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A complete Heyting algebra whose Scott space is non-sober

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Abstract

We prove that (1) for any complete lattice L, the set $\mathcal{D}(L)$ of all nonempty saturated compact subsets of the Scott space of L is a complete Heyting algebra (with the reverse inclusion order); and (2) if the Scott space of a complete lattice L is non-sober, then the Scott space of $\mathcal{D}(L)$ is non-sober. Using these results and Isbell's

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example of a non-sober complete lattice, we deduce that there is a complete Heyting algebra whose Scott space is non-sober, thus giving a positive answer to a problem posed by Achim Jung. We will also prove that a T_0 space is well-filtered iff its upper space (the set $\mathcal{D}(X)$ of all nonempty saturated compact subsets of X equipped with the upper Vietoris topology) is well-filtered, which answers another open problem.

Sobriety and well-filteredness are two of the most important properties for non-Hausdorff topological spaces. The Scott space of every domain (continuous directed complete poset) is sober. Johnstone [8] gave the first example of a dcpo whose Scott space is non-sober. Soon after that, Isbell [6] constructed a complete lattice whose Scott space is non-sober. The general problem in this line is whether each object in a classic class of lattices has a sober Scott space. The Isbell's non-sober complete lattice is not distributive. Thus Achim Jung asked whether there is a distributive complete lattice whose Scott space is non-sober. In this paper we shall give a positive answer to Jung's problem. The main structure we shall use is the poset $\mathcal{D}(X)$ of all nonempty saturated compact subsets of a topological space X equipped with the reverse inclusion order. We first show that for any complete lattice L, the poset $\mathcal{D}(L)$ of all nonempty saturated compact subsets of the Scott space of L is a complete Heyting algebra. Then we prove that for a certain type of T_0 spaces X, if X is non-sober, then the Scott space of $\mathcal{D}(X)$ is non-sober. An immediate conclusion is that for any complete lattice L, if the Scott space of L is non-sober, then the Scott space of the complete Heyting algebra $\mathcal{D}(L)$ is non-sober. Taking L to be Isbell's example, we obtain a complete Heyting algebra whose Scott space is non-sober, thus answering Jung's problem.

Heckmann and Keimel [4] proved that a space X is sober if and only if the upper space $\mathcal{D}(X)$ of X is sober. In [12], Xi and Zhao proved that a space X is well-filtered iff its upper space $\mathcal{D}(X)$ is a d-space. In this paper, they also asked whether it is true that a space is well-filtered if and only if the upper space of X is well-filtered. In the last part of this paper we will give a positive answer to this problem.

1 Preliminaries

A *complete Heyting algebra* is a complete lattice L satisfying the following infinite distributive law:

$$a\wedge\bigvee\{a_i : i\in I\} = \bigvee\{a\wedge a_i : i\in I\}$$

for any $a\in L$ and $\{a_i : i\in I\}\subseteq L$. Such a complete lattice is also called a *frame*. Clearly, every complete Heyting algebra is distributive when viewed as lattice.

An element p of a meet-semilattice S is a prime element if for any $a, b \in S$, $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. A frame A is called *spatial* if every element of A can be expressed as a meet of prime elements. It is well-known that a complete lattice L is a spatial frame iff it is isomorphic to the lattice of all open subsets of some topological space (cf. [9]).

A subset U of a poset $(P, \leq)$ is *Scott open* if i) U is an upper set (that is, $U = \uparrow U = \{x\in P : y\leq x \text{ for some } y\in U\}$), and ii) for any directed subset $D\subseteq P$, $\bigvee D\in U$ implies $D \cap U\neq\emptyset$ whenever $\bigvee D$ exists. The Scott open sets of a poset P form a topology on P, denoted by $\sigma(P)$ and called the *Scott topology* on P. The space $(P, \sigma(P))$ is denoted by ΣP, called the *Scott space* of P.

A poset is called a *directed complete poset* (*dcpo*, for short) if every directed subset of the poset has a supremum. For more about the Scott topology and dcpos, see [1][2].

A subset A of a topological space is called *saturated* if A equals the intersection of all open sets containing it. A T_0 space X is *well-filtered* if for any open set U and filtered family $\mathcal{F}$ of saturated compact subsets of X, $\bigcap\mathcal{F}\subseteq U$ implies $F\subseteq U$ for some $F\in\mathcal{F}$.

