $\mathcal{O}(X) \subseteq \sigma_{\omega^*}(X)$.*
(4) *For any $D \in \mathcal{D}^{\omega^*}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap_{d \in D} \uparrow d \subseteq U$ implies $\uparrow d \subseteq U$ (i.e., $d \in U$) for some $d \in D$.*

Proof. (1) ⇔ (2): Clearly, (2) implies (1). Now we show that (1) implies (2). Let $D \in \mathcal{D}^{\omega^*}(X)$. Then by (1), $\vee D$ exists and the relation $\vee D \in U$ for any open set U of X implies $D \cap U \neq \emptyset$. Therefore, $\overline{D} = \overline{\{\vee D\}}$.

(1) ⇒ (3): Trivial.

(3) ⇒ (4): Suppose that $D \in \mathcal{D}^{\omega^*}(X)$ and $U \in \mathcal{O}(X)$ with $\bigcap_{d \in D} \uparrow d \subseteq U$. Then by condition (3), $\uparrow \vee D = \bigcap_{d \in D} \uparrow d \subseteq U \in \sigma_{\omega^*}(X)$. Therefore, $\vee D \in U$, whence $d \in U$ for some $d \in D$.

(4) ⇒ (2): For each $D \in \mathcal{D}^{\omega^*}(X)$ and $A \in \mathcal{C}(X)$ with $D \subseteq A$, by condition (4), we have $\overline{D} \cap \bigcap_{d \in D} \uparrow d \neq \emptyset$ (otherwise, $\overline{D} \cap \bigcap_{d \in D} \uparrow d = \emptyset$ implies $\bigcap_{d \in D} \uparrow d \subseteq X \setminus \overline{D}$, whence $\uparrow d \subseteq X \setminus \overline{D}$ for some $d \in D$ by condition (4), a contradiction). Select an $x \in \overline{D} \cap \bigcap_{d \in D} \uparrow d$. Then $D \subseteq \downarrow x \subseteq \overline{D}$, and hence $\overline{D} = \downarrow x$. Thus X is an ω^*-d-space. □

By Proposition 4.6, every d-space is an ω^*-d-space, and for any ω^*-dcpo P, $\Sigma_{\omega^*} P$ is an ω^*-d-space. Let $Q = (\mathbb{N}^{(<\omega)}, \subseteq)$. Then Q is an ω^*-dcpo but not a dcpo, whence $\Sigma_{\omega^*} Q$ is an ω^*-d-space but not a d-space.

Definition 4.7. A T_0 space X is called a *countably directed closure space*, or ω^*-**DC** *space* for short, if $\mathsf{Irr}_c(X) = \mathcal{D}_c^{\omega^*}(X)$, that is, for each $A \in \mathsf{Irr}_c(X)$, there exists a countably directed subset D of X such that $A = \overline{D}$.

Now we introduce another countable version of well-filtered spaces.

Definition 4.8. A T_0 space X is called ω^*-*well-filtered*, if for any countably filtered family $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$ (that is, $\{K_i : i \in I\} \in \mathcal{D}^{\omega^*}(\mathsf{K}(X))$) and $U \in \mathcal{O}(X)$, it satisfies

$$\bigcap_{i \in I} K_i \subseteq U \;\Rightarrow\; \exists j \in I, K_j \subseteq U.$$

Clearly, every well-filtered space is ω^*-well-filtered. The converse implication does not hold in general, as shown by the following example.

Example 4.9. Let $P = (\mathbb{N}^{(<\omega)}, \subseteq)$. Then P is countable and every countably directed subset of P has a largest element, whence $\sigma_{\omega^*}(P) = \alpha(P)$. It is well-known that in an Alexandroff space of a poset, the compact saturated sets are exactly the upward closures of finite sets (see, for example, the first paragraph of Section 3.2 of [10]). So $\mathsf{K}(\Sigma_{\omega^*} P) = \{\uparrow F : F \in P^{(<\omega)}\}$. Hence every family of elements of $\mathsf{K}(\Sigma_{\omega^*} P)$ is countable, and consequently, any countably filtered family of elements of $\mathsf{K}(\Sigma_{\omega^*} P)$ has a least element. Therefore, $\Sigma_{\omega^*} P$ is ω^*-well-filtered. Since P is not a dcpo (P is directed but has no largest element), $\Sigma_{\omega^*} P$ is not a d-space, and hence not well-filtered.

Proposition 4.10. *Every ω^*-well-filtered space is an ω^*-d-space.*

Proof. Let X be an ω^*-well-filtered space and $D \in \mathcal{D}^{\omega^*}(X)$. Then $\{\uparrow d : d \in D\} \in \mathcal{D}^{\omega^*}(\mathsf{K}(X))$. By the ω^*-well-filteredness of X, we have $\bigcap_{d \in D} \uparrow d \nsubseteq X \setminus \overline{D}$ or, equivalently, $\bigcap_{d \in D} \uparrow d \cap \overline{D} \neq \emptyset$. Therefore, there is $x \in \bigcap_{d \in D} \uparrow d \cap \overline{D}$, and hence $\overline{D} = \overline{\{x\}}$. □

Proposition 4.11. *For an ω^*-well-filtered X and a countably filtered family $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$, $\bigcap_{i \in I} K_i \in \mathsf{K}(X)$.*

Proof. The proof is similar to that of Proposition 2.13 and so is omitted. □

In the following, using the topological Rudin Lemma, we prove that a T_0 space X is ω^*-well-filtered iff the Smyth power space of X is ω^*-well-filtered iff the Smyth power space of X is an ω^*-d-space. The corresponding results for well-filteredness and ω-well-filteredness are proved in [21, 22, 23].

Theorem 4.12. *For a T_0 space X, the following conditions are equivalent:*

(1) X *is* ω^**-well-filtered.*
(2) $P_S(X)$ *is an* ω^**-d-space.*
(3) $P_S(X)$ *is* ω^**-well-filtered.*

Proof. (1) $\Rightarrow$ (2): Suppose that X is an ω^*-well-filtered space. For any countably filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, by Proposition 4.11, $\bigcap \mathcal{K} \in \mathsf{K}(X)$. Therefore, by Lemma 2.6, $\mathsf{K}(X)$ is an ω^*-dcpo. Clearly, by the ω^*-well-filteredness of X, $\Box U \in \sigma_{\omega^*}(\mathsf{K}(X))$ for any $U \in \mathcal{O}(X)$. Thus $P_S(X)$ is an ω^*-d-space.

