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Stratonovich-Henstock integral for the operator-valued stochastic process

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Abstract

In this paper, we introduce the Stratonovich-Henstock integral of an operator-valued stochastic process with respect to a Q-Wiener process. We also formulate a version of Itô's formula for this integral.

Keywords: *Stratonovich-Henstock Integral, Q-Wiener process, Itô's formula.*

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1. Introduction

The Lebesgue integral, which somehow serves as one of the fundamentals in calculus, is widely used in almost all areas of mathematics. However, the Lebesgue approach to integration requires a strong foundation of measure theory. Hence, one of the notable integrals that were introduced which encompasses the Lebesgue integral is the Henstock integral. This integral was studied independently by Henstock and Kurzweil in the 1950s and later known as the Henstock-Kurzweil integral. The Henstock-Kurzweil integral gives an alternative definition of the Lebesgue integral and avoids an extensive study of measure theory and utilizes only the concepts of δ-fine and coverings. This approach to integration is known as the generalized Riemann approach or Henstock approach. Since then, numerous authors have been interested to study Henstock integration, see [6, 7, 8, 9, 12, 13, 14, 15, 16, 23].

In the classical approach to stochastic integration, the definition of the stochastic integral is almost similar in defining the Lebesgue integral of a measurable function. To reduce the technicalities in the classical way of defining the stochastic integral of the the real-valued process, Henstock approach to stochastic integration had been studied and investigated by some authors, see [2, 17, 18, 24, 25, 26, 27].

In [28], the authors used the generalized Riemann approach to define the Stratonovich integral of a real-valued process with respect to a Brownian motion. This newly defined integral encompasses the classical Stratonovich integral and satisfies the ideal Itô's formula.

In this paper, we define the Stratonovich-Henstock integral of an operator-valued stochastic process with respect to a Hilbert space-valued Q-Wiener process. We also formulate a version of Itôs formula for the Stratonovich-Henstock integral.

2. Preliminaries

Let U and V be separable Hilbert spaces. Denote by $L(U,V)$ the space of all bounded linear operators from U to V, $L(U) := L(U,U)$, $Qu := Q(u)$ if $Q \in L(U,V)$, and $L^2(\Omega, V)$ the space of all square-integrable random variables from Ω to V. An operator $Q \in L(U)$ is said to be nonnegative definite if for every $u \in U$, $\langle Qu, u\rangle_U \geq 0$. If $Q \in L(U)$ is nonnegative definite, then the trace of Q is defined by tr $Q = \sum_{j=1}^{\infty} \langle Qe_j, e_j\rangle_U$, where $\{e_j\}$ is an orthonormal basis (abbrev. as ONB) in U. The trace of Q is welldefined and may be defined in terms of an arbitrary ONB, see [20, p.206]. An operator $Q : U \to U$ is said to be trace-class if tr $[Q] :=$ tr $(QQ^*)^{\frac{1}{2}} < \infty$. Denote by $L_1(U)$ the space of all trace-class operators on U, which is a Banach space [20, p.209] with norm $\|Q\|_1 = \text{tr } [Q]$.

If $Q \in L(U)$ is a symmetric nonnegative definite trace-class operator, then there exists an ONB $\{e_j\} \subset U$ and a sequence of nonnegative real numbers $\{\lambda_j\}$ such that $Qe_j = \lambda_j e_j$ for all $j \in \mathbf{N}$, $\{\lambda_j\} \in \ell^1$, and $\lambda_j \to 0$ as $j \to \infty$, see [20, p.203]. We shall call the sequence of pairs $\{\lambda_j, e_j\}$ an eigensequence defined by Q. The subspace $U_Q := Q^{\frac{1}{2}}U$ of U equipped with the inner product $\langle u, v\rangle_{U_Q} = \left\langle Q^{-1/2}u, Q^{-1/2}v\right\rangle_U$, where $Q^{1/2}$ is being restricted to $[\mathrm{K}er Q^{1/2}]^\perp$ is a separable Hilbert space with $\{\sqrt{\lambda_j}e_j\}$ as its ONB, see [3, p.90], [5, p.23]. Denote by $L_2(U_Q, V)$ the space of all Hilbert-Schmidt operators from U_Q to V, which is a separable Hilbert space with norm $\|S\|_{L_2(U_Q,V)} = \sqrt{\sum_{j=1}^{\infty} \|Sf_j\|_V^2}$, see [19, p.112]. The Hilbert-Schmidt operator $S \in L_2(U_Q, V)$ and the norm $\|S\|_{L_2(U_Q,V)}$ may be defined in terms of an arbitrary ONB, see [3, p.418], [19, p.111]. We note that $L(U, V)$ is properly contained in $L_2(U_Q, V)$, see [5, p.25]. We also note that $L_2(U_Q, V)$ contains genuinely unbounded linear operators from U to V.

