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Maximal Point Spaces of Posets with Relative Lower Topology

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Abstract. In domain theory, by a poset model of a $T_1$ topological space $X$ we usually mean a poset $P$ such that the subspace Max($P$) of the Scott space of $P$ consisting of all maximal points is homeomorphic to $X$. The poset models of $T_1$ spaces have been extensively studied by many authors. In this paper we investigate another type of poset models: lower topology models. The lower topology $\omega(P)$ on a poset $P$ is one of the fundamental intrinsic topologies on the poset, which is generated by the sets of the form $P \setminus \uparrow x, x \in P$. A lower topology poset model (poset LT-model) of a topological space $X$ is a poset $P$ such that the space Max$_{\omega}(P)$ of maximal points of $P$ equipped with the relative lower topology is homeomorphic to $X$. The studies of such new models reveal more links between general $T_1$ spaces and order structures.

The main results proved in this paper include (i) a $T_1$ space is compact if and only if it has a bounded complete algebraic dcpo LT-model; (ii) a $T_1$ space is second-countable if and only if it has an $\omega$-algebraic poset LT-model; (iii) every $T_1$ space has an algebraic dcpo LT-model; (iv) the category of all $T_1$ space is equivalent to a category of bounded complete posets. We will also prove some new results on the lower topology of different types of posets.

1. Introduction

The primary motivation for the study of domains, which was initiated by Dana Scott in the late 1960s, was to search for a denotational semantics of the lambda calculus. Domain theory also provides a platform to study the interlinks between topology and order. One of the most important topologies in domain theory is the Scott topology: a topology on a poset with respect to which every directed subset converges to its supremum. In general, the Scott space of a poset is only $T_0$. However, if we take the set Max$_{\sigma}(P)$ of maximal points of $P$ with the relative Scott topology, a more abundant number of spaces can be obtained. A poset model of a topological space $X$ is a poset $P$ with a homeomorphism $\phi : X \to \text{Max}_{\sigma}(P)$. Spaces with a domain model enjoy many favourable properties and have been studied by many authors. See \cite{1,11,12,17,19} for more details.

Zhao \cite{24} and Erné \cite{3} independently proved that every $T_1$ space has a bounded complete algebraic poset model. Therefore, the $T_1$ spaces are exactly those spaces which have a poset model. Recently, Xi...
and Zhao [25] further proved that every $T_1$ space has a directed complete poset model. The Xi-Zhao dcpo models have been used in several other recent work ([8] [20] [22] [26]).

Note that the poset models based on other topologies, such as Lawson topology and the strong Scott topology have also been studied by other people ([12] [16] [27]).

Besides the Scott topology, there are also other intrinsic topologies defined on a poset $P$, one of them is the lower topology of which $\{P \setminus \uparrow x : x \in P\}$ forms a subbase, denoted by $\omega(P)$. We write $\Omega P = (P, \omega(P))$.

The set $\text{Max}(P)$ of maximal points of $P$ with the relative lower topology will be denoted by $\text{Max}_{\omega}(P)$.

A natural question arising here is: which topological spaces are homeomorphic to $\text{Max}_{\omega}(P)$ for some poset $P$.

We call a poset $P$ a lower topology poset model (poset LT-model, for short) of a space $X$ if $\text{Max}_{\omega}(P)$ is homeomorphic to $X$. This notion is not new, and was originally called the totally space by Kamimura and Tang [10] (see Section 5). They proved that a space $X$ is second-countable compact $T_1$ if and only if it has a bounded complete $\omega$-algebraic dcpo LT-model.

Recently Hui Li and Qingguo Li also studied such model [14] and obtained the following:

1. Every $T_1$ space has a bounded complete algebraic poset LT-model;
2. A $T_1$ space has a dcpo LT-model if and only if it has a local dcpo LT-model, where a local dcpo is a poset that every upper bounded directed subset has a supremum.

As every bounded complete poset is a local dcpo, thus combing the above results (1) and (2), one can immediately deduce that every $T_1$ space has a dcpo LT-model (note that the authors did not state this most important result on LT-models explicitly in [14]). We have also obtained this result independently and presented at the Third Pan-Pacific International Conference on Topology and Applications. But here we shall focus on the new results on the lower topology model listed in the abstract.