(2) $\Rightarrow$ (3): Suppose that $\{\mathcal{K}_i : i \in I\} \subseteq \mathsf{K}(P_S(X))$ is countably filtered, $\mathcal{U} \in \mathcal{O}(P_S(X))$, and $\bigcap_{i\in I} \mathcal{K}_i \subseteq \mathcal{U}$. Then $\{\mathcal{K}_i : i \in I\}$ is directed in $\mathsf{K}(P_S(X))$ (with the specialization order, i.e., the Smyth order), whence $\{\mathcal{K}_i : i \in I\} \in \mathsf{Irr}(P_S(P_S(X)))$. If $\mathcal{K}_i \not\subseteq \mathcal{U}$ for all $i \in I$, then by Lemma 3.1, $\mathsf{K}(X) \setminus \mathcal{U}$ contains an irreducible closed subset $\mathcal{A}$ that still meets all $\mathcal{K}_i$ $(i \in I)$. For each $i \in I$, let $K_i = \bigcup \uparrow_{\mathsf{K}(X)}(\mathcal{A} \cap \mathcal{K}_i)$ $(= \bigcup(\mathcal{A} \cap \mathcal{K}_i))$. Then by Lemma 2.7, $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$ is countably filtered, and $K_i \in \mathcal{A}$ for all $i \in I$ since $\mathcal{A} = \downarrow_{\mathsf{K}(X)}\mathcal{A}$. Let $K = \bigcap_{i\in I} K_i$. Then $K \in \mathsf{K}(X)$ and $K = \bigvee_{\mathsf{K}(X)}\{K_i : i \in I\} \in \mathcal{A}$ by Lemma 2.6 and condition (2). We claim that $K \in \bigcap_{i\in I} \mathcal{K}_i$. Suppose, on the contrary, that $K \notin \bigcap_{i\in I} \mathcal{K}_i$. Then there is a $j \in I$ such that $K \notin \mathcal{K}_j$. Select a $G \in \mathcal{A} \cap \mathcal{K}_j$. As $G \in \mathcal{K}_j = \uparrow_{\mathsf{K}(X)}\mathcal{K}_j$ and $K \notin \mathcal{K}_j$, we have $G \not\sqsubseteq K$, i.e., $K \not\subseteq G$. It follows that $g \in K_i = \bigcup(\mathcal{A} \cap \mathcal{K}_i)$ for all $i \in I$ and $G \notin \Diamond_{\mathsf{K}(K)}\overline{\{g\}}$. For each $i \in I$, by $g \in K_i = \bigcup(\mathcal{A} \cap \mathcal{K}_i)$, there is a $K_i^g \in \mathcal{A} \cap \mathcal{K}_i$ such that $g \in K_i^g$, whence $K_i^g \in \Diamond_{\mathsf{K}(K)}\overline{\{g\}} \cap \mathcal{A} \cap \mathcal{K}_i$. Thus $\Diamond_{\mathsf{K}(K)}\overline{\{g\}} \cap \mathcal{A} \cap \mathcal{K}_i \neq \emptyset$ for all $i \in I$. By the minimality of $\mathcal{A}$, we have $\mathcal{A} = \Diamond_{\mathsf{K}(K)}\overline{\{g\}} \cap \mathcal{A}$, and consequently, $G \in \mathcal{A} \cap \mathcal{K}_j = \Diamond_{\mathsf{K}(K)}\overline{\{g\}} \cap \mathcal{A} \cap \mathcal{K}_j$, which is a contradiction with $G \notin \Diamond_{\mathsf{K}(K)}\overline{\{g\}}$. Thus $K \in \bigcap_{i\in I} \mathcal{K}_i \subseteq \mathcal{U} \subseteq \mathsf{K}(X) \setminus \mathcal{A}$, being a contradiction with $K \in \mathcal{A}$. Therefore, $P_S(X)$ is ω^*-well-filtered.

(3) $\Rightarrow$ (1): Suppose that $\mathcal{K} \subseteq \mathsf{K}(X)$ is countably filtered, $U \in \mathcal{O}(X)$, and $\bigcap \mathcal{K} \subseteq U$. Let $\widetilde{\mathcal{K}} = \{\uparrow_{\mathsf{K}(X)}K : K \in \mathcal{K}\}$. Then $\widetilde{\mathcal{K}} \subseteq \mathsf{K}(P_S(X))$ is countably filtered and $\bigcap \widetilde{\mathcal{K}} \subseteq \Box U$. By the ω^*-well-filteredness of $P_S(X)$, there is a $K \in \mathcal{K}$ such that $\uparrow_{\mathsf{K}(X)}K \subseteq \Box U$, whence $K \subseteq U$, proving that X is ω^*-well-filtered. $\Box$

Definition 4.13. A T_0 space X is called a countably directed closure space, ω^*-DC *space* for short, if $\mathsf{Irr}_c(X) = \mathcal{D}_c^{\omega^*}(X)$, that is, for each $A \in \mathsf{Irr}_c(X)$, there exists a countably directed subset of X such that $A = \overline{D}$.

By Remark 2.8, Proposition 4.6 and Proposition 4.10, we deduce the following result.

Proposition 4.14. *For any* T_0 *space* X*, the following conditions are equivalent:*

(1) X *is sober.*
(2) X *is an* ω^*-DC *and* ω^**-well-filtered space.*
(3) X *is an* ω^*-DC *and* ω^**-d-space.*

Finally, it is worth mentioning that some discussions in Section 4 can be carried out for a general (infinite) cardinal number $\aleph$. Explicitly, for an arbitrary cardinal number $\aleph$, a nonempty subset D of a poset P is called $\aleph$-*directed* if every $E \in D^{(<\aleph)}$ has an upper bound in D. The set of all $\aleph$-directed sets of P is denoted by $\mathcal{D}^{\aleph^*}(P)$. The poset P is called an $\aleph$-*directed complete poset*, or $\aleph^*$-*dcpo* for short, if for any $D \in \mathcal{D}^{\aleph^*}(P)$, $\vee D$ exists in P. So one get the ω^*-directedness by taking $\aleph = \omega_1$ and the directedness by taking $\aleph = \omega$. We would gain in generality this way. In fact, instead of mere cardinals, one may even consider arbitrary subset selections consisting of (some) directed subsets of a poset P (cf. [2]) and carry out some similar discussions for an arbitrary subset selection of P.

5. First-countability of sobrifications and ω-Rudin spaces

In this section, we prove that if the sobrification of a T_0 space X is first-countable, then X is a ω-Rudin space. Hence every ω-well-filtered space having a first-countable sobrification is sober.

We first give two useful lemmas. Our proof arguments (and also the proof argument of Theorem 5.6 below) are inspired by those of [15, Theorem 3.1] and [1, Proposition 4.1] (see also [22, Theorem 4.1]), such constructions originate from M. Schröder [18].

Lemma 5.1. *Let X be a T_0 space and $A \in \mathcal{C}(X)$. For $\{U_n : n \in \mathbb{N}\} \subseteq \mathcal{O}(X)$ with $U_1 \supseteq U_2 \supseteq ... \supseteq U_n \supseteq U_{n+1} \supseteq ...$, if $A \in m(\{U_n : n \in \mathbb{N}\})$ and $x_n \in U_n \cap A$ for each $n \in \mathbb{N}$, then any subset of $\{x_n : n \in \mathbb{N}\}$ is compact.*

Proof. Suppose E is a nonempty subset of $\subseteq \{x_n : n \in \mathbb{N}\}$ and $\{V_i : i \in I\}$ is an open cover of E, that is, $E \subseteq \bigcup_{i\in I} V_i$.

Case 1. $E \cap (X \setminus V_j)$ is finite for some $j \in I$.