Let $Q : U \to U$ be a symmetric nonnegative definite trace-class operator, $\{\lambda_j, e_j\}$ be an eigensequence defined by Q, and $\{B_j\}$ be a sequence of independent Brownian motions (abbrev. as BM) defined on a filtered probability space $(\Omega, F, \{F_t\}, \mathbf{P})$. A process $\tilde{W}_t := \sum_{j=1}^{\infty} \sqrt{\lambda_j} B_j(t) e_j$, is called a Q-Wiener process in U, with the series converging in $L^2(\Omega, U)$. For every $u \in U$, denote $\tilde{W}_t(u) := \sum_{j=1}^{\infty} \sqrt{\lambda_j} B_j(t) \langle e_j, u\rangle_U$, with the series converging in $L^2(\Omega, R)$. Since the operator Q is assumed to be symmetric nonnegative definite trace-class, there exists a U-valued process W, known as a U-valued Q-Wiener process, such that $\tilde{W}_t(u)(\omega) = \langle W_t(\omega), u\rangle_U$ $\mathbf{P}$-almost surely (abbrev. as $\mathbf{P}$-a.s.). It should be noted that the process W is a multi-dimentional BM, and if we assume that $\lambda_j > 0$ for all j, $\frac{W_t(e_j)}{\sqrt{\lambda_j}}$, $j = 1, 2, \ldots,$ is a sequence of real-valued BM defined on $(\Omega, F, \{F_t\}, \mathbf{P})$, see [3, p.87].

A filtration $\{F_t\}$ on a probability space $(\Omega, F, \mathbf{P})$ is called normal if (i) F_0 contains all elements $A \in F$ such that $\mathbf{P}(A) = 0$, and (ii) $F_t = F_{t+} := \bigcap_{s>t} F_s$ for all $t \in [0, T]$. A Q-Wiener process W_t, $t \in [0, T]$ is called a Q-

Wiener process with respect to a filtration $\{F_t\}$ if (i) W_t is adapted to $\{F_t\}$, $t \in [0,T]$ and (ii) $W_t - W_s$ is independent of F_s for all $0 \leq s \leq t \leq T$. A U-valued Q-Wiener process $W(t)$, $t \in [0,T]$, is a Q-Wiener process with respect to a normal filtration, see [19, p.16]. From now onwards, a filtered probability space $(\Omega, F, \{F_t\}, \mathbf{P})$ shall mean a probability space equipped with a normal filtration.

3. Stratonovich-Henstock Integral

In this section, we shall define the Stratonovich-Henstock integral of an operator-valued stochastic process and enumerate some of its standard properties. This type of integral was first used in [28] for the real-valued stochastic process with respect to a Brownian motion. Throughout this paper, the given closed interval $[0,T]$ is nondegenerate, i.e. $0 < T$ and can be replaced with any closed interval $[a,b]$. If no confusion arises, we may write $(D)\sum$ instead of $\sum_{i=1}^{n}$ for the given finite collection D. By a gauge, we mean a positive function. From now onwards, assume that U and V are separable Hilberts spaces, $Q : U \to U$ is a symmetric nonnegative definite trace-class operator, $\{\lambda_j, e_j\}$ is an eigensequence defined by Q, and W is a U-valued Q-Wiener process.

Definition 3.1. *Let δ be a gauge on $[0,T)$. A finite collection $D = \{([\xi_i, v_i], \xi_i)\}_{i=1}^{n}$ of inteval-point pairs is a δ-fine belated partial division of $[0,T]$ if $\{[\xi_i, v_i]\}_{i=1}^{n}$ is a collection of non-overlapping subintervals of $[0,T]$ and each $[\xi_i, v_i]$ is δ-fine belated, i.e., $[\xi_i, v_i] \subset [\xi_i, \xi_i + \delta(\xi_i))$*

The term partial division is used in the above definition to emphasize that the finite collection of non-overlapping subintervals of $[0,T]$ may not cover the entire interval $[0,T]$. Using the Vitali covering theorem, the following concept can be defined.

Definition 3.2. *Given $\eta > 0$, a given δ-fine belated partial division $D = \{([\xi, v], \xi)\}$ is said to be a (δ, η)-fine belated partial division of $[0,T]$ if it fails to cover $[0,T]$ by at most length η, that is,*

$$\left| T - (D)\sum (v - \xi) \right| \leq \eta.$$

Definition 3.3. *Let $f : [0,T] \times \Omega \to L(U,V)$ be an adapted process. We say that f is Stratonovich-Henstock integrable, or $\mathcal{SH}$-integrable, on $[0,T]$*

with respect to W provided there exists a random variable $A : \Omega \to V$ such that for every $\gamma > 0$ and every $\epsilon > 0$, there is a gauge δ on $[0,T)$ and a constant $\eta > 0$ such that

$$\mathbf{P}\left\{\omega \in \Omega : \left\| \sum_{i=1}^{n} f_{\xi_i^*}\Big(W_{v_i} - W_{\xi_i}\Big) - A \right\|_V \geq \gamma \right\} < \epsilon$$

whenever $D = \left\{\Big([\xi_i, v_i], \xi_i\Big)\right\}_{i=1}^{n}$ is (δ, η)-fine belated partial division of $[0,T]$, where $\xi_i^* = \frac{\xi_i + v_i}{2}$.

In this case, f is $\mathcal{SH}$-integrable to A on $[0,T]$ and A is called the $\mathcal{SH}$-integral of f. For simplicity, we can denote

$$S(D, f, \delta, \eta) = \sum_{i=1}^{n} f_{\xi_i^*}\Big(W_{v_i} - W_{\xi_i}\Big) \text{ and } A := (\mathcal{SH}) \int_0^T f_t \circ dW_t.$$

When no confusion arises, let $\xi_i^* = \frac{\xi_i + v_i}{2}$, for a given subinterval $[\xi, v] \subset [0,T]$.