A nonempty subset A of a space X is *irreducible* if for any closed subsets F_1, F_2 of X, $F\subseteq F_1\cup F_2$ implies $F\subseteq F_1$ or $F\subseteq F_2$. Obviously, the closure of every singleton set is irreducible. A space X is called *sober* if every irreducible closed set of X is the closure of a unique singleton set. It is well known that every sober space is well-filtered. Johnstone [8] constructed the first example of a dcpo whose Scott space is non-sober. Isbell [6] constructed a complete lattice whose Scott space is non-sober. Kou [5] constructed the first example of a dcpo whose Scott space is well-filtered but non-sober, answering a problem posed by Heckmann in[3].

The *specialization order* $\leq_\tau$ on a T_0 space (X, τ) is defined by $x\leq_\tau y$ iff $x\in cl(\{y\})$, where $cl(\{y\})$ is the closure of set $\{y\}$. A space (X, τ) is called a *d-space* (or *monotone convergence space*) if $(X, \leq_\tau)$ is a dcpo and $\tau\subseteq\sigma((X, \leq_\tau))$ (cf. [1]).

2 The existence of a complete Heyting algebra whose Scott space is non-sober

For any topological space X, following Heckmann and Keimel [4], we shall use $\mathcal{D}(X)$ to denote the set of all nonempty compact saturated subsets of X. The *upper Vietoris topology* on $\mathcal{D}(X)$ is the topology that has $\{\Box U : U \in \mathcal{O}(X)\}$ as a base, where $\Box U = \{K \in \mathcal{D}(X) : K \subseteq U\}$. The set $\mathcal{D}(X)$ equipped with the upper Vietoris topology is called the *Smyth power space* or *upper space* of X (cf. [4][10]).

The specialization order on the upper space $\mathcal{D}(X)$ is the reverse inclusion order $\supseteq$. In what follows, this is the order we will be concerned in this section.

For a poset P, we shall use $\mathcal{D}(P)$ to denote the poset of all nonempty compact saturated subsets of the Scott space $(P, \sigma(P))$.

A space X is called *coherent* if the intersection of any two compact saturated subsets in X is compact.

Lemma 1. *For any complete lattice L, $\mathcal{D}(L)$ is a complete Heyting algebra.*

Proof. The Scott space ΣL of L is well-filtered by Xi and Lawson[11], and is coherent by Jia and Jung [7]. We now show that the poset $\mathcal{D}(L)$ is a complete Heyting algebra.

(i) Since $(L, \sigma(L))$ is well-filtered, we have that $\mathcal{D}(L)$ is closed under filtered intersections (Remark 3(3) of [12]), thus it is a dcpo, in which the infimum of a directed subset $\mathcal{K}$ of $\mathcal{D}(L)$ is its intersection.

(ii) Also, ΣL is coherent and every member of $\mathcal{D}(L)$ contains the top element 1_L of L, so the intersection $K_1 \cap K_2$ of any two members K_1, K_2 of $\mathcal{D}(L)$ is again a member of $\mathcal{D}(L)$, which equals their join $K_1 \vee K_2$. Also, $\mathcal{D}(L)$ has L as the least element, and $\{1_L\}$ as the top element. It follows that $\mathcal{D}(L)$ is both a dcpo and a join semilattice, and has a least element. Therefore $\mathcal{D}(L)$ is a complete lattice. In addition, for any $K_1, K_2 \in \mathcal{D}(L)$, the meet $K_1 \wedge K_2$ of K_1, K_2 in $\mathcal{D}(L)$ clearly equals $K_1 \cup K_2$. Now for any subfamily $\{K_i : i \in I\} \subseteq \mathcal{D}(L)$, from (i) and (ii) we can deduce that

$$\bigvee\{K_i : i \in I\} = \bigcap\{K_i : i \in I\}.$$

Then for any $K \in \mathcal{D}(L)$ and $\{K_i : i \in I\} \subseteq \mathcal{D}(L)$, we have

$$\begin{aligned} K \wedge \textstyle\bigvee_{i \in I} K_i &= K \cup (\textstyle\bigcap_{i \in I} K_i) \\ &= \textstyle\bigcap_{i \in I} (K \cup K_i) \\ &= \textstyle\bigvee_{i \in I} (K \wedge K_i). \end{aligned}$$

Hence $\mathcal{D}(L)$ is a complete Heyting algebra. $\Box$

For any T_0 space (X,τ), let $\xi_X : X \to \mathcal{D}(X)$ be the *canonical mapping* given by

$$\xi_X(x) = \uparrow x = \{y \in X : x \leq_\tau y\}.$$

It is easy to see that $\xi_X : (X, \leq_\tau) \longrightarrow (\mathcal{D}(X), \supseteq)$ is an order embedding.