Then there is $I_j \in I^{(<\omega)}$ such that $E \cap (X \setminus V_j) \subseteq \bigcup_{i\in I_j} V_i$, and hence $E \subseteq V_j \cup \bigcup_{i\in I_j} V_i$.

Case 2. $E \cap (X \setminus V_i)$ is infinite for all $i \in I$.

For each $i \in I$, since $U_1 \supseteq U_2 \supseteq ... \supseteq U_n \supseteq U_{n+1} \supseteq ...$ and $x_n \in U_n \cap A$ for each $n \in \mathbb{N}$, we have that $U_n \cap A \cap (X \setminus V_i) \neq \emptyset$ for all $n \in \mathbb{N}$, and hence $A \cap (X \setminus V_i) = A$ by the minimality of A. It follows that $A \subseteq \bigcap_{i\in I}(X \setminus V_i) = X \setminus \bigcup_{i\in I} V_i$. Therefore, $E \subseteq A \cap \bigcup_{i\in I} V_i = \emptyset$, which is impossible.

By Case 1 and Case 2, E is compact. □

Lemma 5.2. *Let X be a T_0 space and $A \in \mathsf{Irr}_c(X)$. For any open neighborhood base $\{\Diamond U_i : i \in I\}$ of A in X^s, $A \in m(\{U_i : i \in I\})$.*

Proof. Clearly, $A \in M(\{U_i : i \in I\})$. Suppose $B \in \mathcal{C}(X)$ and $B \subseteq A$. If $B \neq A$, then $A \cap (X \setminus B) \neq \emptyset$, and hence $A \in \Diamond(X \setminus B)$. Since $\{\Diamond U_i : i \in I\}$ is an open neighborhood base at A in X^s, there is $j \in I$ such that $\Diamond U_j \subseteq \Diamond(X \setminus B)$ or, equivalently, $U_j \subseteq X \setminus B$. Therefore, $U_j \cap B = \emptyset$. So $B \notin M(\{U_i : i \in I\})$. Thus $A \in m(\{U_i : i \in I\})$. □

Proposition 5.3. *For a T_0 space X, the following two conditions are equivalent:*

(1) *X is second-countable.*

(2) *X^s is second-countable.*

Proof. Clearly, the correspondence $U \leftrightarrow \Diamond U$ is a lattice isomorphism between $\mathcal{O}(X)$ and $\mathcal{O}(X^s)$ (recall that $X^s = P_H(\mathsf{Irr}_c(X))$). Whence, for a $\mathcal{B} \subseteq \mathcal{O}(X)$, $\mathcal{B}$ is a base (or a subbase) of X iff $\Diamond\mathcal{B} = \{\Diamond U : U \in \mathcal{B}\}$ is a base (or a subbase) of X^s. Therefore, X is second-countable iff X^s is second-countable. □

For a countable T_0 space X, it is easy to see that X is second-countable iff X is first-countable. Suppose that X is first-countable. Let $X = \{x_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, by the first-countability of X, there is a countable base $\mathcal{B}_n$ at x_n. Let $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$. Then $\mathcal{B}$ is a countable base of X. Thus X is second-countable. Therefore, by Proposition 5.3, we deduce the following corollary.

Corollary 5.4. *If a T_0 space X is first-countable and $|X| \leq \omega$, then X^s is second-countable.*

Since first-countability is a hereditary property, by Remark 2.5, we have the following result.

Proposition 5.5. *For a T_0 space X, if X^s is first-countable, then X is first-countable.*

The converse of Proposition 5.5 does not hold in general, as shown in Example 5.12 below. It is worth noting that the Scott topology on a countable complete lattice may not be first-countable, see [22, Example 4.8].

Now we give one of the main results of this paper.

Theorem 5.6. *Every T_0 space with a first-countable sobrification is an ω-Rudin space.*

Proof. Suppose that X is a T_0 space and X^s is first-countable. Now we show that X is an ω-Rudin space. Let $A \in \mathsf{Irr}_c(X)$. By the first-countability of X^s, there is an open neighborhood base $\{\Diamond U_n : n \in \mathbb{N}\}$ of A such that

$$\Diamond U_1 \supseteq \Diamond U_2 \supseteq \ldots \supseteq \Diamond U_n \supseteq \ldots,$$

or equivalently, $\ldots \subseteq U_n \subseteq \ldots \subseteq U_2 \subseteq U_1$. By Lemma 5.2, $A \in m(\{U_n : n \in \mathbb{N}\})$. For each $n \in \mathbb{N}$, choose an $x_n \in U_n \cap A$, and let $K_n = \uparrow\{x_m : m \geq n\}$. Then $K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq \ldots$, and $\{K_n : n \in \mathbb{N}\} \subseteq \mathsf{K}(X)$

by Lemma 5.1. Clearly, $A \in M(\{K_n : n \in \mathbb{N}\})$. For any $B \in \mathcal{C}(X)$, if B is a proper subset of A, that is, $A \cap (X \setminus B) = A \setminus B \neq \emptyset$, then $A \in \Diamond(X \setminus B) \in \mathcal{O}(X^s)$ $(= \mathcal{O}(P_H(\mathsf{Irr}_c(X))))$. Therefore, $\Diamond U_m \subseteq \Diamond(X \setminus B)$ for some $m \in \mathbb{N}$, and hence $U_m \subseteq X \setminus B$ or, equivalently, $U_m \cap B = \emptyset$. Thus $B \notin M(\{K_n : n \in \mathbb{N}\})$, proving $A \in m(\{K_n : n \in \mathbb{N}\})$. So X is an ω-Rudin space. $\square$

By Proposition 5.3, Corollary 5.4 and Theorem 5.6, we deduce the following two results.

Corollary 5.7. *Every second-countable T_0 space is an ω-Rudin space.*

Corollary 5.8. *Every countable first-countable T_0 space is an ω-Rudin space.*

Theorem 5.9. *Every ω-well-filtered space with a first-countable sobrification is sober.*

Proof. For $A \in \mathsf{Irr}_c(X)$, by Theorem 5.6, there is a countable filtered family $\{K_n : n \in \mathbb{N}\} \subseteq \mathsf{K}(X)$ such that $A \in m(\{K_n : n \in \mathbb{N}\})$. Since X is ω-well-filtered, $\bigcap_{n\in\mathbb{N}} K_n \not\subseteq X \setminus A$, that is, $\bigcap_{n\in\mathbb{N}} K_n \cap A \neq \emptyset$. Choose $x \in \bigcap_{n\in\mathbb{N}} K_n \cap A$. Then $\overline{\{x\}} \in M(\{K_n : n \in \mathbb{N}\})$ and $\overline{\{x\}} \subseteq A$. By the minimality of A, we have $A = \overline{\{x\}}$. Thus X is sober. $\square$

By Corollary 5.7, Corollary 5.8 and Theorem 5.9, we get the following two corollaries.

Corollary 5.10. *Every second-countable ω-well-filtered space is sober.*

Corollary 5.11. *Every countable first-countable ω-well-filtered space is sober.*

In Theorem 5.6 and Theorem 5.9, the first-countability of X^s can not be weakened to that of X as shown in the following example (see also Example 6.9 in Section 6).