Example 3.4. *Let $f : [0,T] \times \Omega \to L(U, \mathbf{R})$ defined by $f_t(\omega) = \langle W_t, \cdot \rangle_U$, for all $t \in [0,T]$ and $\omega \in \Omega$. Then f is $\mathcal{SH}$-integrable on $[0,T]$ and*

$$(\mathcal{SH}) \int_0^T f_t \circ dW_t = \frac{1}{2} \|W_T\|_U^2.$$

Proof. We need to consider the following claims:

Claim 1. Let $D = \{[\xi, v]\}$ be a finite collection of non-overlapping subintervals of $[0,T]$ and $\gamma > 0$. Then $\mathbf{P}\{\omega \in \Omega : |I| \geq \lambda\} \leq \dfrac{M}{2\gamma^2}(D) \sum (v - \xi)^2$, where $I = (D) \sum \left\{ \langle W_\xi, W_{v_i} - W_{\xi_i} \rangle_U - \dfrac{1}{2}\Big(\|W_v\|_U^2 - \|W_\xi\|_U^2 \Big) + \dfrac{1}{2}(v - \xi)\text{tr}Q \right\}$ and $M = \sum_{j=1}^{\infty} \lambda_j^2 < \infty$.

By Chebyshevs inequality and [21, Example 1, Claim 1], we have

$$\begin{aligned}
&\mathbf{P}\{\omega \in \Omega : |I| \geq \gamma\} \\
= \;& \mathbf{P}\left\{\omega \in \Omega : \left|(D) \sum \left\{ \|W_v - W_\xi\|_U^2 - (v - \xi)\text{tr}Q \right\}\right| \geq 2\gamma \right\} \\
\leq \;& \frac{1}{4\gamma^2} \mathbf{E}\left[\left|(D) \sum \left\{ \|W_v - W_\xi\|_U^2 - (v - \xi)\text{tr}Q \right\}\right|^2 \right] \\
= \;& \frac{M}{2\gamma^2} (D) \sum (v - \xi)^2.
\end{aligned}$$

This proves the first claim.

Claim 2. f is $\mathcal{SH}$-integrable to $\dfrac{1}{2}\left\|W_T\right\|_U^2$ on $[0,T]$.

Let $\gamma > 0$ and $\epsilon > 0$. Choose a constant function δ on $[0,T)$ defined by $\delta(\xi) = \dfrac{\epsilon\gamma^2}{2^7 MT}$ and a number $\eta = \dfrac{\sqrt{\epsilon}\gamma}{2^3 \mathrm{tr}\, Q}$. Let $D = \{([\xi, v], \xi)\}$ be a (δ,η)-fine belated partial division of $[0,T]$. Let D^c be the collection of all subintervals of $[0,T]$ which are not included in D. Then

$$\begin{aligned}
&\mathbf{P}\left\{\omega\in\Omega : \left|(D)\sum f_{\xi_i^*}(W_v - W_\xi) - \frac{1}{2}\left\|W_T\right\|_U^2\right| \geq \gamma\right\}\\
= \;&\mathbf{P}\left\{\omega\in\Omega : \left|(D)\sum\{I_1 + I_2 + I_3\} - (D^c)\sum\frac{1}{2}\left\{\left\|W_v\right\|_U^2 - \left\|W_\xi\right\|_U^2\right\}\right| \geq \gamma\right\},
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \langle W_{\xi^*}, W_v - W_{\xi^*}\rangle_U - \tfrac{1}{2}(\|W_v\|_U^2 - \|W_{\xi^*}\|_U^2) + \tfrac{1}{2}(v - \xi^*)\mathrm{tr}\, Q\\
I_2 &= \langle W_\xi, W_{\xi^*} - W_\xi\rangle_U - \tfrac{1}{2}(\|W_{\xi^*}\|_U^2 - \|W_\xi\|_U^2) + \tfrac{1}{2}(\xi^* - \xi)\mathrm{tr}\, Q\\
I_3 &= \|W_{\xi^*} - W_\xi\|_U^2 - (\xi^* - \xi)\mathrm{tr}\, Q.
\end{aligned}$$

It can be shown that f is Itô-Henstock integrable (see [21, Example 1]) to $F := \dfrac{1}{2}\left\|W_T\right\|_U^2 - \dfrac{1}{2}T\,\mathrm{tr}\, Q$ on $[0,T]$ and it is known that F is $AC^2[0,T]$ (see [10, Definition 3.9, Theorem 3.10]. It follows that

$$\begin{aligned}
&\mathbf{E}\left[\left|(D^c)\sum\left(\|W_v\|_U^2 - \|W_\xi\|_U^2\right)\right|^2\right]\\
\leq\;& 2\mathbf{E}\left[\left|(D^c)\sum\left\{\|W_v\|_U^2 - \|W_\xi\|_U^2 - (v-\xi)\mathrm{tr}\, Q\right\}\right|^2\right]\\
&+2(\mathrm{tr}\, Q)^2\mathbf{E}\left[\left|(D^c)\sum(v-\xi)\right|^2\right]\\
<\;& \frac{\epsilon\gamma^2}{2^5} + 2(\mathrm{tr}\, Q)^2\frac{\epsilon\gamma^2}{2^6(\mathrm{tr}\, Q)^2} = \frac{\epsilon\gamma^2}{2^5}.
\end{aligned}$$

By Claim 1, we have

$$\mathbf{P}\left\{\omega\in\Omega : \left|(D)\sum I_1\right| \geq \lambda\right\} \leq \frac{M}{2\gamma^2}(D)\sum(v - \xi^*)^2$$

and

$$\mathbf{P}\left\{\omega\in\Omega : \left|(D)\sum I_2\right| \geq \lambda\right\} \leq \frac{M}{2\gamma^2}(D)\sum(\xi^* - \xi)^2.$$