To emphasize the codomain, we shall use ξ^σ_X to denote the corresponding mapping $\xi^\sigma_X : (X,\tau) \longrightarrow (\mathcal{D}(X), \sigma(\mathcal{D}(X)))$, where $\xi^\sigma_X(x) = \uparrow x$ for each $x \in X$.

Theorem 1. *Let X be a T_0 space such that*

(i) the upper Vietoris topology on $\mathcal{D}(X)$ is contained in $\sigma(\mathcal{D}(X))$ (that is, the upper Vietoris topology is weaker than the Scott topology);

(ii) the mapping $\xi^\sigma_X : (X,\tau) \longrightarrow (\mathcal{D}(X), \sigma(\mathcal{D}(X)))$ is continuous; and

(iii) $(\mathcal{D}(X), \sigma(\mathcal{D}(X)))$ is sober.

Then X is sober.

Proof. Let F be a closed irreducible subset of X. Then, as ξ^σ_X is continuous, $\xi^\sigma_X(F)$ is an irreducible subset of $(\mathcal{D}(X), \sigma(\mathcal{D}(X)))$. Therefore, there exists $K \in \mathcal{D}(X)$ such that

$$cl_{\sigma(\mathcal{D}(X))}(\xi^\sigma_X(F)) = \downarrow_{\mathcal{D}(X)} K (= \{A \in \mathcal{D}(X) : K \subseteq A\}),$$

where $cl_{\sigma(\mathcal{D}(X))}$ is the closure operator in the Scott space $(\mathcal{D}(X), \sigma(\mathcal{D}(X)))$.

Claim 1. Every element of K is an upper bound of F in the poset $(X, \leq_\tau)$.

In fact, let $k \in K$ and $x \in F$. Then $\uparrow k \in \mathcal{D}(X)$ and $\uparrow k \subseteq K$. In addition, $\uparrow x = \xi^\sigma_X(x) \in \downarrow_{\mathcal{D}(X)} K$, so $\uparrow x \supseteq K$. Hence $\uparrow x \supseteq K \supseteq \uparrow k$, which implies $x \leq_\tau k$.

Claim 2. K has a least element.

If, on the contrary, for each $k \in K$, there is $s(k) \in K$ such that $k \not\leq_\tau s(k)$. Then

$$K \subseteq \bigcup \{X \setminus \downarrow s(k) : k \in K\}.$$

Thus $K \in \Box \bigcup \{X \setminus \downarrow s(k) : k \in K\} \in \sigma(\mathcal{D}(X))$ (by assumption (i)), implying

$$K \in cl_{\sigma(\mathcal{D}(X))}(\xi^\sigma_X(F)) \cap \Box \bigcup \{X \setminus \downarrow s(k) : k \in K\}.$$

Hence

$$cl_{\sigma(\mathcal{D}(X))}(\xi^\sigma_X(F)) \cap \Box \bigcup \{X \setminus \downarrow s(k) : k \in K\} \neq \emptyset.$$

Therefore $\xi^\sigma_X(F) \cap \Box \bigcup \{X \setminus \downarrow s(k) : k \in K\} \neq \emptyset$. Hence there exists $y \in F$ with $\uparrow y \subseteq \bigcup \{X \setminus \downarrow s(k) : k \in K\}$. It follows that $y \not\leq_\tau s(k)$ for some $s(k) \in K$. But this contradicts Claim 1 (every element of K is an upper bound of F).

Therefore K has a least element, say s. Then $K = \uparrow s$.

Claim 3. $F = cl_X(\{s\})$. As s is an upper bound of F and F is closed, we only need to confirm that $s \in F$.

Assume, on the contrary, that $s\notin F$. Then $K\subseteq X\backslash F$, so

$$K\in cl_{\sigma(\mathcal{D}(X))}(\xi^\sigma_X(F))\cap\Box(X\backslash F)$$

and $\Box(X\backslash F)\in\sigma(\mathcal{D}(X))$.