Example 5.12. Let ω_1 be the first uncountable ordinal number and $P = [0, \omega_1)$. Then

(a) $\mathcal{C}(\Sigma P) = \{\downarrow t : t \in P\} \cup \{\emptyset, P\}$.
(b) ΣP is compact since P has a least element 0.
(c) ΣP is first-countable.
(d) $(\Sigma P)^s$ is not first-countable. In fact, it is easy to verify that $(\Sigma P)^s$ is homeomorphic to $\Sigma[0, \omega_1]$. Since sup of a countable family of countable ordinal numbers is still a countable ordinal number, $\Sigma[0, \omega_1]$ has no countable base at the point ω_1.
(e) P is an ω-dcpo but not a dcpo (note that P is directed and $\vee P$ does not exist). So ΣP is an ω-d-space but not a d-space, and hence not a sober space.
(f) $\mathsf{K}(\Sigma P) = \{\uparrow x : x \in P\}$. For $K \in \mathsf{K}(\Sigma P)$, we have $\inf K \in K$, and hence $K = \uparrow \inf K$.
(g) ΣP is a Rudin space. One can easily check that $\mathsf{Irr}_c(\Sigma P) = \{\downarrow x : x \in P\} \cup \{P\}$. Clearly, $\downarrow x$ is a Rudin set for each $x \in P$. Now we show that P is a Rudin set. First, $\{\uparrow s : s \in P\}$ is filtered. Second, $P \in M(\{\uparrow s : s \in P\})$. For a closed subset B of ΣP, if $B \neq P$, then $B = \downarrow t$ for some $t \in P$, and hence $\uparrow(t+1) \cap \downarrow t = \emptyset$. Thus $B \notin M(\{\uparrow s : s \in P\})$, proving that P is a Rudin set.
(h) ΣP is not an ω-Rudin space. We prove that the irreducible closed set P is not an ω-Rudin set. For any countable filtered family $\{\uparrow \alpha_n : n \in \mathbb{N}\} \subseteq \mathsf{K}(\Sigma P)$, let $\beta = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then β is still a countable ordinal number. Clearly, $\downarrow \beta \in M(\{\uparrow \alpha_n : n \in \mathbb{N}\})$ and $P \neq \downarrow \beta$. Therefore, $P \notin m(\{\uparrow \alpha_n : n \in \mathbb{N}\})$. Thus P is not an ω-Rudin set, and hence ΣP is not an ω-Rudin space.
(i) ΣP is ω-well-filtered. If $\{\uparrow x_n : n \in \mathbb{N}\} \subseteq \mathsf{K}(\Sigma P)$ is countable filtered family and $U \in \sigma(P)$ with $\bigcap_{n\in\mathbb{N}} \uparrow x_n \subseteq U$, then $\{x_n : i \in \mathbb{N}\}$ is a countable subset of $P = [0, \omega_1)$. Since sup of a countable family of countable ordinal numbers is still a countable ordinal number, we have $\beta = \sup\{x_n : n \in \mathbb{N}\} \in P$, and hence $\uparrow \beta = \bigcap_{n\in\mathbb{N}} \uparrow x_n \subseteq U$. Therefore, $\beta \in U$, and consequently, $x_n \in U$ for some $n \in \mathbb{N}$ or, equivalently, $\uparrow x_n \subseteq U$, proving that ΣP is ω-well-filtered.

6. First-countability and well-filtered determined spaces

In this section, we prove that any continuous mapping $f : X \to Y$ from a first-countable T_0 space X to an ω-well-filtered space Y maps any irreducible subset of X to a Rudin set of Y, hence any first-countable ω-well-filtered space is a Rudin space. We also show that any first-countable T_0 space is well-filtered determined. In [22], it was proved that in a first-countable ω-well-filtered T_0 space X, all irreducible closed subsets of X are directed (see [22, Theorem 4.1]). In the following we will strengthen this result by proving that in a first-countable ω-well-filtered space X, every irreducible closed subset of X is countably directed.

First, we give the following key lemma.

Lemma 6.1. *Let X be a first-countable T_0 space and $A \in \mathsf{Irr}(X)$. Then for any $\{a_n : n \in \mathbb{N}\} \subseteq \overline{A}$, there exists a sequence $(c_n)_{n\in\mathbb{N}}$ in A such that the following three conditions hold:*

(1) $\forall n \in \mathbb{N}, K_n = \uparrow\{c_m : m \geq n\} \in \mathsf{K}(X)$.
(2) $\bigcap_{n\in\mathbb{N}} K_n = \bigcap_{n\in\mathbb{N}} \uparrow c_n$.
(3) $\bigcap_{n\in\mathbb{N}} K_n \subseteq \bigcap_{n\in\mathbb{N}} \uparrow a_n$.

Proof. For each $x \in X$, since X is first-countable, there is an open neighborhood base $\{U_n(x) : n \in \mathbb{N}\}$ of x such that

$$U_1(x) \supseteq U_2(x) \supseteq \ldots \supseteq U_k(x) \supseteq \ldots,$$

that is, $\{U_n(x) : n \in \mathbb{N}\}$ is a decreasing sequence of open subsets.

For any $m \in \mathbb{N}$ and any $\{(l_i, k_i) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq m\}$, since $\{a_n : n \in \mathbb{N}\} \subseteq \overline{A}$ and $A \in \mathsf{Irr}(X)$, we have $\bigcap\limits_{i=1}^{m} U_{l_i}(a_{k_i}) \cap A \neq \emptyset$.

Choose $c_1 \in U_1(a_1) \cap A$. Now suppose we already have a set $\{c_1, c_2, \ldots, c_{n-1}\}$ such that for each $2 \leq i \leq n-1$,

$$c_i \in \bigcap_{j=1}^{i-1} U_i(c_j) \cap \bigcap_{j=1}^{i} U_i(a_j) \cap A.$$

By the irreducibility of A, we have $\bigcap\limits_{j=1}^{n-1} U_n(c_j) \cap \bigcap\limits_{j=1}^{n} U_n(a_j) \cap A \neq \emptyset$. So we can choose $c_n \in \bigcap\limits_{j=1}^{n-1} U_n(c_j) \cap \bigcap\limits_{j=1}^{n} U_n(a_j) \cap A$.

For each $n \in \mathbb{N}$, let $K_n = \uparrow\{c_m : m \geq n\}$. Now we verify that the sequence $(c_n)_{n\in\mathbb{N}}$ satisfies the three conditions in Lemma 6.1.

Claim 1: $\{K_n : n \in \mathbb{N}\} \subseteq \mathsf{K}(X)$ and $\{K_n : n \in \mathbb{N}\}$ is a decreasing sequence.