Using [21, Example 1, Claim 1], we have

$$\mathbf{P}\left\{\omega \in \Omega : \left|(D)\sum I_3\right| \geq \lambda\right\} \leq \frac{2M}{\gamma^2}(D)\sum(\xi^* - \xi)^2$$

By Chebyshevs inequality, we have

$$\begin{aligned}
& \mathbf{P}\left\{\omega \in \Omega : \left|(D)\sum f_{\xi_i^*}(W_v - W_\xi) - \frac{1}{2}\|W_T\|_U^2\right| \geq \gamma\right\} \\
\leq\ & \mathbf{P}\left\{\omega \in \Omega : \left|(D)\sum I_1\right| \geq \frac{\gamma}{2^3}\right\} + \mathbf{P}\left\{\omega \in \Omega : \left|(D)\sum I_2\right| \geq \frac{\gamma}{2^3}\right\} \\
& +\mathbf{P}\left\{\omega \in \Omega : \left|(D)\sum I_3\right| \geq \frac{\gamma}{2^3}\right\} \\
& \quad +\mathbf{P}\left\{\omega \in \Omega : \left|(D^c)\sum \frac{1}{2}(\|W_v\|_U^2 - \|W_\xi\|_U^2)\right| \geq \frac{\gamma}{2^3}\right\} \\
\leq\ & \frac{2^5 M}{\gamma^2}(D)\sum(v - \xi^*)^2 + \frac{2^5 M}{\gamma^2}(D)\sum(\xi^* - \xi)^2 \\
& +\frac{2^7 M}{\gamma^2}(D)\sum(\xi^* - \xi)^2 + \frac{\epsilon}{2} \\
<\ & \frac{\epsilon}{4} + \frac{\epsilon}{4} + \epsilon + \frac{\epsilon}{2} = 2\epsilon.
\end{aligned}$$

Consequently, $(\mathcal{SH})\int_0^T f_t \circ dW_t = \frac{1}{2}\|W_T\|_U^2$. □

The following results show that the $\mathcal{SH}$-integral possesses the standard properties of an integral. The proofs are omitted since they are standard in Henstock-Kurzweil integration.

(1) The $\mathcal{SH}$-integral of an adapted process $f : [0,T] \times \Omega \to L(U,V)$ is unique almost surely.

(2) Let $\alpha \in R$. If f and g are $\mathcal{SH}$-integrable on $[0,T]$, then

   (i) $f + g$ is $\mathcal{SH}$-integrable on $[0,T]$, and

$$(\mathcal{SH})\int_0^T (f+g) \circ dW_t = (\mathcal{SH})\int_0^T f_t \circ dW_t + (\mathcal{SH})\int_0^T f_t \circ dW_t;$$

   (ii) αf is $\mathcal{SH}$-integrable on $[0,T]$, and

$$(\mathcal{SH})\int_0^T (\alpha f_t) \circ dW_t = \alpha \cdot (\mathcal{SH})\int_0^T f_t \circ dW_t.$$

(3) If $f : [0,T] \times \Omega \to L(U,V)$ is $\mathcal{SH}$-integrable on $[0,a]$ and $[a,T]$ where $a \in (0,T)$, then f is $\mathcal{SH}$-integrable on $[0,T]$ and

$$(\mathcal{SH}) \int_0^T f_t \circ dW_t = (\mathcal{SH}) \int_0^a f_t \circ dW_t + (\mathcal{SH}) \int_a^T f_t \circ dW_t.$$

(4) If $f : [0,T] \times \Omega \to L(U,V)$ is $\mathcal{SH}$-integrable on $[0,T]$, then f is also $\mathcal{SH}$-integrable on any subinterval $[c,d]$ of $[0,T]$

(5) (*Sequential definition*). A process $f : [0,T] \times \Omega \to L(U,V)$ is $\mathcal{SH}$-integrable on $[0,T]$ if and only if there exist a random variable $A : \Omega \to V$, a decreasing sequence of gauges $\{\delta_n\}$ on $[0,T)$, and a decreasing sequence of positive numbers $\{\eta_n\}$ such that for any (δ_n, η_n)-fine belated partial division D_n of $[0,T]$, we have

$$\lim_{n\to\infty} S(f, D_n, \delta_n, \eta_n) = A \text{ in probability.}$$

In this case, $A := (\mathcal{SH}) \int_0^T f_t \circ dW_t$.

(6) (*Cauchy criterion*). A process $f : [0,T] \times \Omega \to L(U,V)$ is $\mathcal{SH}$-integrable on $[0,T]$ if and only if for every $\gamma > 0$ and $\epsilon > 0$, there exist a gauge δ on $[0,T)$ and a constant $\eta > 0$ such that for any two (δ,η)-fine belated partial divisions D_1 and D_2 of $[0,T]$, we have

$$\mathbf{P}\left\{\omega \in \Omega : \|S(D_1, f, \delta, \eta) - S(D_2, f, \delta, \eta)\|_V \geq \gamma\right\} < \epsilon.$$

(7) (*Saks-Henstock lemma*). Let f be $\mathcal{SH}$-integrable on $[0,T]$ and define $F[u,v] := (\mathcal{SH}) \int_u^v f_t \circ dW_t$ for any $[u,v] \subseteq [0,T]$. Then for any $\gamma > 0$ and $\epsilon > 0$, there exist a gauge δ on $[0,T)$ and a constant $\eta > 0$ such that

$$\mathbf{P}\left\{\omega \in \Omega : \left\|(D)\sum\left\{f_{\xi_i^*}(W_v - W_\xi) - F[\xi, v]\right\}\right\|_V \geq \gamma\right\} < \epsilon$$

whenever $D = \{([\xi, v], \xi)\}_{i=1}^n$ is a (δ,η)-fine belated partial division of $[0,T]$.

