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Tangrams: On Attention and Error

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Even in mathematics, a lack of attention may result in a publication with errors. This article postulates a reason why errors are made in mathematics and their consequences. The discussion revolves around the possibility of using certain numbers of Tangram sets to form squares of sides of different lengths. Implications for pedagogy are drawn to avoid and to capitalise on errors.

Keywords: tangram, proof, errors, attention, fractals

Errors in Mathematics Articles

Singh (1997) describes the agony that Andrew Wiles went through when it was discovered that there was an error in his proof of Fermat's Last Theorem. Though the story had a happy ending when the proof was fixed, it seemed at the point of discovery that the error was "fatal". Indeed, the error was fatal in the sense that that particular section of the proof was logically flawed but it was not fatal in the sense that the theorem itself was wrong. Thus, a fix was obtained after a year, from the retrieval of earlier rejected attempts.

In an earlier issue of this journal *The Mathematician Educator*, Toh, Tay and Tong (2021) debunked a common error that $\frac{dsinx^{\circ}}{dx} = \frac{dsinx}{dx}$. Indeed, the correction of mathematical errors is a rich area in the genre of *mathematician educator* writing. Thus, in the first article of the first issue of *The Mathematician Educator*, Tay (2020) states that one pedagogical affordance of reading mathematics journal papers is that "[t]he student can also learn that errors lurk in mathematics texts."

In that same article, Tay (2020) remarked "When schoolteachers understand that it can be *proven* that two tangram sets cannot be assembled into a square, then they are *mathematician* educators." Imagine my embarrassment when I cited this to a new colleague who said that he could make a square from two tangram sets as in Figure 1 below!

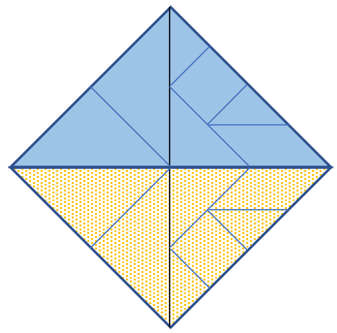

Figure 1. A square from two tangram sets

This paper pivots from this incident to discuss the different results of attending to different aspects of a matter, the nature of mathematical errors with regard to *fatalness*, and of course, pedagogical implications.

Tangrams and Attention

Let us first describe precisely what a tangram set is. A tangram set consists of 7 polygons with the following properties:

1. There are two right isosceles triangles each with hypotenuse of length 1.
2. There are two right isosceles triangles each with hypotenuse of length $\frac{1}{2}$.
3. There is one right isosceles triangle with hypotenuse of length $\frac{\sqrt{2}}{2}$.
4. There is one square of side $\frac{\sqrt{2}}{4}$.
5. There is one parallelogram with adjacent sides of length $\frac{1}{2}$ and $\frac{\sqrt{2}}{4}$.

Together, the seven pieces fit exactly to form a square of side 1 as shown in Figure 2 below.

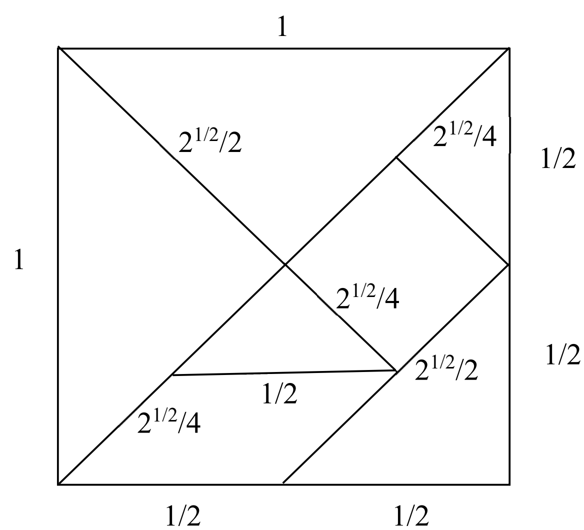

Figure 2. Dimensions of one tangram set

The tangram incident started when a colleague mentioned how he used tangrams to teach school children and I seemed to hear him saying "I would give them two tangram sets and ask them to make a square. I would then tell them that it is impossible to do so." As a mathematician educator, I was immediately prompted by the latter sentence and asked, "Did you tell them that you could *prove* that it cannot be done?" My colleague replied, "Really, you can prove that it can't be done?" and I said, "Yes, using Galois Theory." In my mind, I had reasoned that if it cannot be done, it would be proved by showing that the length of the side of the required square could not be achieved by the sum of the lengths of any combination of sides of two sets of tangrams. I was sure that a simple application of field extensions would suffice without actually going through the calculations. While my colleague's attention was on the fact that it could not be done, my attention was on proclaiming the 'aesthetic' aspect of mathematics of being able to prove deductively that something *cannot* happen.