Suppose that $n \in \mathbb{N}$ and $\{U_i : i \in I\}$ is an open cover of K_n, i.e., $K_n \subseteq \bigcup_{i\in I} U_i$. Then there is $i_0 \in I$ such that $c_n \in U_{i_0}$, and thus there is $m \geq n$ such that $c_n \in U_m(c_n) \subseteq U_{i_0}$. It follows that $c_k \in U_k(c_n) \subseteq U_m(c_n) \subseteq U_{i_0}$ for all $k \geq m$. Thus $\{c_k : k \geq m\} \subseteq U_{i_0}$. For each c_k, where $n \leq k < m$, choose a U_{i_k} such that $c_k \in U_{i_k}$. Then the finite family $\{U_{i_k} : n \leq k < m\} \cup \{U_{i_0}\}$ covers K_n. So K_n is compact. Clearly, $K_n \supseteq K_{n+1}$ for all $n \in \mathbb{N}$, that is, $\{K_n : n \in \mathbb{N}\}$ is a decreasing sequence.

Claim 2: $\bigcap_{n\in\mathbb{N}} K_n = \bigcap_{n\in\mathbb{N}} \uparrow c_n$.

Clearly, $\bigcap_{n\in\mathbb{N}} \uparrow c_n \subseteq \bigcap_{n\in\mathbb{N}} K_n$. Now we show $\bigcap_{n\in\mathbb{N}} K_n \subseteq \uparrow c_m$ for every $m \in \mathbb{N}$. Suppose $U \in \mathcal{O}(X)$ with $c_m \in U$. Then $U_{n(m)}(c_m) \subseteq U$ for some $n(m) \in \mathbb{N}$. For any $l \geq \max\{m, n(m)\}$, we have $K_l = \uparrow\{c_k : k \geq l\} \subseteq \bigcup_{k\geq l} U_k(c_m) \subseteq U_{n(m)}(c_m) \subseteq U$. It follows that $\bigcap_{n\in\mathbb{N}} K_n \subseteq \bigcap\{U : c_m \in U \in \mathcal{O}(X)\} = \uparrow c_m$. Thus $\bigcap_{n\in\mathbb{N}} K_n \subseteq \bigcap_{n\in\mathbb{N}} \uparrow c_n$.

Claim 3: $\bigcap_{n\in\mathbb{N}} K_n \subseteq \bigcap_{n\in\mathbb{N}} \uparrow a_n$.

For $m \in \mathbb{N}$ and $W \in \mathcal{O}(X)$ with $a_m \in W$, we have $U_{k(m)}(a_m) \subseteq W$ for some $k(m) \in \mathbb{N}$. For any $l \geq \max\{m, k(m)\}$, we have $K_l = \uparrow\{c_k : k \geq l\} \subseteq \bigcup_{k\geq l} U_k(a_m) \subseteq U_{k(m)}(a_m) \subseteq W$, and consequently, $\uparrow K_l \subseteq W$. It follows that $\bigcap_{n\in\mathbb{N}} K_n \subseteq \bigcap\{W : a_m \in W \in \mathcal{O}(X)\} = \uparrow a_m$. Thus $\bigcap_{n\in\mathbb{N}} K_n \subseteq \bigcap_{n\in\mathbb{N}} \uparrow a_n$.

Therefore, the sequence $(c_n)_{n\in\mathbb{N}}$ satisfies the all three conditions in Lemma 6.1. □

We shall call the countable family $\{K_n : n \in \mathbb{N}\} \subseteq \mathsf{K}(X)$ in the above proof *a decreasing sequence of compact saturated subsets related to* $\{a_n : n \in \mathbb{N}\}$ *via the sequence* $(c_n)_{n\in\mathbb{N}}$.

Remark 6.2. Suppose that X is a first-countable T_0 space and $f : X \to Y$ is a continuous mapping from X to a T_0 space Y. Let $A \in \mathsf{Irr}(X)$ and $\{a_n : n \in \mathbb{N}\} \subseteq \overline{A}$. Then any sequence $(c_n)_{n\in\mathbb{N}}$ of A obtained in the proof of Lemma 6.1 and $\{K_n = \uparrow\{c_m : m \geq n\} : n \in \mathbb{N}\}$ satisfy the following additional conditions:

(a) $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) = \bigcap_{n\in\mathbb{N}} \uparrow f(c_n)$.
(b) $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \subseteq \bigcap_{n\in\mathbb{N}} \uparrow f(a_n)$.

Proof. By the continuity of f, f is order-preserving (with the specialization orders of X and Y), whence $\uparrow f(K_n) = \uparrow\{f(c_m) : m \geq n\}$ for each $n \in \mathbb{N}$.

First, we show that $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) = \bigcap_{n\in\mathbb{N}} \uparrow f(c_n)$. Clearly, $\bigcap_{n\in\mathbb{N}} \uparrow f(c_n) \subseteq \bigcap_{n\in\mathbb{N}} \uparrow f(K_n)$. Now we show $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \subseteq \uparrow f(c_m)$ for each $m \in \mathbb{N}$. Suppose $V \in \mathcal{O}(Y)$ with $f(c_m) \in V$. Then $c_m \in f^{-1}(V) \in \mathcal{O}(X)$, whence $U_{n(m)}(c_m) \subseteq f^{-1}(V)$ for some $n(m) \in \mathbb{N}$. For any $l \geq \max\{m, n(m)\}$, we have $K_l = \uparrow\{c_k : k \geq l\} \subseteq \bigcup_{k\geq l} U_k(c_m) \subseteq U_{n(m)}(c_m) \subseteq f^{-1}(V)$, and consequently, $\uparrow f(K_l) \subseteq V$. It follows that $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \subseteq \bigcap\{V : f(c_m) \in V \in \mathcal{O}(Y)\} = \uparrow f(c_m)$. Thus $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \subseteq \bigcap_{n\in\mathbb{N}} \uparrow f(c_n)$.

Second, we verify that $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \subseteq \bigcap_{n\in\mathbb{N}} \uparrow f(a_n)$. For $m \in \mathbb{N}$ and $W \in \mathcal{O}(Y)$ with $f(a_m) \in W$, we have $a_m \in f^{-1}(W) \in \mathcal{O}(X)$, whence $U_{k(m)}(a_m) \subseteq f^{-1}(W)$ for some $k(m) \in \mathbb{N}$. For any $l \geq \max\{m, k(m)\}$, we have $K_l = \uparrow\{c_k : k \geq l\} \subseteq \bigcup_{k\geq l} U_k(a_m) \subseteq U_{k(m)}(a_m) \subseteq f^{-1}(W)$, and consequently, $\uparrow f(K_l) \subseteq W$. It follows that $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \subseteq \bigcap\{W : f(a_m) \in W \in \mathcal{O}(Y)\} = \uparrow f(a_m)$. Thus $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \subseteq \bigcap_{n\in\mathbb{N}} \uparrow f(a_n)$. $\square$

Definition 6.3. Let X be a T_0 space and $f : X \to Y$ a continuous mapping from X to a T_0 space Y, and let $A \in \mathsf{Irr}(X)$ and $\{a_n : n \in \mathbb{N}\} \subseteq \overline{A}$. A countable family $\{K_n : n \in \mathbb{N}\} \subseteq \mathsf{K}(X)$ is said to be

(i) a *decreasing sequence of compact saturated subsets related to* $\{a_n : n \in \mathbb{N}\}$, provided that there is a sequence $(c_n)_{n\in\mathbb{N}}$ in A such that $K_n = \uparrow\{c_m : m \geq n\}$ for each $n \in \mathbb{N}$ and conditions (1)–(3) of Lemma 6.1 hold. Let $\mathcal{K}^\omega(\{a_n : n \in \mathbb{N}\})$ denote the set of all decreasing sequences of compact saturated subsets related to $\{a_n : n \in \mathbb{N}\}$ and let $\mathcal{K}^\omega(A) = \bigcup\{\mathcal{K}^\omega(\{a_n : n \in \mathbb{N}\}) : \{a_n : n \in \mathbb{N}\} \subseteq \overline{A}\}$.