Next, we show that the $\mathcal{SH}$-integral is $\mathbf{P}$-a.s. continuous and adapted. We shall consider first the following concept.

Definition 3.5. *A process $F : [0,T] \times \Omega \to V$ is said to be $AC^p[0,T]$ if for every $\epsilon > 0$ and $\gamma > 0$, there exists a constant $\eta > 0$ such that for any finite collection $D = \{[\xi, v]\}$ of non-overlapping subintervals of $[0,T]$ with $(D)\sum(v-\xi) < \eta$, we have $\mathbf{P}\left\{\omega \in \Omega : \left\|(D)\sum(F_v - F_\xi)\right\|_V \geq \gamma\right\} < \epsilon$.*

The following lemma can be proved using the same argument used in [21, Lemma 3].

Lemma 3.6. *Let $f : [0,T] \times \Omega \to L(U,V)$ be an $\mathcal{SH}$-integrable process. Then for every $\gamma > 0$ and $\epsilon > 0$, there exists a gauge δ on $[0,T)$ and a constant $\eta > 0$ such that*

$$\mathbf{P}\left\{\omega \in \Omega : \left\|(D)\sum f_{\xi^*}\Big(W_v - W_\xi\Big)\right\|_V \geq \gamma\right\} < \epsilon$$

for any δ-fine belated partial division $D = \{([\xi, v], \xi)\}$ of $[0,T]$ with

$$(D)\sum(v-\xi) \leq \eta.$$

Theorem 3.7. *Let $f : [0,T] \times \Omega \to L(U,V)$ be an $\mathcal{SH}$-integrable on $[0,T]$ and define $F_t := (\mathcal{SH})\int_0^t f_s \circ dW_s$. Then F is $AC^p[0,T]$.*

Proof. Let $\epsilon > 0$ and $\gamma > 0$. By Lemma 3.6, there exist a gauge δ on $[0,T)$ and a positive number η such that

$$\mathbf{P}\left\{\omega \in \Omega : \left\|(D)\sum f_{\xi^*}(W_v - W_\xi))\right\|_V \geq \frac{\gamma}{2}\right\} < \frac{\epsilon}{2}.$$

for any δ-fine belated partial division $D = \{([\xi, v], \xi)\}$ of $[0,T]$ with

$$(D)\sum(v-\xi) \leq \eta.$$

For every $[u,v] \subset [0,T]$, let $F[u,v] := F_v - F_u$. Let $\{[a_j, b_j]\}_{j=1}^m$ be a finite collection of non-overlapping subintervals of $[0,T]$ with $\sum_{j=1}^m (b_j - a_j) \leq \eta$. Note that f is also $\mathcal{SH}$-integrable on each $[a_j, b_j]$. Hence, for each $j = 1, 2, \ldots, m$, there exists a gauge $\delta_j > 0$ and a constant η_j such that if D_j is a (δ_j, η_j)-fine belated partial division of $[a_j, b_j]$, we have

$$\mathbf{P}\left\{\omega \in \Omega : \left\|S(f, D_j\delta_j, \eta_j) - F[a_j, b_j]\right\}\right\|_V \geq \frac{\gamma}{2^{m+1}}\right\} < \frac{\epsilon}{2^{j+1}}.$$

We can choose $\{\delta_j\}_{j=1}^m$ and $\{\eta_j\}_{j=1}^m$ such that $\delta_j(\xi) \le \delta(\xi)$ for all j and $\sum_{j=1}^m \eta_j \le \eta$. Let $P = D_1 \cup D_2 \cup \cdots \cup D_m$, which is a δ-fine belated partial division of $[0,T]$ with

$$(P)\sum(v-\xi) \le \sum_{j=1}^m (b_j - a_j) \le \eta.$$

This implies

$$\mathbf{P}\Big\{\omega \in \Omega : \Big\|(P)\sum f_{\xi^*}(W_v - W_\xi))\Big\|_V \ge \frac{\gamma}{2}\Big\} < \frac{\epsilon}{2}.$$

Hence,

$$\begin{aligned}
\mathbf{P} \quad & \Big\{\omega \in \Omega : \Big\|\textstyle\sum_{j=1}^m F[a_j,b_j]\Big\|_V \ge \gamma\Big\} \\
& \le \mathbf{P}\Big\{\omega \in \Omega : \Big\|\textstyle\sum_{j=1}^m \Big\{S(f, D_j\delta_j, \eta_j)\Big\}\Big\|_V \ge \frac{\gamma}{2}\Big\} \\
& + \mathbf{P}\Big\{\omega \in \Omega : \Big\|\textstyle\sum_{j=1}^m \Big\{F[a_j,b_j] - S(f, D_j\delta_j, \eta_j)\Big\}\Big\|_V \ge \frac{\gamma}{2}\Big\} \\
& < \tfrac{\epsilon}{2} + \textstyle\sum_{j=1}^m \mathbf{P}\Big\{\omega \in \Omega : \Big\|F[a_j,b_j] - S(f, D_j, \delta_j, \eta_j)\Big\|_V \ge \frac{\gamma}{2^{m+1}}\Big\} \\
& \le \tfrac{\epsilon}{2} + \textstyle\sum_{j=1}^m \frac{\epsilon}{2^{j+1}} \le \frac{\epsilon}{2} + \sum_{j=1}^\infty \frac{\epsilon}{2^{j+1}} \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus, F is $AC^P[0,T]$. □