Therefore $\xi^\sigma_X(F)\cap\Box(X\backslash F)\neq\emptyset$, which is impossible.

Hence $s\in F$, thus $F=\downarrow s=cl_X(\{s\})$.

All these together show that (X,τ) is sober. □

Proposition 1. *(1) For every T_0 space X, the mapping $\xi_X : X\longrightarrow\mathcal{D}(X)$ (the upper space of X) is a topological embedding.*

(2) For any poset P, the mapping

$$\xi^\sigma_P : (P,\sigma(P))\longrightarrow(\mathcal{D}(P),\sigma(\mathcal{D}(P)))$$

is continuous, i.e., it preserves all existing directed suprema.

(3) Every well-filtered space is a d-space.

(4) A T_0 space X is well-filtered iff $\mathcal{D}(X)$ is a dcpo and the upper Vietoris topology on $\mathcal{D}(X)$ is contained in $\sigma(\mathcal{D}(X))$ (equivalently, the upper space $\mathcal{D}(X)$ is a d-space).

Proof. (1) See [4].

(2) This follows from a straightforward verification.

(3) See Corollary 3.3 of [13].

(4) See Proposition 3 of [12]. □

In general, the well-filteredness of X is stronger than the condition that the upper Vietoris topology on $\mathcal{D}(X)$ is contained in $\sigma(\mathcal{D}(D))$. For example, consider the poset $\mathbb{N}$ of natural numbers with the usual order. Then every element in $\mathbb{N}$ is compact and so $\mathbb{N}$ is an algebraic poset. Hence $\Sigma\mathbb{N}$ ($\sigma(\mathbb{N})$ equals the *Alexandroff topology* on $\mathbb{N}$) is locally compact and $\mathcal{D}(\mathbb{N})=\{\uparrow n : n\in\mathbb{N}\}$, which is isomorphic to $\mathbb{N}$. Therefore $\mathcal{D}(\mathbb{N})$ is not a dcpo. Now we have that the upper Vietoris topology on $\mathcal{D}(\mathbb{N})$ equals the Scott topology $\sigma(\mathcal{D}(\mathbb{N}))$ (and also equals the Alexandroff topology on $\mathcal{D}(\mathbb{N})$). But $\Sigma\mathbb{N}$ is not well-filtered.

Example 1. Let X be any non-countable set and τ be the co-countable topology on X. Then (X,τ) is a T_1 space. Clearly, the nonempty compact (saturated) subsets of (X,τ) are exactly the nonempty finite subsets of X, that is, $\mathcal{D}(X)=\{F : F \text{ is a nonempty finite subset of } X\}$. Every directed subset $\mathcal{E}$ of $\mathcal{D}(X)$ has a largest element (which is the intersection of $\mathcal{E}$), so (X,τ) is well-filtered but non-sober (X is an irreducible closed set but not the closure of any singleton set). Clearly $\mathcal{D}(X)$ is a dcpo and every element in $\mathcal{D}(X)$ is compact. Hence $\mathcal{D}(X)$ is an algebraic domain and $\sigma(\mathcal{D}(X))$ (which equals the Alexandroff topology) is sober. For (X,τ), the conditions (i) and (iii) in Theorem 1 are satisfied, but the assumption (ii) does not hold. In this case, the sobriety of $(\mathcal{D}(X), \sigma(\mathcal{D}(X)))$ does not imply the sobriety of (X,τ).

By Proposition 1 (4), Theorem 1 can be restated as the follows.

Theorem 2. *Let (X,τ) be a T_0 space such that*

(i) X is well-filtered;

(ii) the mapping $\xi_X^\sigma : (X,\tau)\longrightarrow(\mathcal{D}(X), \sigma(\mathcal{D}(X)))$ is continuous; and

(iii) $(\mathcal{D}(X), \sigma(\mathcal{D}(X)))$ is sober.

Then X is sober.

By Theorem 2 and Proposition 1, we deduce the following.

Corollary 1. *For a dcpo P, if $(P,\sigma(P))$ is well-filtered and $(\mathcal{D}(P), \sigma(\mathcal{D}(P)))$ is sober (equivalently, the upper space $\mathcal{D}(P)$ is a d-space and $(\mathcal{D}(P), \sigma(\mathcal{D}(P)))$ is sober), then $(P,\sigma(P))$ is sober.*

By Xi and Lawson [11], for any complete lattice L, $(L,\sigma(L))$ is well-filtered. Now applying Corollary 1, we obtain the following.