(ii) a *decreasing sequence of compact saturated subsets related to* $\{a_n : n \in \mathbb{N}\}$ *and* f, provided that there is a sequence $(c_n)_{n\in\mathbb{N}}$ in A such that $K_n = \uparrow\{c_m : m \geq n\}$ for each $n \in \mathbb{N}$ and both conditions (1)–(3) of Lemma 6.1 and conditions (a), (b) of Remark 6.2 hold. Let $\mathcal{K}^\omega_f(\{a_n : n \in \mathbb{N}\})$ denote the set of all decreasing sequences of compact saturated subsets related to $\{a_n : n \in \mathbb{N}\}$ and f, and let $\mathcal{K}^\omega_f(A) = \bigcup\{\mathcal{K}^\omega_f(\{a_n : n \in \mathbb{N}\}) : \{a_n : n \in \mathbb{N}\} \subseteq \overline{A}\}$.

Proposition 6.4. *Let X be an ω-well-filtered T_0 space, Y a T_0 space and $f : X \to Y$ a continuous mapping. Then for any $A \in \mathsf{Irr}(X)$ and $\{a_n : n \in \mathbb{N}\} \subseteq \overline{A}$, we have $\{f(a_n) : n \in \mathbb{N}\} \subseteq \overline{f(A)} \in \mathsf{Irr}_c(Y)$, and f induces a mapping $f^* : \mathcal{K}^\omega(\{a_n : n \in \mathbb{N}\}) \to \mathcal{K}^\omega(\{f(a_n) : n \in \mathbb{N}\})$, where $f^*(\{K_n : n \in \mathbb{N}\} = \{\uparrow f(K_n) : n \in \mathbb{N}\}$ for each $\{K_n : n \in \mathbb{N}\} \in \mathcal{K}^\omega(\{a_n : n \in \mathbb{N}\})$. Whence, f induces a natural mapping $f_A : \mathcal{K}^\omega(A) \to \mathcal{K}^\omega(\overline{f(A)}), \{K_n : n \in \mathbb{N}\} \mapsto \{\uparrow f(K_n) : n \in \mathbb{N}\}$.*

Proof. By the continuity of f and Lemma 2.3, $\{f(a_n) : n \in \mathbb{N}\} \subseteq \overline{f(A)} \in \mathsf{Irr}_c(Y)$ and f is order-preserving. Suppose $\{K_n : n \in \mathbb{N}\} \in \mathcal{K}^\omega(\{a_n : n \in \mathbb{N}\})$. Then there is a sequence $(c_n)_{n\in\mathbb{N}}$ in A such that $K_n = \uparrow\{c_m : m \geq n\}$ for all $n \in \mathbb{N}$ and conditions (1)–(3) of Lemma 6.1 hold. Since f is order-preserving, $\uparrow f(K_n) = \uparrow\{f(c_m) : m \geq n\}$ for each $n \in \mathbb{N}$. Now we verify that $\{\uparrow f(K_n) : n \in \mathbb{N}\} \in \mathcal{K}^\omega(\{f(a_n) : n \in \mathbb{N}\})$; more precisely, $\{\uparrow f(K_n) : n \in \mathbb{N}\}$ is a decreasing sequence of compact saturated subsets related to $\{f(a_n) : n \in \mathbb{N}\}$ via the sequence $(f(c_n))_{n\in\mathbb{N}}$.

Claim 1: $\forall n \in \mathbb{N}, \uparrow f(K_n) = \uparrow\{f(c_m) : m \geq n\} \in \mathsf{K}(Y)$.

For each $n \in \mathbb{N}$, since $K_n \in \mathsf{K}(X)$ and f is continuous, we have $\uparrow f(K_n) \in \mathsf{K}(Y)$. Clearly, $\{\uparrow f(K_n) : n \in \mathbb{N}\}$ is a decreasing sequences of compact saturated subsets of Y.

Claim 2: $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) = \bigcap_{n\in\mathbb{N}} \uparrow f(c_n)$.

Clearly, $\bigcap_{n\in\mathbb{N}} \uparrow f(c_n) \subseteq \bigcap_{n\in\mathbb{N}} \uparrow f(K_n)$. On the other hand, by $\bigcap_{n\in\mathbb{N}} K_n = \bigcap_{n\in\mathbb{N}} \uparrow c_n$ and Proposition 2.14, we have $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) = \uparrow f(\bigcap_{n\in\mathbb{N}} K_n) = \uparrow f(\bigcap_{n\in\mathbb{N}} \uparrow c_n) \subseteq \bigcap_{n\in\mathbb{N}} \uparrow f(c_n)$. Thus $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) = \bigcap_{n\in\mathbb{N}} \uparrow f(c_n)$.

Claim 3: $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \subseteq \bigcap_{n\in\mathbb{N}} \uparrow f(a_n)$.

By $\bigcap_{n\in\mathbb{N}} K_n \subseteq \bigcap_{n\in\mathbb{N}} \uparrow a_n$ and Proposition 2.14, $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) = \uparrow f(\bigcap_{n\in\mathbb{N}} K_n) \subseteq \uparrow f(\bigcap_{n\in\mathbb{N}} \uparrow a_n) \subseteq \bigcap_{n\in\mathbb{N}} \uparrow f(a_n)$.

By Claims 1–3, $\{\uparrow f(K_n) : n \in \mathbb{N}\}$ is a decreasing sequence of compact saturated subsets related to $\{f(a_n) : n \in \mathbb{N}\}$ via the sequence $(f(c_n))_{n\in\mathbb{N}}$ of $f(A)$. □

Proposition 6.5. *Suppose that X is a first-countable T_0 space, Y is an ω-well-filtered T_0 space and $f : X \to Y$ is a continuous mapping. Then for any $A \in \mathsf{Irr}(X)$ and $\{a_n : n \in \mathbb{N}\} \subseteq \overline{A}$, $\bigcap_{n\in\mathbb{N}} \uparrow f(a_n) \bigcap \overline{f(A)} \neq \emptyset$.*

