PEDAGOGICAL CONTENT KNOWLEDGE AND THE TEACHING OF AREA: A CASE STUDY OF THREE PRIMARY TEACHERS

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ABSTRACT
This research, adopting a case study paradigm, investigated the pedagogical content knowledge (PCK) of three established primary mathematics teachers in Singapore. The objectives of the study were two fold: (i) to examine the nature of teacher’s PCK in the teaching of the concept area at the upper primary levels; (ii) to identify how teachers’ PCK influenced pupils’ understanding and learning. The participants were selected for their commitment to teaching
using a wide repertoire of mathematical representations. They were observed in their teaching and post-lesson interviews were conducted to verify the rationale for their actions. The findings showed that the teachers used real life examples to illustrate abstract concept of area. They proceeded from simple to complex knowledge, and linked other topics and concepts for cohesion, non-fragmented teaching. Multiple approaches involving various tasks and representations to depict the concept of area, were incorporated. The teachers displayed extensive and perceptive understanding to teach mathematical concepts and procedures, and not simply apply algorithms. This research helped provide insights into the teaching of area.

Information pertaining to mathematical communication between pupil-pupil, and teacher-pupil — reasoning, thinking, justifying, was surfaced. This could be expanded to novice and even older teachers interested in teaching for conceptual understanding, initiate teachers’ professional development, and inform on mathematics teachers education.

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Since Shulman first brought it to the fore in his presidential address to the American Education Research Association, it is accepted fact that teachers with good pedagogical content knowledge (PCK) makes good teachers. However because of its complex nature, teachers’ PCK is difficult to characterize. It transcended knowing something for oneself. It is a particular form of subject-matter knowledge most germane to its teachability; the most useful forms of representations to facilitate comprehension by others (Shulman, 1986). Shulman considered this to be the unique province of teachers, their unique form of professional understanding peculiar only to teachers.
Research (Shulman, 1986; Schwab, 1964) has shown that it is very difficult to quantify teachers’ PCK. To teach a particular concept for understanding, teachers have to know how to present a given set of concepts, the different modes of representing those concepts, the appropriate analogies and examples to use that may make the concepts more meaningful to pupils.

To help us make sense of teachers’ PCK, a framework was constructed to analyze this complex knowledge. In this paper we will describe the different aspects of the framework to teach the concept area to primary pupils. The work reported in this paper is grounded on the teaching of three primary teachers.

**RESEARCH QUESTION:**
This paper addresses the following research question.

What kind of knowledge do teachers use to select authentic tasks and meaningful representations for instruction?

**FRAMEWORK UNDERPINNING THIS STUDY**

**Teachers’ Pedagogical Content Knowledge**

After much literature review, discussions, and dialogues with colleagues on PCK, we came up with a framework which focused almost exclusively on questions from the perspective of the teachers; on teaching, the learners, and how learners learn. The framework incorporated facets of the teacher, pupil, and mathematical concepts, procedures and approaches. Some of the questions raised were: i) what are the sources of teachers’ knowledge, ii) what do teachers know, and when do they come to know about it, iii) how are representations and tasks selected to promote conceptual understanding, and iv) how do teachers make use of pupils’ prior knowledge and
experiences to rectify their misconceptions. Such elements were reflected in our framework to provide structure for the analysis of PCK by focusing on aspects emphasized by Shulman, which served as guides in designing better education. In the later phases of analysis and report writing, the framework helped to organize our observation of trends and analysis of the role of PCK in teaching.

Figure 2.1  Framework of Teachers’ Pedagogical Content Knowledge
Multiple Representations

The relevance and importance of instructional representations is established in mathematics instruction (Ball, 1993a; Lampert, 1989). Probably the most significant findings from research that informed practice were those involving the use of multiple representations of mathematical concepts. Bruner (1966) and Dienes (1960) were early proponents of this approach, and researchers since then had continued to explore the effects of multiple representations on pupils’ learning of mathematics. Some representations are comparatively better for portraying particular aspects of a situation. Representations which captured the vividness of real world contexts supported a higher level of problem-solving. Thus, multiple representations are necessary, together with a developmental approach to the ways teachers and pupils could use them (Verngaud, 1987).

In light of mounting research evidence that pupil learning is more complex than was previously thought; there is all the more reason to investigate this particular knowledge, PCK, which we believe, had the potential to be especially significant. Hence, this study would expand current research to investigate the practice of established teachers, to examine their PCK, which is not only subject-matter specific, but goes beyond knowledge of subject-matter per se to the dimension of subject matter knowledge for teaching (Shulman, 1986). We would look into the established teachers’ PCK in the context of teaching area, and how they acquired and enhanced such knowledge that influenced teachers' effectiveness. Information on the established teachers’ conceptions of teaching, their PCK, and their teaching behaviors would be sought.