Mason (2021) wrote a very interesting paper showing the engaged reader how by shifting attention to various aspects of a series of patterns, one is able to obtain different representations of a number as an infinite series. As an example, let us see how by shifting attention, we can conclude from Figure 3 below that $\frac{2}{5} + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \left(\frac{2}{5}\right)^5 + \cdots = \frac{2}{3}$.

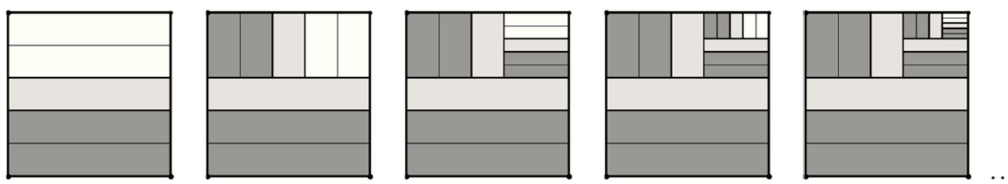

Figure 3. $\frac{2}{5} + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \left(\frac{2}{5}\right)^5 + \cdots = \frac{2}{3}$ (Mason, 2021)

We see that for each square, regions are coloured in three possible ways: black, grey and white.

In the first square, we pay attention first to the black region and note that the black region is $\frac{2}{5}$ of the whole square. We next shift our attention to both the black and grey regions and note that the black region is $\frac{2}{3}$ of the combined black and grey regions. Finally, for the first square, we pay attention to the white region and note that it is $\frac{2}{5}$ of the whole square.

In the second square, we see that the white region from the first square has been divided into 5 smaller rectangles: 2 black, 1 grey, 2 white. Nothing has changed for the black and grey regions from the first square. So we pay attention *only* to the 5 smaller rectangles. Let us call this the 'available space'. We pay attention first to the new black region and note that the new black region is $\frac{2}{5}$ of the available space. We next shift our attention to both the new black and grey regions and note that the new black region is $\frac{2}{3}$ of the combined new black and grey regions. Finally, we pay attention to the new white region and note that it is $\frac{2}{5}$ of the available space.

By now, we see that the 'available space' of the third figure is obtained from the white region of the second figure. Similar arguments lead us to the conclusion that the new black region is

$\frac{2}{3}$ of the combined new black and grey regions. And we carry on in a similar manner with each new figure.

By the end of our '…' process, we can imagine the original square with only black and grey regions, the white regions being systematically replaced by them. Now revert attention to the ratio of new black to new grey regions at each step – it is 2:1. Thus, in the 'final' square, the black region is exactly $\frac{2}{3}$ of the whole square. Revert attention next to *how* the black region is formed at each step (it is $\frac{2}{5}$ of the 'available' space) and *how* each available space is formed (it is $\frac{2}{5}$ of the 'available' space of the previous figure). Thus, in the first figure, the black region is $\frac{2}{5} \times 1$, in the second figure, the black region is $\frac{2}{5} \times 1 + \frac{2}{5} \times \frac{2}{5}$, in the third figure, the black region is $\frac{2}{5} \times 1 + \frac{2}{5} \times \frac{2}{5} + \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$, and so on. Thus, equating the two ways we obtain the total area of the black region in the 'final' figure, we have $\frac{2}{5} + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \left(\frac{2}{5}\right)^5 + \cdots = \frac{2}{3}$.

The elaboration above hopefully explains how that by paying attention to one aspect of an issue, we 'block out' other aspects leading to a clearer and more insightful view of the salient features of the aspect in focus.

Tangrams and Fixing Errors

In this article, we highlight how paying attention to one aspect can unfortunately lead to inadvertent errors in the aspects not paid attention to. The tangram incident is one such example. Thus, we find that the focus on the *provability* of the "impossibility of making a square from two tangram sets" made the author, the readers, and even the reviewers gloss over the *truth* of the statement that "it is impossible to make a square from two tangram sets". It was fortunate that after the conversation that showed the falsity of the statement (see Figure 1 again), I was in time to correct the error in another article that was in the last stages of publication (Tay & Kaur, 2021) by simply changing "two tangram sets" to "three tangram sets". In retrospect, I am quite sure that I must have misheard my colleague's statement and that he actually said three instead of two. This type of error is just plain embarrassing but is not considered to be *fatal* because it can be easily fixed. We will just change 2 to 3 and prove the statement in what follows.