Proof. By Lemma 6.1 and Remark 6.2, there exists a sequence $(c_n)_{n\in\mathbb{N}}$ in A such that $(c_n)_{n\in\mathbb{N}}$ and $\{K_n = \uparrow\{c_m : m \geq n\} : n \in \mathbb{N}\}$ satisfy conditions (1)–(3) of Lemma 6.1 and conditions (a), (b) of Remark 6.2. As f is continuous, $\{\uparrow f(K_n) : n \in \mathbb{N}\}$ is a decreasing sequence of $\mathsf{K}(Y)$. By Proposition 2.13, $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \in \mathsf{K}(Y)$. Now we show that $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \cap \overline{f(A)} \neq \emptyset$. Assume, on the contrary, that $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \cap \overline{f(A)} = \emptyset$ or, equivalently, $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \subseteq Y \setminus \overline{f(A)}$. Then by the ω-well-filteredness of Y, $\uparrow f(K_n) \subseteq Y \setminus \overline{f(A)}$ for some $n \in \mathbb{N}$, which is in contradiction with $\{c_m : m \geq n\} \subseteq A \cap K_n$. Thus $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \cap \overline{f(A)} \neq \emptyset$. As $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \subseteq \bigcap_{n\in\mathbb{N}} \uparrow f(a_n)$, we have $\bigcap_{n\in\mathbb{N}} \uparrow f(a_n) \cap \overline{f(A)} \neq \emptyset$. □

Corollary 6.6. *In a first-countable ω-well-filtered T_0 space X, every irreducible closed subset of X is countably directed. Therefore, X is an ω^*-**DC** space.*

Proof. Applying Proposition 6.5 to $A \in \mathsf{Irr}_c(X)$ and the identity $id_X : X \longrightarrow X$. □

By Remark 2.8, Proposition 4.14 and Corollary 6.6, we get the following result.

Theorem 6.7. *For a first-countable T_0 space X, the following conditions are equivalent:*

(1) *X is a sober space.*
(2) *X is a well-filtered space.*
(3) *X is an ω-well-filtered d-space.*
(4) *X is an ω-well-filtered ω^*-d-space.*

A first-countable d-space may not be sober as shown in the following example.

Example 6.8. Let X be a countably infinite set and X_{cof} the space equipped with the *co-finite topology* (the empty set and the complements of finite subsets of X are open). Then

(a) $\mathcal{C}(X_{cof}) = \{\emptyset, X\} \cup X^{(<\omega)}$, X_{cof} is T_1 and hence a d-space.
(b) $\mathsf{K}(X_{cof}) = 2^X \setminus \{\emptyset\}$.
(c) X_{cof} is first-countable.
(d) X_{cof} is locally compact and hence a Rudin space by Proposition 3.6.
(e) X_{cof} is non-sober. $\mathcal{K}_X = \{X \setminus F : F \in X^{(<\omega)}\} \subseteq \mathsf{K}(X_{cof})$ is countable filtered and $\bigcap \mathcal{K}_X = X \setminus \bigcup X^{(<\omega)} = X \setminus X = \emptyset$, but $X \setminus F \neq \emptyset$ for all $F \in X^{(<\omega)}$. Thus X_{cof} is not ω-well-filtered, and consequently, X_{cof} is not well-filtered and not sober by Remark 2.8.

A topological space Y is said to be a *Noetherian space* if every open subset is compact (see [8, Definition 9.7.1]). As $\mathsf{K}(X_{cof}) = 2^X \setminus \{\emptyset\}$, the space X_{cof} is a Noetherian space. So Example 6.8 shows that a Noetherian first-countable T_0 space (and hence a locally compact first-countable T_0 space) need not be ω-well-filtered.

The following example shows that a first-countable ω-well-filtered T_0 space need not be sober. So in Theorem 6.7, condition (4) (and so condition (3)) cannot be weakened to the condition that X is only an ω-well-filtered space.

Example 6.9. Let L be the complete chain $[0, \omega_1]$. Then

(a) $\sigma(L) \neq \sigma_\omega(L)$. Since sups of all countable families of countable ordinal numbers are still countable ordinal numbers, we have that $\{\omega_1\} \in \sigma_\omega(L)$ but $\{\omega_1\} \notin \sigma(L)$ (note that $\omega_1 = \sup\, [0, \omega_1)$).
(b) $\Sigma_\omega L$ is first-countable.

(c) $\sigma(L) \neq \sigma_{\omega^*}(L)$. It is easy to check that $[\omega, \omega_1] \in \sigma_{\omega^*}(L)$ but $[\omega, \omega_1] \notin \sigma(L)$ (note that $\omega = \sup \mathbb{N}$).

(d) $\mathsf{K}(\Sigma_\omega L) = \{\uparrow\alpha : \alpha \in [0, \omega_1]\}$. In fact, for $K \in \mathsf{K}$, we have $\inf K \in K$, and hence $K = \uparrow\inf K$.

(e) $\Sigma_\omega L$ is not an ω-Rudin space. It is easy to check that $[0, \omega_1) \in \mathsf{Irr}_c(\Sigma_\omega L)$ (note that $\{\omega_1\} \in \sigma_\omega(L)$). If $[0, \omega_1) \in \mathsf{RD}_\omega(\Sigma_\omega L)$, then by (d), there is a countable subset $\{\alpha_n : n \in \mathbb{N}\} \subseteq [0, \omega_1)$ such that $[0, \omega_1) \in m(\{\uparrow\alpha_n : n \in \mathbb{N}\})$. Let $\beta = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\beta \in [0, \omega_1)$, and hence $\downarrow\beta \in \mathcal{C}(\Sigma_\omega L)$ and $\downarrow\beta \in M(\{\uparrow\alpha_n : n \in \mathbb{N}\})$, which is in contradiction with $[0, \omega_1) \in m(\{\uparrow\alpha_n : n \in \mathbb{N}\})$.

(f) $\Sigma_\omega L$ is ω-well-filtered. Suppose that $\{\uparrow\alpha_n : n \in \mathbb{N}\} \subseteq \mathsf{K}(\Sigma_\omega L)$ is countable filtered and $U \in \sigma_\omega(L)$ with $\bigcap_{n\in\mathbb{N}} \uparrow\alpha_n \subseteq U$. Let $\alpha = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\{\alpha_n : n \in \mathbb{N}\}$ is a countable directed subset of L and $\alpha \in U$ since $\uparrow\alpha = \bigcap_{n\in\mathbb{N}} \uparrow\alpha_n \subseteq U$. It follows that $\alpha_n \in U$ or, equivalently, $\uparrow\alpha_n \subseteq U$ for some $n \in \mathbb{N}$. Thus $\Sigma_\omega L$ is ω-well-filtered, and hence an ω-d-space.

(g) $\Sigma_\omega L$ is not well-filtered. $\{\uparrow t : t \in [0, \omega_1)\} \subseteq \mathsf{K}(\Sigma_\omega L)$ is filtered and $\bigcap_{t\in[0,\omega_1)} \uparrow t = \{\omega_1\} \in \sigma_\omega(L)$, but $\uparrow t \nsubseteq \{\omega_1\}$ for all $t \in [0, \omega_1)$. Therefore, $\Sigma_\omega L$ is not well-filtered, and hence it is not sober.

(h) $\Sigma_\omega L$ is not a d-space. $[0, \omega_1) \in \mathsf{Irr}_c(\Sigma_\omega L)$ and $[0, \omega_1)$ is directed, but $[0, \omega_1) \neq \mathrm{cl}_{\sigma_\omega(L)}\{\alpha\} = [0, \alpha]$ for all $\alpha \in L$. Thus $\Sigma_\omega L$ is not a d-space.