DESIGN AND METHODS
In this research, the case study was employed to understand in depth the PCK of established teachers. It enabled a description of classroom instruction to be depicted for the reader’s understanding of the established teachers’ PCK that was under study, where the discovery of new meaning was illuminated, thereby extending the reader’s experience or confirming what was known. The aim of the case studies was to describe the unit of analysis in depth, with detail, and in context (Stake, 1978). Case study is highly appropriate for this research as it facilitated responses to personal and environmental cues, and adaptation to multiple factors and situations. According to Lincoln & Guba (1985), human instruments allowed propositional and tacit knowledge to be used instantly to process and summarize data, seek clarification from respondents, and explore atypical responses.

The participating teachers were selected as they were committed to teach for conceptual understanding in mathematics, possessed a wide repertoire of relevant and meaningful mathematical representations, and were able to create and select tasks and activities that facilitated pupils’ conceptual growth and understanding. In addition, they were effective in anticipating and rectifying pupils’ misconceptions in mathematics learning.

Data from multiple sources (classroom observations, interviews and a questionnaire survey) formed the corpus of data for this study. Data was collected in the three studies that spanned two whole years. Study 1, or the Pilot Study, spanned a whole year. It included eighteen classroom observations and numerous informal interviews. Each classroom observation was followed by an interview to probe and verify doubts and interpretations pertaining to the lesson taught. Participating teachers were also interviewed over the telephone frequently to verify interpretations of the observed lessons on the teachers’ thinking processes, ideas and beliefs, as
well as to find out if the teachers’ thinking about the various domains, subject matter knowledge and PCK had changed from previous years, and how their teaching had taken shape. The subsequent rounds of data collection (Study 2 and 3) were carried out in the following year. Each of the three established teachers was observed twice for the teaching of area. Again, post-lesson interviews were conducted to clarify doubts and ascertain what was intended by the teachers. The primary focus was on each teacher’s perspective on using PCK. The questionnaire survey was administered after the second observation. The participants completed a questionnaire to collect factual data about their educational history and preferences, mathematics courses attended, in-service teaching experiences, teaching experiences and classroom practices. The tasks, discourse and analysis carried out during classroom observations were then written in vignettes to describe, explain, and compare the teachers’ PCK during instruction. The participants’ verbatim accounts of conversations were presented as often as possible in the write-ups to illustrate the participants’ meanings and to combat researcher bias (Lincoln and Guba, 1985). Subsequently, the data from the various sources were analyzed. A cross-case analysis in which themes that cut across them were examined was used to write up the three cases. The individual cases were then used to explain differences within these themes.

RESULTS

Teachers’ Use of Authentic Tasks and Meaningful Representations

The established teachers used authentic tasks involving significant mathematics and real-world data. They employed various teaching strategies and instructional representations to enable pupils to learn mathematics in a meaningful and practical manner.

Teaching Vignette: Cho
In a lesson on area, Cho got the pupils to experiment and to look for a pattern that would result in a convenient way or rule to follow when looking for areas of squares or rectangles. Cho asked the pupils to list as many rectangles with an area of 36cm$^2$ as possible. She asked, “How will you know that you have an exhaustive list?” “What can you use to display all your answers systematically?” To link area to perimeter, Cho asked the pupils to list 2 sets of rectangles or squares with similar area and perimeter. Next Cho challenged the pupils to look for a pattern and to come up with a rule for rectangles and squares with similar areas and perimeters. The pupils tabulated their results systematically. However, most pupils were not able to change one variable at a time, keeping the rest constant. Some pupils found that as the breadth varied with different lengths, different patterns resulted. With a constant length, as the breadth increased, the area increased and then became the same as the perimeter, as in a length of 4 or a length of 6.

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<th>Length/ concrete materials</th>
<th>Breadth/ cm</th>
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<th>Perimeter/ cm</th>
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Then, as the breadth increased for the same length, the area increased too. At times, the rectangles did not have similar areas and perimeters, as the breadth varied, as in length of 5.

Cho had deliberately chosen a problem that would provide opportunities for pupils to make connections with concepts they had explored in finding the perimeter. They made connections between perimeter and area that involved the development of concepts, procedures, and skills. Pupils needed to understand that the perimeter was the distance around a section or region, and
the area was space or surface of the section or region. Also, they needed to know that perimeter and area were not directly related; that is, two rectangles could have different perimeters but the same area. The pupils were engaged in the task intellectually because it challenged them to search for something, to reason and to verify their findings. Being able to justify an answer and to show that a problem was solved were critical components of mathematical reasoning and problem solving. As such, the pupils showed that they could apply ideas and language from their earlier work in the various mathematics topics. They communicated mathematically and drew attention to patterns through the use of a shared mathematical language. The task had allowed the pupils to develop their understanding of the concept of area and its relationship to perimeter.