> **Proposition** It is not possible to construct a square using three tangram sets.
>
> **Proof** Suppose it were possible. Then, since the total area of 3 tangram sets is 3, the side of the resulting square must be $\sqrt{3}$. The side must be made up of sides of pieces of the tangram sets joined end to end. We observe that the sides of the pieces are of the form *a* or $b\sqrt{2}$, where *a* and *b* are rational numbers. The sum of any number of sides would then be of the form $a + b\sqrt{2}$. Thus, we have

$$a + b\sqrt{2} = \sqrt{3}$$
$$a^2 + 2ab\sqrt{2} + 2b^2 = 3$$
$$2ab\sqrt{2} = 3 - a^2 - 2b^2.$$

Since $\sqrt{2}$ is irrational, we must have $ab = 0$ and $3 - a^2 - 2b^2 = 0$. If $a = 0$, we have $3 = 2b^2$, which is impossible because *b* is rational. If $b = 0$, we have $3 = a^2$, which is again impossible because *a* is rational.

Thus, the supposition is wrong and it is not possible to construct a square using three tangram sets. □

We can see that if we use a similar argument, we can prove the necessity for the following generalisation:

It is possible to construct a square using *n* tangram sets if and only if $n = 2^m$, for $m \in \mathbb{Z}_0^+$.

The sufficiency is proved by stacking 2^k rows of 2^k squares of side 1 when $n = 2^{2k}$ and stacking 2^k rows of 2^k squares of side $\sqrt{2}$ when $n = 2^{2k+1}$.

But wait! After leaving the writing of this article at this point at the end of the day, I mulled over the incomplete 'proof' of the 'generalisation' and felt uneasy. So I paid attention to (focused on) the generalisation. It seemed that I had generalised the theorem by extending the possible squares of lengths 1 and $\sqrt{2}$ to squares of side $\left(\sqrt{2}\right)^m$. I needed to pay more attention to this 'jump'. I thus ran through some numbers which were NOT of the form 2^m to see if they would form squares. It did not take long to see that the squares, i.e., m^2 would form squares – simply stack *m* rows of *m* squares of side 1. Still on focus mode, it did not take long to check that $2m^2$ tangram sets would also do - stack *m* rows of *m* squares of side $\sqrt{2}$.

I decided that to avoid any more errors, I should write out the generalisation and the proof in complete detail as follows.

Proposition It is possible to construct a square using *n* tangram sets if and only if $n = m^2$ or $n = 2m^2$ for $m \in \mathbb{Z}_0^+$.

Proof Suppose $n = m^2$ or $n = 2m^2$ for $m \in \mathbb{Z}_0^+$. Then for the former, we form m^2 squares of side 1 from the m^2 tangram sets and stack *m* rows of *m* squares of side 1 to form a square of side *m*. For the latter, we form m^2 squares of side $\sqrt{2}$ from the $2m^2$ tangram sets and stack *m* rows of *m* squares of side $\sqrt{2}$ to form a square of side $\sqrt{2}m$.

Suppose $n \neq m^2$ and $n \neq 2m^2$ for $m \in \mathbb{Z}_0^+$. Then, since the total area of *n* tangram sets is *n*, the side of the resulting square must be $\sqrt{n}$. The side must be made up of sides of pieces of the tangram sets joined end to end. We observe that the sides of the pieces are of the form *a* or $b\sqrt{2}$, where *a* and *b* are rational numbers. The sum of any number of sides would then be of the form $a + b\sqrt{2}$. Thus, we have

$$a + b\sqrt{2} = \sqrt{n}$$
$$a^2 + 2ab\sqrt{2} + 2b^2 = n$$
$$2ab\sqrt{2} = n - a^2 - 2b^2.$$

Since $\sqrt{2}$ is irrational, we must have $ab = 0$ and $n - a^2 - 2b^2 = 0$. If $a = 0$, we have $n = 2b^2$, which contradicts the assumption that $n \neq 2m^2$. (Note: $2b^2$ is an integer implies that b is an integer.) If $b = 0$, we have $n = a^2$, which contradicts the assumption that $n \neq m^2$. Thus, the supposition is wrong and it is not possible to construct a square using n tangram sets if $n \neq m^2$ and $n \neq 2m^2$ for $m \in \mathbb{Z}_0^+$.