Theorem 3.8. *Let $f : [0,T] \times \Omega \to L(U,V)$ be an $\mathcal{SH}$-integrable on $[0,T]$ and define $F_t := (\mathcal{SH})\int_0^t f_s \circ dW_s$. Then F is P-a.s. continuous and adapted.*

Proof. In view of Theorem 3.7, F is P-a.s. continuos on $[0,T]$. Let $t \in [0,T]$. We note that f is also $\mathcal{SH}$-integrable on $[0,t]$. In view of the sequential definition, there exist a decreasing sequence of gauges $\{\delta_n\}$ on $[0,T)$, and a decreasing sequence of positive numbers $\{\eta_n\}$ such that for any (δ_n, η_n)-fine belated partial division D_n of $[0,T]$, we have

$$\lim_{n\to\infty} S(f, D_n, \delta_n, \eta_n) = A \text{ in probability},$$

in which the limit holds almost surely for some subsequence. Since f and W are adapted, for $0 \le \xi \le v \le t$, $S(f, D_n, \delta_n, \eta_n) = (D_n)\sum f_{\xi^*}\Big(W_v - W_\xi\Big)$ is F_t-measurable. Accordingly, F_t is F_t-measurable. □

4. Ideal Itô's Formula

In this section, we present a version of Itô's formula for the Stratonovich-Henstock integral, which turns out to be "ideal" [28] in the sense that the "tail" term has been removed. As a consequence, we can give the relationship between the Itô-Henstock and Stratonovich-Henstock integrals for the operator-valued stochastic process.

For the discussions on Fréchet derivatives, regulated mappings, primitives, billinear mappings, Riemann integrability on Banach spaces, and Taylor's formula, one may refer to [1], [4], [11], and [22].

We may use an analogous argument in [22, Lemma 4.7] to prove the following lemma.

Lemma 4.1. *Let $f : [0,T] \times \Omega \to L(U,V)$ be an $\mathcal{SH}$-integrable process. Given $\gamma > 0$ and $\epsilon > 0$, there exist a gauge δ on $[0,T)$ and a constant $\eta > 0$ such that if $D = \{([\xi, v], \xi)$ is a (δ, η)-fine belated partial division of $[0,T]$, then*

$$\mathbf{P}\left\{\omega \in \Omega : \left\| S(D \cup D^c, f, \delta, \eta) - (\mathcal{SH}) \int_0^T f_t \circ dW_t \right\|_V \geq \gamma \right\} < \epsilon,$$

where $\{[\xi, v] : ([\xi, v], \xi) \in D^c\}$ is the collection of all subintervals of $[0,T]$ which are not included in the set $\{[\xi, v] : ([\xi, v], \xi) \in D\}$ and

$$S(D \cup D^c, f, \delta, \eta) := (D \cup D^c) \sum f_{\xi^*} \Big(W_v - W_\xi \Big).$$

In view of Lemma 4.1, we have the following result.

Lemma 4.2. *Let $f : [0,T] \times \Omega \to L(U,V)$ be an $\mathcal{SH}$-integrable process. Then there exist a sequence of gauges $\{\delta_n\}$ on $[0,T]$ and a sequence of positive constants $\{\eta_n\}$ such that*

$$\lim_{n \to \infty} S(D_n \cup D_n^c, f, \delta_n, \eta_n) = (\mathcal{SH}) \int_0^T f_t \circ dW_t \quad \text{in probability}$$

where D_n is any (δ_n, η_n)-fine belated partial division of $[0,T]$.

The following result is an enhancement of [11, Lemma 4.18] since the function f is assumed to be continuous.

Lemma 4.3. *Let $f : U \to L(U \times U, V)$ be a continuous function. Then for every $\epsilon > 0$, there exists a $\delta > 0$ such that for any partition $P = \{[\xi, v]\}$ of $[0, T]$ with $d(P) < \delta$,*

$$\|\overline{S}(f, P, \delta)\|_{L^2(\Omega, V)} < \epsilon,$$

where

$$\overline{S}(f, P, \delta) := (P) \sum \left\{ \int_0^1 (1-t)\Big(f(\overline{W}) - f(W_\xi)\Big)(W_\xi - W_{\xi^*})^{(2)} dt \right\},$$

$\overline{W} := W_{\xi^*} + t(W_\xi - W_{\xi^*})$, and $t \in [0, 1]$.

Proof. Let $\epsilon > 0$ be given. Following the same argument as in [11, Lemma 4.18], we have $\mathbf{E}\left[\|W_{\xi^*} - W_\xi\|_U^4\right] = (\xi^* - \xi)^2 \left(2\sum_{j=1}^{\infty} \lambda_j^2 + (\operatorname{tr} Q)^2\right)$. Since $\operatorname{tr} Q < \infty$, let $M \in R^+$ such that $2\sum_{j=1}^{\infty} \lambda_j^2 + (\operatorname{tr} Q)^2 \leq M$. Since f is continuous, there exists $\delta_1 > 0$, $\delta_1 < \epsilon$ such that if $\|u - w\|_U < \delta_1$, then $\|f(u) - f(w)\|_{L(U,V)} < \dfrac{\epsilon}{T\sqrt{M}}$. Also, since W is continuous on a compact interval $[0, T]$, there exists $\delta > 0$ such that if $|v - \xi| < \delta$, then $\|W_v - W_\xi\|_U < \delta_1$. Let $P = \{[\xi, v]\}$ of $[0, T]$ be a partition of $[0, T]$ with $d(P) < \delta$. Since $|\xi^* - \xi| < \delta$, $\left\|\overline{W} - W_\xi\right\|_U = (1 - t)\|W_{\xi^*} - W_\xi\|_U < \delta_1$ for $t \in [0, 1]$. It follows that $\left\|f(\overline{W}) - f(W_\xi)\right\|_{L(U,V)} < \dfrac{\epsilon}{T\sqrt{M}}$. Hence,