Theorem 3. *For any complete lattice L, if $(L,\sigma(L))$ is non-sober, then $(\mathcal{D}(L), \sigma(\mathcal{D}(L)))$ is non-sober.*

Now we are ready to answer Jung's problem mentioned in the introduction.

Example 2. In [6], Isbell constructed a complete lattice whose Scott topology is non-sober, thus answered a question posed by Johnstone in [8]. Isbell's complete lattice is not distributive. In one of his recent talk in Singapore, Achim Jung asked whether there is a distributive complete lattice whose Scott topology is non-sober. We now can give a positive answer to this problem. Take M be the complete lattice constructed by Isbell and let $L = \mathcal{D}(M)$. Then by Lemma 1, L is a complete Heyting algebra. Since the Scott space of M is non-sober, by Theorem 3, the Scott space of L is non-sober. Hence L is a complete Heyting algebra whose Scott space is non-sober.

Remark 1. For any T_0 space (X, τ), the poset $(\mathcal{D}(X), \supseteq)$ is a meet-semilattice, where the meet of $K_1, K_2 \in \mathcal{D}(X)$ equals $K_1 \cup K_2$. Then clearly every principal filter $\uparrow x = \{y \in X : x \leq_\tau y\}$ is a prime element of $\mathcal{D}(X)$. In addition, for any $K \in \mathcal{D}(X)$,

$$K = \bigwedge\{\uparrow x : x \in K\},$$

showing that every element of $\mathcal{D}(X)$ can be expressed as a meet of prime elements. Hence by Lemma 1, $(\mathcal{D}(L), \supseteq)$ is actually a spatial frame for any complete lattice L (see [9] for more about spatial frames). Thus the non-sober complete Heyting algebra L obtained in Example 2 is also a spatial frame.

3 Well-filteredness of upper spaces

In this section, the symbol $\mathcal{D}(X)$ will denote the upper space of the topological space X.

In [4], it is proved that a space X is sober iff the upper space $\mathcal{D}(X)$ is sober. In [12], it is proved that a T_0 space X is well-filtered if and only if its upper space is a d-space. In that paper, the question was asked whether the upper space $\mathcal{D}(X)$ will be well-filtered if X is well-filtered.

We now give a positive answer to the above problem.

Lemma 2. ([4]) *Let X be a topological space and $\mathcal{A}$ an irreducible subset of the upper space $\mathcal{D}(X)$. Then every closed set $C \subseteq X$ that meets all members of $\mathcal{A}$ contains a minimal irreducible closed subset A that still meets all members of $\mathcal{A}$.*

The following result can be verified straightforwardly (see e.g. [**9**, page 128] or the proof of Lemma 3.1 in [7]).

Lemma 3. *If $\mathcal{K} \subseteq \mathcal{D}(X)$ is a nonempty compact subset of $\mathcal{D}(X)$, then $\bigcup \mathcal{K} \in \mathcal{D}(X)$.*

Theorem 4. *A topological space X is well-filtered iff its upper space $\mathcal{D}(X)$ is well-filtered.*

Proof. By Proposition 1 (3)(4), we only need to show that if X is well-filtered, then so is $\mathcal{D}(X)$. Let $\{\mathcal{K}_t : t \in T\}$ be a filtered family of saturated compact subsets of $\mathcal{D}(X)$, $\mathcal{U} = \bigcup\{\Box U_i : i \in I\}$ an open set of $\mathcal{D}(X)$ such that

$$\bigcap\{\mathcal{K}_t : t \in T\} \subseteq \mathcal{U}.$$

Suppose that $\mathcal{K}_t \not\subseteq \mathcal{U}$ for all t, that is, $\mathcal{K}_t \cap (\mathcal{D}(X) \setminus \mathcal{U}) \neq \emptyset$. Then as $\{\mathcal{K}_t : t \in T\}$ is an irreducible subset of the space $\mathcal{D}(X)$, by Lemma 2, there is a minimal closed irreducible subset $\mathcal{C} \subseteq \mathcal{D}(X) \setminus \mathcal{U}$ that meets every $\mathcal{K}_t (t \in T)$.