The proof is now complete. □

The elaboration above shows that errors in mathematics can easily occur when we are sloppy in rigour and that some of these sloppy errors are due to a lack or a diversion of attention. Two solutions exist. The first is to be careful and pay attention to all aspects, which Kahneman (2011) described as System 2 thinking, all the time. This however, is very tiring. The other is to have a habit of checking our work, somewhat akin to Pólya's (1954) fourth stage of "Looking Back". The optimal solution may be a conservative use of System 2 thinking together with checking as an academic habit.

The optimal aspect of not being on System 2 thinking all the time is the conservation of creativity and attention. Knowing that one needs to move along, and accepting that errors may occur as long as they are not fatal, allows the mathematician to be optimally productive. Indeed, sometimes errors fortuitously produce better results than intended, perhaps a better theorem or a better construction of a proof. The following anecdote ends this section to illustrate the fortuitous result of an error.

Once I was working on writing a code for the Koch curve on Microsoft Excel. The Koch curve is a fractal formed by iterations of 4 contraction transformations:

1. A scaling of size 1/3.
2. A scaling of size 1/3 followed by a translation along the *x*-axis of length 2/3.
3. A scaling of size 1/3 followed by an anti-clockwise rotation of 60° about the origin and a translation along the *x*-axis of length 1/3.
4. A scaling of size 1/3 followed by a translation along the x-axis of length −1/3, a clockwise rotation of 60° about the origin, and a translation along the *x*-axis of length 2/3.

The result would look like the shape in Figure 4 below.

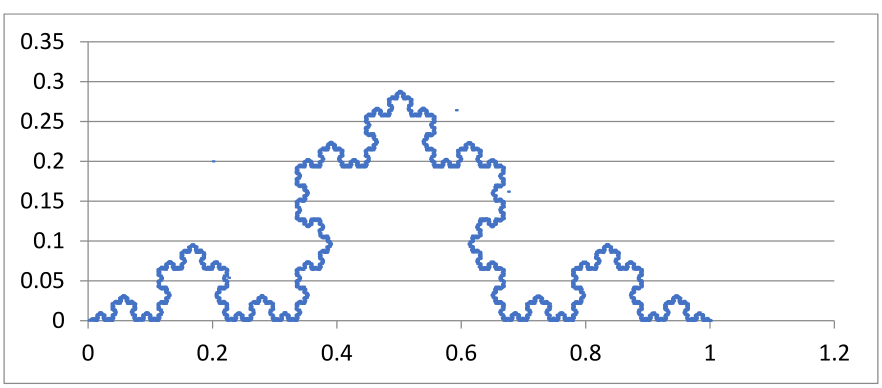

Figure 4. The Koch curve

The codes for the *x*-coordinate and *y*-coordinate in Excel for the probabilistic program would respectively be:

```
=IF(C1<1,A1/3,IF(C1<2,(A1-SQRT(3)*B1+2)/6,IF(C1<3,(A1+SQRT(3)*B1+3)/6,(A1+2)/3)))
=IF(C1<1,B1/3,IF(C1<2,(SQRT(3)*A1+B1)/6,IF(C1<3,(B1-SQRT(3)*A1+SQRT(3))/6,B1/3)))
```

Certainly, there is "room for error"! But in coding, I find it productive to just go ahead and see what happens rather than meticulously check each symbol in the code before running. Thus, I coded quickly and obtained the following in Figure 5 instead.

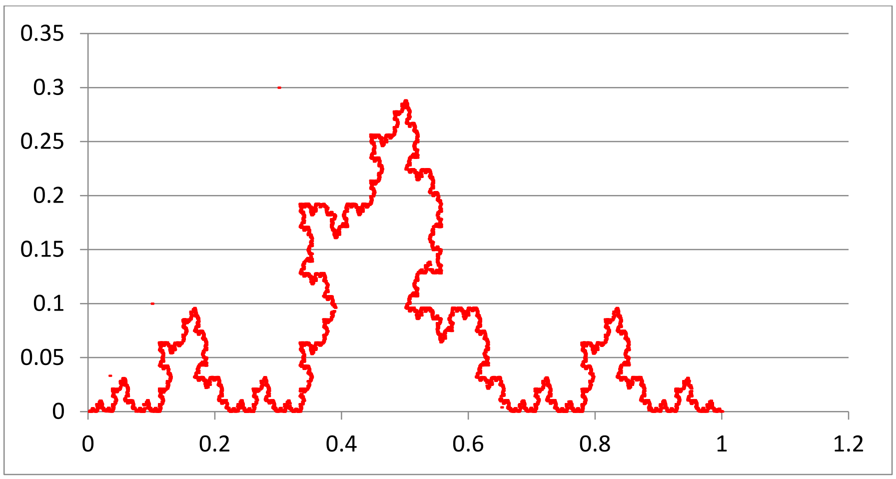

Figure 5. The Fire curve

I think most would agree that this shape, which I named Fire, has its own natural beauty and may even look a bit nicer than the Koch curve. On examining, I saw that my error was in Transformation 4 - I had coded an anti-clockwise rotation of 120° about the origin and a translation along the *x*-axis of length 2/3. An error had resulted in a more natural looking shape and indeed, such natural looking shapes are sought after in computer generated graphics. Figure 6 shows four iterations of a deterministic program for both curves. Note that they seemingly start with a common first iteration "_/_".