To lead the children to think and explore further, Cho asked the pupil to draw the different changing shapes of the rectangles as the areas and perimeters increased, and to look for a pattern and relationship between them.

![Diagram](https://via.placeholder.com/150)

Length = 4cm

From their drawings, the pupils realized that the figure had the same area and perimeter when L = B, that is, when Perimeter = L X 4 and Area = L X L. Hence, in this particular case, Perimeter = area, as L = 4cm, as length was multiplied 4 times for perimeter, and length was multiplied by length which happened to be 4 also. The pupils could point out that the area increased less slowly then the perimeter until the sides had the same length of 4cm. Thereafter, the pupils noticed that the area increased faster than the perimeter. However, they could not come up with an explanation for it. Cho led the pupils to look at the size of the rectangles that they had drawn.
Prior to a length of 4cm, the length was multiplied by a number smaller than 4 for area. However, after a length of 4cm, to find area, the length was multiplied by a number bigger than 4, so it increased at a faster rate. Comparatively, the areas increased at a faster rate as multiplication was involved, whereas for perimeter, only addition was involved, so the increase was slower. For example, with a measurement of 4cm by 6cm, the area was found by multiplying 4 by 6, an increase of six fold, whereas the perimeter was found by adding 4 to 4 and then to 6 and 6, where the increase was slower. Cho had more subject matter knowledge than most primary school teachers through her on-going fascination with the quantitative aspects of life. Her evident pleasure in numbers and her creativity in finding opportunities to conjecture about them from the real world had prompted other teachers to look for her for insights into mathematics instruction. Thus, her representations were used with the flexibility high subject matter knowledge provided, allowing her to help pupils struggle with the complexities of real problems not typically found in the pre-made materials on which less confident teachers tended to rely. Previously, she tended to use representations to develop traditional disciplinary knowledge. Now, she further required her pupils to think deeply about the dimensions of a problem and empowered them to solve problems in a community setting with the tools she had helped them develop. Cho intended to help pupils learn to conjecture, invent, and solve problems. The pupils asked, “What will happen if we increase only the breadth?”, “What if we do not increase the breadth but the length only?”, “Do you see a pattern?”, “Can you predict the next one?”, “What is alike and what is different about your method from mine?” After a quick introduction to a new topic, Cho focused on problem solving. The pupils were really working intensely and thinking; they created their own problems, explored extensions and elaborations, and were willing to do a lot of “what if” thinking. They always explained and not just gave an
answer, but knew where the answers came from, and how they got the answer. Cho intended to help the pupils learn mathematics in a fundamentally connected way.

Teaching Vignette: Sen

To use multiple representations that were meaningful so as to promote pupils mathematical learning and understanding, Sen led her Primary 5 pupils, who had already learned how to find the area of a square, to find the area of a triangle by folding a square into half. By halving the square or rectangle, the pupils could relate to halving the area as well; and hence arrive at the area of the isosceles triangle to be $\frac{1}{2} \times L \times B$ or in the terms of a triangle without memorizing the formula.

She helped them connect between areas of squares and triangles. The folding and overlapping enabled the pupils to observe that the area of the square was halved, $\div 2$ or $\times \frac{1}{2}$, when the square was transformed by folding to a triangle. When asked about the area of the triangle, the pupils having observed the actual process, replied that the area of the rectangle had been folded into two congruent parts, transforming the square into a triangle, so the area was halved too; $\frac{1}{2} \times L \times B$. This enabled them to derive the formula of the isosceles and right angled triangles.

Next, Sen brought up the following question to allow the children to apply what they had learned. She asked, “Is the area of A the same as that of B in the following diagram? Why?”
Teaching Vignette: Flo

Through her many years of teaching, Flo knew that pupils always had misconceptions and difficulty finding the perpendicular heights of obtuse and isosceles triangles whilst finding the area of obtuse triangles and non-right angled triangles, or to be more precise, when the perpendicular height was not one of the edges of the given triangle. Hence, to troubleshoot that, she had a worksheet on various shapes of scalene, isosceles and obtuse triangles as shown below.