$$\begin{aligned}
&\left\|\overline{S}(f, P, \delta)\right\|_{L^2(\Omega, V)} \\
\leq\ & (P)\sum \int_0^1 (1-t)\sqrt{\mathbf{E}\left[\left\|f(\overline{W}) - f(W_\xi)\right\|^2_{L(U\times U,V)} \left\|(W_\xi^* - W_\xi)\right\|_U^4\right]}\, dt \\
\leq\ & \frac{1}{2}(D)\sum \sqrt{\left(\frac{\epsilon}{T\sqrt{M}}\right)^2 (\xi^* - \xi)^2 M} \\
\leq\ & \frac{1}{2}\cdot\frac{\epsilon}{T}(D)\sum(\xi^* - \xi) \leq \frac{1}{2}\cdot\frac{\epsilon}{T}\cdot T < \epsilon.
\end{aligned}$$

□

Lemma 4.4. *Let $f : U \to L(U \times U, V)$ be a continuous function. Then there exists a sequence of positive numbers $\{\delta_n\}$ such that for any partition $P_n = \{[\xi, v]\}$ of $[0, T]$ with $d(P_n) < \delta$,*

$$\lim_{n\to\infty} \overline{S}(f, P_n, \delta_n) = 0 \text{ in probability.}$$

where

$$\overline{S}(f,P_n,\delta_n) := (P_n)\sum\Big\{\int_0^1 (1-t)\Big(f(\overline{W})-f(W_\xi)\Big)(W_\xi - W_{\xi^*})^{(2)}dt\Big\}.$$

Similarly, the following result holds.

Lemma 4.5. *Let $f: U \to L(U\times U, V)$ be a continuous function. Then there exists a sequence of positive numbers $\{\delta_n\}$ such that for any partition $P_n = \{[\xi, v]\}$ of $[0,T]$ with $d(P_n) < \delta$,*

$$\lim_{n\to\infty} \widetilde{S}(f,P_n,\delta_n) = 0 \textit{ in probability.}$$

where

$$\widetilde{S}(f,P,\delta) := (P_n)\sum\Big\{\int_0^1 (1-t)\Big(f(\widetilde{W})-f(W_{\xi^*})\Big)(W_v - W_{\xi^*})^{(2)}dt\Big\},$$

$\widetilde{W} := W_{\xi^*} + t(W_v - W_{\xi^*})$ *and* $t\in[0,1]$.

We shall now state the ideal Itô's formula for the Stratonovich-Henstock integral and give its proof using the above-mentioned lemmas.

Theorem 4.6. *(Itô's formula). Let $f: U\to V$ be a function such that the first and second Fréchet derivatives of f are continuous on U. Suppose that*

(*i*) $f'(W_t)$ *is $\mathcal{SH}$-integrable on $[0,T]$;*

(*ii*) $E\Big[\|f''(W_t)\|^2_{L_2(U_Q\times U_Q, V)}\Big]$ *is bounded on $[0,T]$; and*

(*iii*) $M_t := \sum_{j=1}^{\infty} f''(W_t)(e_j,e_j)$ *is Riemann-integrable on $[0,T]$.*

Then

$$f(W_T) - f(W_0) = (\mathcal{SH})\int_0^T f'(W_t)\circ dW_t$$

for almost all $\omega\in\Omega$.

Proof. By Taylor's formula, for any $v > \xi^*$, we have

$$\begin{aligned} f(W_v) - f(W_{\xi^*}) &= f'(W_{\xi^*})(W_v - W_{\xi^*}) + \frac{1}{2} f''(W_{\xi^*})(W_v - W_{\xi^*})^{(2)} \\ &\quad + \int_0^1 (1-t)(f''(\widetilde{W}) - f''(W_{\xi^*}))(W_v - W_{\xi^*})^{(2)} dt \end{aligned}$$

where $\widetilde{W} := W_{\xi^*} + t(W_v - W_{\xi^*})$. Similarly, for any $\xi^* > \xi$, we have

$$\begin{aligned} f(W_\xi) - f(W_{\xi^*}) &= f'(W_{\xi^*})(W_\xi - W_{\xi^*}) + \frac{1}{2} f''(W_\xi)(W_\xi - W_{\xi^*})^{(2)} \\ &\quad + \int_0^1 (1-t)(f''(\overline{W}) - f''(W_\xi))(W_\xi - W_{\xi^*})^{(2)} dt \end{aligned}$$

where $\overline{W} := W_{\xi^*} + t(W_\xi - W_{\xi^*})$. It follows that

$$($$

$$W_v) - f(W_\xi) = f'(W_{\xi^*})(W_v - W_\xi)$$

$$+\tfrac{1}{2} f''(W_{\xi^*})(W_v - W_{\xi^*})^{(2)} - \tfrac{1}{2} f''(W_\xi)(W_\xi - W_{\xi^*})^{(2)}$$

$$+ \textstyle\int_0^1 (1-t)(f''(\widetilde{W}) - f''(W_{\xi^*}))(W_v - W_{\xi^*})^{(2)} dt$$

$$- \textstyle\int_0^1 (1-t)(f''(\overline{W}) - f''(W_\xi))(W_\xi - W_{\xi^*})^{(2)} dt$$