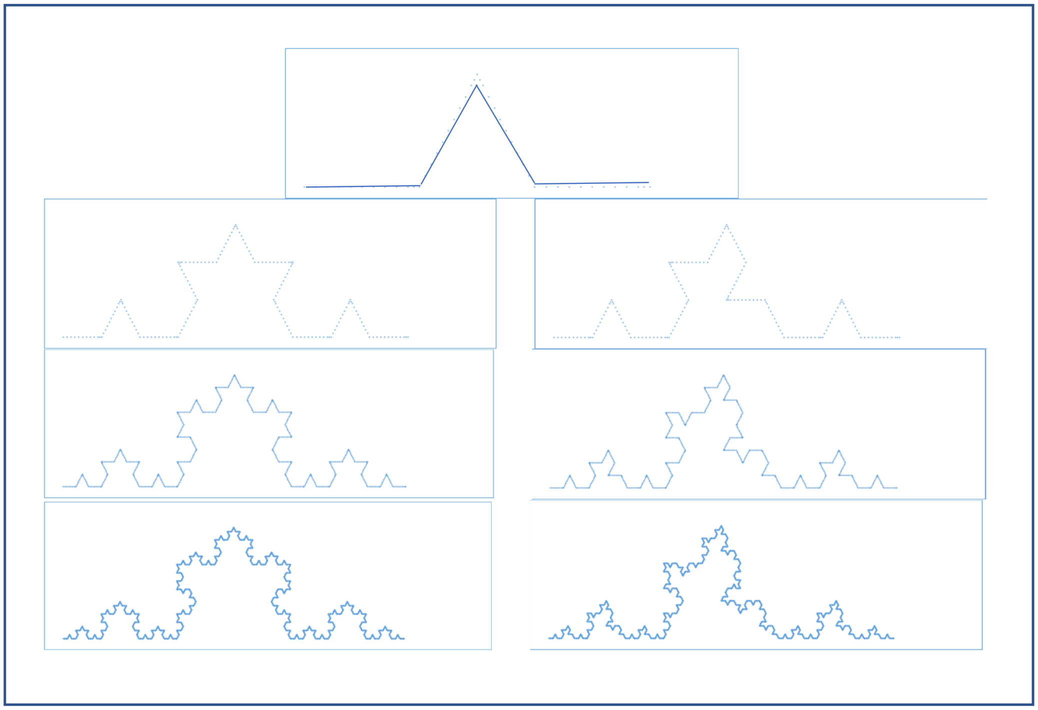

Figure 6. The Koch curve and the Fire curve in 4 iterations

Pedagogical implications

This article examines an embarrassing error made by the author that was published in a journal. It reflects on why errors are made and even unnoticed. A key reason is the aspect of attention. The author was only concerned with the provability of a statement and did not pay attention to the truth of the statement. The mathematician educator is one who should focus the attention of his or her learners on certain aspects of the mathematics at hand to derive the maximum insight. An elaboration of one of Mason's patterns gives an example of how attention to different aspects at different times aids learning. An elaboration of an attempt to fix the tangram problem also shows how important it is to check one's own work.

Yet, to try to completely avoid errors by paying full attention to all aspects of a problem is untenable and unproductive. Thus, it is suggested to mathematician educators to find an optimal approach of care in attention complemented with a habit of checking one's work. The same would be good to encourage in one's students. Errors which give rise to fortuitous results should also be celebrated and examined for possible productive directions.

Finally, following the suggestion of a reviewer regarding the tangram problem, we can see that the substance of the argument is that $a + b\sqrt{2} = \sqrt{3}$ has no solutions. The mathematician educator may take this opportunity to mention another possible generalisation which is that no linear combination of $\sqrt{p_i}$ for distinct primes p_i can be a rational number either. One can easily see how to relax these conditions further by merely placing coprimality conditions on the p_i's. Generalisations in mathematics are very important, and even the simplest examples can lead the student surprisingly far and would be an opportunity too good to miss.

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