She asked her pupils to label the base of each triangle, and then the height. The pupils had difficulty with the heights as they were not as clear cut as those of right angled triangles. She got round this by asking them to tie a weight (their erasers) to the end of a string. To find the height perpendicular to the base of the triangle, the pupils were to pin the string with the small weight to the corner of the triangle opposite the labeled base, and then hold the paper vertically to allow the string to ‘drop’ freely, perpendicular to the base. Next, she elicited from the children that the string actually acted as the perpendicular height from the attached corner to touch the base of the triangle. If the eraser fell outside the triangle, the pupils were asked to extend the base of the triangle in the given figure with a dotted line to meet the string perpendicularly. Throughout the whole activity, the teacher asked, “Why do you choose the corner opposite the base and not the two corners of the triangle along the chosen base?” “What does the string acts as when dropped from the opposite corner?” “If the height drops inside the triangle, please show me where the base is?”, “Show me using your fingers exactly from which part to which part the base of the triangle is?” “If the height drops outside the triangle, will the base be the same as the figure’s?”
Flo knew that to help her pupils learn area of triangles involved developing concepts, procedures, and skills. Pupils needed to understand that the height of a triangle was the length that was perpendicular to its base. The pupils were actively discussing and reasoning logically, and were engaged in the task intellectually. The task allowed the pupils to justify their answers and to develop their understandings of the concept of area of triangles and the relationship between the perpendicular height and its base. As for given bases which were not horizontal to the edge of their desks, the pupils were unsure of what to do next. Flo asked them for suggestions. One pupil suggested turning the edge of the triangle till it was parallel to the edge of their desks before they began using the string and weight. Another pupil further suggested that they wrote the word ‘base’ on the triangle before starting to work on them. Flo asked them to try out all their suggestions and to share with the class which worked best later. She then asked, “Must the height of a triangle always be inside a triangle?”, “Can the height of the triangle drop from a corner of the triangle to be outside the triangle?” Some pupils answered yes, others said no. Flo got them to continue to conjecture and come up with various shapes of triangles. They then experimented with all shapes and sizes of triangles using the string, tabulated their results systematically, and concluded that for obtuse triangles, the height was outside the triangle, while for acute and right angled triangles, the perpendicular heights were always inside or on one of the edges of the triangles. The use of manipulatives gave the pupils an opportunity to interact with the physical objects so as to construct, modify, and integrate ideas. Flo had provided a context for them to discuss among themselves, and to challenge and defend their ideas in their groups. She helped her pupils to connect their thoughts with the mathematical definition and provided further information. Flo got to know how pupils learned mathematics and about their misconceptions
through observations and interviews. Through the years, she had revised her assumptions and learned to interpret pupils’ words, written work and ways of thinking.

To summarize, the teaching vignettes by the three established teachers were used to illustrate the use of authentic tasks and meaningful representations that incorporated significant mathematics.

**DISCUSSION, IMPLICATIONS AND CONCLUSIONS**

The established teachers were able to connect mathematical concepts and procedures, to make connections from simple knowledge to complex knowledge with underlying connections among various mathematical operations and sub domains. As such, their teaching was a cohesive, whole body of knowledge to prevent pupils from learning in fragmented parts. Furthermore, they possessed and displayed an extensive, perceptive and profound understanding which enabled the pupils to learn significant and meaningful mathematics. The teachers revealed and represented this understanding in their teaching and pupils’ learning of mathematics.

They employed authentic tasks involving significant mathematics. In addition, meaningful and practical representations and explanations were incorporated. Frequently, they encouraged and accepted different ideas and approaches to solutions from their pupils, to give them a flexible understanding of the discipline. The teachers incorporated simple but powerful basic mathematical concepts, principles and procedures to enable pupils to solve problems and carry out activities related to everyday situation to allow the pupils to see how meaningful and practical mathematics could be. There was frequent revisit and reinforcement of these basic ideas. Besides, the teachers displayed knowledge pertaining to the level they were teaching.
which facilitated the revision of concepts learned earlier effortlessly and to provide them with a foundation that would equip them properly.

All in all, the teaching of established teachers had connectedness, where basic ideas were revisited and reinforced, and longitudinal coherence was upheld. The established teachers were able to reveal and represent connections among mathematical concepts and procedures in their daily instructions. They had extensive understanding that allowed for connection of a topic with topics of similar or less conceptual power. Also, they possessed depth in understanding which enabled them to connect a topic with those of greater conceptual power. Their thoroughness facilitated the connections between concepts and across mathematical topics. Besides, they were able to respond appropriately to pupils’ questions, co-constructed mathematical ideas and representations with the pupils by linking their prior knowledge with what was to be learned, design and engage pupils in learning tasks and activities incorporating significant mathematics, and orchestrate mathematical discourse in the classroom. These established teachers believed that pupils should be engaged in tasks that involved interrelationships among mathematical concepts and procedures so that mathematics would be learned in a practical and meaningful manner and not in an isolated way. If the connections occurred frequently enough, pupils’ understanding of mathematical content and connections would be expanded, thereby influencing pupils’ beliefs. Only through such relational understanding could pupils transfer their learning to solve mathematical problems successfully in new situations.
REFERENCES