For each $t \in T$, let $K_t = \bigcup (\mathcal{K}_t \cap \mathcal{C})$. As $\mathcal{K}_t \cap \mathcal{C}$ is nonempty and compact in $\mathcal{D}(X)$, by Lemma 3 we have that $K_t \in \mathcal{D}(X)$. Also $\{K_t : t \in T\}$ is a filtered family of members of $\mathcal{D}(X)$, thus $K = \bigcap \{K_t : t \in T\}$ is a member of $\mathcal{D}(X)$ because X is well-filtered.

Claim 1. $K \notin \mathcal{U}$.

Assume, on the contrary, that $K \in \mathcal{U}$. Then $K \in \Box U_i$ for some $i \in I$, so $K = \bigcap \{K_t : t \in T\} \subseteq U_i$. Thus, as X is well-filtered, $K_t \subseteq U_i$ holds for some $t \in T$. Then $\emptyset \neq \mathcal{K}_t \cap \mathcal{C} \subseteq \Box U_i \subseteq \mathcal{U}$, contradicting $\mathcal{C} \subseteq \mathcal{D}(X) \setminus \mathcal{U}$.

Claim 2. $K \in \bigcap \{\uparrow_{\mathcal{D}(X)} (\mathcal{K}_t \cap \mathcal{C}) : t \in T\}$.

Suppose, on the contrary, that $K \notin \bigcap \{\uparrow_{\mathcal{D}(X)} (\mathcal{K}_t \cap \mathcal{C}) : t \in T\}$. Then there is $t_0 \in T$ such that $K \notin \uparrow_{\mathcal{D}(X)} (\mathcal{K}_{t_0} \cap \mathcal{C})$. Thus, for any $G \in \mathcal{K}_{t_0} \cap \mathcal{C}$, there exists $e(G) \in K \setminus G$. Then $G \cap \downarrow e(G) = \emptyset$ (Note that G is a saturated compact set). Now for any $G \in \mathcal{K}_{t_0} \cap \mathcal{C}$ and any $t \in T$, since $e(G) \in K$ (so $e(G) \in K_t$), we have $e(G) \in \bigcup (\mathcal{K}_t \cap \mathcal{C})$. Thus there exists $H_t \in \mathcal{K}_t \cap \mathcal{C}$ such that $e(G) \in H_t$, implying

$$H_t \in \mathcal{K}_t \cap \mathcal{C} \cap \diamond (\downarrow e(G)).$$

It follows that

$$\mathcal{K}_t \cap \mathcal{C} \cap \diamond (\downarrow e(G)) \neq \emptyset, \text{ for all } t \in T.$$

By the minimality of $\mathcal{C}$, we have $\mathcal{C} \cap \diamond (\downarrow e(G)) = \mathcal{C}$, which implies $\mathcal{C} \subseteq \diamond (\downarrow e(G))$.

Therefore $\mathcal{C} \subseteq \bigcap \{\diamond (\downarrow e(G)) : G \in \mathcal{K}_{t_0} \cap \mathcal{C}\}$. Note that for any $G \in \mathcal{K}_{t_0} \cap \mathcal{C}$, $G \notin \diamond (\downarrow e(G))$. Hence

$$\emptyset \neq \mathcal{K}_{t_0} \cap \mathcal{C} = \mathcal{K}_{t_0} \cap \mathcal{C} \cap \bigcap \{\diamond (\downarrow e(G)) : G \in \mathcal{K}_{t_0} \cap \mathcal{C}\} = \emptyset.$$

This contradiction confirms Claim 2.

Now $K \in \bigcap \{\uparrow_{\mathcal{D}(X)} (\mathcal{K}_t \cap \mathcal{C}) : t \in T\} \subseteq \bigcap \{\mathcal{K}_t : t \in T\} \subseteq \mathcal{U}$, which implies $K \in \mathcal{U}$. But this contradicts Claim 1.

All these together deduce that there must be some $t_0 \in T$ such that $\mathcal{K}_{t_0} \subseteq \mathcal{U}$.

Hence $\mathcal{D}(X)$ is well-filtered. The proof is completed. $\Box$

The following result collects some of the equivalent conditions for a space to be well-filtered.

Theorem 5. *For any T_0 space X, the following statements are equivalent:*

(1) X is well-filtered.

(2) The upper space $\mathcal{D}(X)$ of X is a d-space.

(3) The upper space $\mathcal{D}(X)$ of X is well-filtered.

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