By Lemma 4.2, there exist a sequence of gauges $\{\delta'_n\}$ on $[0,T]$ and a sequence of positive numbers $\{\eta'_n\}$ such that for any (δ'_n, η'_n)-fine belated partial division D'_n of $[0,T]$,

$$(D'_n \cup {D'_n}^c) \sum f'(W_{\xi^*})(W_v - W_\xi) \longrightarrow (\mathcal{SH}) \int_0^T f'(W_t) \circ dW_t$$

in probability. By Lemma 4.5, there exists a sequence of positive numbers $\{\widetilde{\delta_n}\}$ such that for any partition $\widetilde{P_n} = \{[\xi, v]\}$ of $[0,T]$ with $d(\widetilde{P_n}) < \widetilde{\delta_n}$,

$$(\widetilde{P_n}) \sum \left\{ \int_0^1 (1-t)\Big(f''(\widetilde{W}) - f''(W_{\xi^*})\Big)(W_v - W_{\xi^*})^{(2)} dt \right\} \longrightarrow 0$$

in probability. Similarly, by Lemma 4.4, there exists a sequence of positive numbers $\{\overline{\delta_n}\}$ such that for any partition $\overline{P_n} = \{[\xi, v]\}$ of $[0,T]$ with $d(\overline{P_n}) < \overline{\delta_n}$,

$$(\overline{P_n}) \sum \left\{ \int_0^1 (1-t)\Big(f''(\overline{W}) - f''(W_\xi)\Big)(W_\xi - W_{\xi^*})^{(2)} dt \right\} \longrightarrow 0$$

in probability. In view of [11, Lemma 4.20], there exists a sequence of positive numbers $\{\widehat{\delta_n}\}$ such that for any partition $\widehat{P_n} = \{[\xi, v]\}$ of $[0, T]$ with $d(\widehat{P_n}) < \widehat{\delta_n}$,

$$\lim_{n\to\infty}(\widehat{P_n})\sum\Big\{f''(W_{\xi^*})(W_v - W_{\xi^*})^{(2)} - f''(W_{\xi^*})(W_\xi - W_{\xi^*})^{(2)}\Big\} = 0$$

in probability. Choose $\delta_n(\xi) < \min\{\delta_n'(\xi), \widetilde{\delta_n}, \overline{\delta_n}, \widehat{\delta_n}\}$ for all $\xi \in [0, T)$ and $\eta_n < \min\{\eta_n', \widetilde{\delta_n}, \overline{\delta_n}, \widehat{\delta_n}\}$. Let $D_n = \{([\xi, v], \xi)\}$ be a (δ_n, η_n)-fine belated partial division of $[0, T]$, which is also a (δ_n', η_n')-fine belated partial division of $[0, T]$. Also, $(D_n \cup D_n^c)$ is a partition of $[0, T]$ with $d(D_n \cup D_n^c) < \widetilde{\delta_n}, \overline{\delta_n}, \widehat{\delta_n}$. Hence,

$$(D_n \cup D_n^c)\sum\{f(W_v) - f(W_\xi)\} \longrightarrow f(W_T) - f(W_0),$$

$$(D_n \cup D_n^c)\sum f_{\xi^*}'(W_v - W_\xi) \longrightarrow (\mathcal{SH})\int_0^T f'(W_t)\circ dW_t,$$

$$(D_n \cup D_n^c)\sum\left\{\int_0^1 (1-t)\Big(f''(\widetilde{W}) - f''(W_{\xi^*})\Big)(W_v - W_{\xi^*})^{(2)} dt\right\} \longrightarrow 0,$$

$$(D_n \cup D_n^c)\sum\left\{\int_0^1 (1-t)\Big(f''(\overline{W}) - f''(W_\xi)\Big)(W_\xi - W_{\xi^*})^{(2)} dt\right\} \longrightarrow 0,$$

and

$$(D_n \cup D_n^c)\sum\Big\{f''(W_{\xi^*})(W_v - W_{\xi^*})^{(2)} - f''(W_\xi)(W_\xi - W_{\xi^*})^{(2)}\Big\} \longrightarrow 0$$

in probability. The assertion holds for some subsequence. □

Corollary 4.7. *Let $f : U \to V$ be a function such that the first and second Fréchet derivatives of f are continuous on U. Suppose that*

(i) *$f'(W_t)$ is $\mathcal{IH}$-integrable and $\mathcal{SH}$-integrable on $[0, T]$;*

(ii) *$E\left[\|f''(W_t)\|^2_{L_2(U_Q\times U_Q, V)}\right]$ is bounded on $[0, T]$; and*

(ii) *$M_t := \sum_{j=1}^{\infty} f''(W_t)(e_j, e_j)$ is Riemann-integrable on $[0, T]$.*

Then

$$(\mathcal{SH})\int_0^T f'(W_t)\circ dW_t = (\mathcal{IH})\int_0^T f'(W_t)\; dW_t + \frac{1}{2}(R)\int_0^T M_t\; dt$$

for almost all $\omega \in \Omega$.

5. Conclusion and Recommendations

In this paper, we establish a version of Itô's formula for the Stratonovich-Henstock integral of an operator-valued stochastic process with respect to a Hilbert space-valued Q-Wiener process, which turns out to be ideal since the tail term has been removed. A worthwhile direction for further investigation is to give a version of fundamental theorem for the Stratonovich-Henstock integral and establish different versions of convergence theorems.

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