(i) $\Sigma_\omega L$ is not an ω^*-d-space. In fact, by (a), $\{\omega_1\} \in \sigma_\omega(L)$ but $\{\omega_1\} \notin \sigma_{\omega^*}(L)$, and hence by Proposition 4.6, $\Sigma_\omega L$ is not an ω^*-d-space.

Now we give the second main result of this paper that any continuous mapping $f : X \to Y$ from a first-countable T_0 space X to an ω-well-filtered space Y maps any irreducible subset of X to a Rudin set of Y.

Theorem 6.10. *Let X be a first-countable T_0 space and Y an ω-well-filtered T_0 space. Then for any continuous mapping $f : X \to Y$ and $A \in \mathsf{Irr}(X)$, $\overline{f(A)} \in \mathbf{RD}(Y)$.*

Proof. Let $\mathcal{K}_A = \{\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) : \{K_n : n \in \mathbb{N}\} \in \mathcal{K}_f^\omega(A)\}$. Then by Proposition 2.13, Lemma 6.1 and Remark 6.2, we have

1° $\mathcal{K}_A \neq \emptyset$ and $\mathcal{K}_A \subseteq \mathsf{K}(Y)$.

2° $\uparrow f(a) \in \mathcal{K}_A$ for all $a \in A$.

For $\{a_n : n \in \mathbb{N}\} \subseteq A$ with $a_n \equiv a$, as carried out in the proof of Lemma 6.1, choose $c_n \equiv a$ for all $n \in \mathbb{N}$. Then $K_1 = K_2 = ... = K_n = ... = \uparrow a$, and hence $\uparrow f(a) = \uparrow f(\uparrow a) = \bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \in \mathcal{K}_A$.

3° $\mathcal{K}_A$ is filtered.

Suppose that $\{K_n : n \in \mathbb{N}\}, \{G_n : n \in \mathbb{N}\} \in \mathcal{K}_f^\omega(A)$. Then there are two countable subsets $\{a_n : n \in \mathbb{N}\}$ and $\{b_n : n \in \mathbb{N}\}$ of $\overline{A}$ such that $\{K_n : n \in \mathbb{N}\} \in \mathcal{K}_f^\omega(\{a_n : n \in \mathbb{N}\})$ and $\{G_n : n \in \mathbb{N}\} \in \mathcal{K}_f^\omega(\{a_n : n \in \mathbb{N}\})$; whence, there are two sequences $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ in A such that $K_n = \uparrow\{c_m : m \geq n\}$ and $G_n = \uparrow\{d_m : m \geq n\}$ for all $n \in \mathbb{N}$, and $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ satisfy conditions (1)–(3) of Lemma 6.1 and conditions (a), (b) of Remark 6.2, respectively. Consider $\{s_n : n \in \mathbb{N}\} = \{c_1, d_1, c_2, d_2, ..., c_n, d_n, ...\} \subseteq A$, that is,

$$s_n = \begin{cases} c_k & n = 2k - 1 \\ d_k & n = 2k. \end{cases}$$

Then by Lemma 6.1 and Remark 6.2, there is $\{H_n : n \in \mathbb{N}\} \in \mathcal{K}_f^\omega(\{s_n : n \in \mathbb{N}\}) \subseteq \mathcal{K}_f^\omega(A)$, and hence $\bigcap_{n\in\mathbb{N}} \uparrow f(H_n) \subseteq \bigcap_{n\in\mathbb{N}} \uparrow f(s_n) = \bigcap_{n\in\mathbb{N}} \uparrow f(c_n) \cap \bigcap_{n\in\mathbb{N}} \uparrow f(d_n) = \bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \cap \bigcap_{n\in\mathbb{N}} \uparrow f(G_n)$. Thus $\mathcal{K}_A$ is filtered.

4° $\overline{f(A)} \in M(\mathcal{K}_A)$.

For $\{K_n : n \in \mathbb{N}\} \in \mathcal{K}_f^\omega(A)$, we show $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \cap \overline{f(A)} \neq \emptyset$. Assume, on the contrary, that $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \cap \overline{f(A)} = \emptyset$ or, equivalently, $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \subseteq Y \setminus \overline{f(A)}$. As $\{\uparrow f(K_n) : n \in \mathbb{N}\} \subseteq \mathsf{K}(Y)$ is a decreasing family and Y is ω-well-filtered, we have $\uparrow f(K_m) \subseteq Y \setminus \overline{f(A)}$ for some $m \in \mathbb{N}$, which is in contradiction with $A \cap K_m \neq \emptyset$. Therefore, $\bigcap_{n\in\mathbb{N}} \uparrow f(K_n) \cap \overline{f(A)} \neq \emptyset$. Thus $\overline{f(A)} \in M(\mathcal{K}_A)$.

5° $\overline{f(A)} \in m(\mathcal{K}_A)$.

If B is a closed subset with $B \in M(\mathcal{K}_A)$, then for each $a \in A$, by 2°, we have $\uparrow f(a) \cap B \neq \emptyset$, and hence $f(a) \in B$. It follows that $f(A) \subseteq B$, and hence $\overline{f(A)} \subseteq B$. Thus $\overline{f(A)} \in m(\mathcal{K}_A)$.

By 1°, 3° and 5°, $\overline{f(A)} \in \mathbf{RD}(Y)$. □

Corollary 6.11. *Every first-countable ω-well-filtered T_0 space is a Rudin space.*

Proof. Applying Theorem 6.10 to $A \in \mathsf{Irr}_c(X)$ and the identity $id_X : X \longrightarrow X$. $\square$

By Theorem 6.10, we get another main result of this paper.

Theorem 6.12. *Every first-countable T_0 space is a well-filtered determined space.*

Proof. Let X be a first-countable T_0 space and $A \in \mathsf{Irr}_c(X)$. We need to show $A \in \mathsf{WD}(X)$. Suppose that $f : X \longrightarrow Y$ is a continuous mapping from X to a well-filtered space Y. By Theorem 6.10, $\overline{f(A)} \in \mathbf{RD}(Y)$, whence by the well-filteredness of Y and Proposition 3.4, there is a (unique) $y_A \in Y$ such that $\overline{f(A)} = \overline{\{y_A\}}$. Thus $A \in \mathsf{WD}(X)$. $\square$

Corollary 6.13. ([22, Theorem 4.2]) *Every first-countable well-filtered T_0 space is sober.*

Remark 6.14. (1) In [16, Example 4.15], a well-filtered space, non-Rudin space X was given. By Corollary 6.13, X is not first-countable.

(2) Let L be the complete chain $[0, \omega_1]$. Then by Example 6.9, $\Sigma_\omega L$ is a first-countable ω-well-filtered space but not a sober space. Therefore, by by Proposition 3.11 and Corollary 6.11, $\Sigma_\omega L$ is a Rudin space but is not ω-well-filtered determined. So first-countability does not imply ω-well-filtered determinedness in general.

Finally, by Theorem 5.6, Corollary 6.11 and Theorem 6.12, we naturally pose the following problem.

Problem 6.15. Is every first-countable T_0 space a Rudin space?

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