In Section 3, we prove that a topological space $X$ is second-countable if and only if $\Omega(C^*(X,\leq))$ is second-countable; $X$ is compact if and only if $\Omega(C^*(X,\leq))$ is compact, where $C^*(X)$ is the set of all nonempty closed sets. The main results proved in Sections 4 include (i) $T_1$ spaces are precisely the spaces that have a poset LT-model; a $T_1$ space $X$ is second-countable if and only if it has an $\omega$-algebraic poset LT-model; (ii) A $T_1$ space $X$ is compact if and only if it has a bounded complete dcpo LT-model. In Section 5, we prove that every $T_1$ space has an algebraic dcpo LT-model, which strengthens the result deduced from [14]. In Section 6, based on the results in the previous sections we explore the existence of functors derived from lower topology models. We prove that the category of all $T_1$ spaces is equivalent to a category of bounded complete posets. This result indicates some advantages of considering lower topology models.

2. Preliminaries

We first recall some basic notions and results to be used later. We refer readers to [2] [5] [7] for more details.

For a set $X$, the family of all finite subsets of $X$ will be denoted by $X^{(<\omega)}$.

Let $P$ be a poset. A nonempty subset $D$ of $P$ is directed if every two elements of $D$ have an upper bound in $D$. A poset $P$ is a directed complete poset, or dcpo for short, if for any directed subset $D \subseteq P$, $\forall D$ exists.

A poset $P$ is bounded complete if for any $A \subseteq P$, $\forall A$ exists whenever $A$ has an upper bound in $P$, or equivalently, $\bigwedge A$ exists whenever $A \neq \emptyset$.

For $x, y \in P$, $x$ is way-below $y$, denoted by $x \ll y$, if for any directed subset $D$ of $P$ for which $\forall D$ exists, $y \ll \forall D$ implies $x \ll d$ for some $d \in D$. Denote $\forall x = \{y \in P : x \ll y\}$ and $\downarrow x = \{y \in P : y \ll x\}$. A poset $P$ is continuous, if for any $x \in P$, the set $\downarrow x$ is directed and $x = \forall \downarrow x$. A continuous dcpo is also called a domain.

An element $x$ in a poset $P$ is compact if $x \ll x$, and we use $K(P)$ to denote the set of all compact elements of $P$. A poset $P$ is algebraic, if for any $x \in P$, the set $K(P) \land x$ is directed and $x = \forall (K(P) \land x)$. An algebraic poset $P$ is called $\omega$-algebraic if $K(P)$ is countable.

A subset $U$ of a poset $P$ is Scott open if (i) $U = \uparrow U$ and (ii) for any directed subset $D$ of $P$ for which $\forall D$ exists, $\forall D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of $P$ form a topology, and we call this topology
the Scott topology on $P$ and denote it by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the Scott space of $P$. In an algebraic dcpo $P$, the family $\{x : x \in K(P)\}$ forms a base for the Scott topology on $P$.

For any $T_0$ space $X$, the specialization order $\leq$ on $X$ is defined by $x \leq y$ if and only if $x \in \text{cl}(\{y\})$, where cl is the closure operator. A subset $S$ of $X$ is saturated if $S = \uparrow S$ with respect to the specialization order.

**Remark 2.1.** For each poset $P$, the specialization order on $\Sigma P$ (resp., $\Omega P$) is exactly the (resp., dual of) partial order on $P$.

**Definition 2.2.** A $T_0$ space $X$ is called well-filtered if for any filtered family $\{Q_i : i \in \Delta\}$ of compact saturated subsets of $X$ and any open set $U \subseteq X$, $\bigcap_{i \in \Delta} Q_i \subseteq U$ implies $Q_{i_0} \subseteq U$ for some $i_0 \in \Delta$.

**Definition 2.3.** A nonempty subset $A$ of a topological space $X$ is irreducible if for any closed sets $F_1, F_2$ of $X$, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. A $T_0$ space $X$ is sober, if for any irreducible closed set $F$ of $X$ there is a point $x \in X$ such that $F = \text{cl}(\{x\})$.

### 3. The Lower Topology

The standard name ‘lower topology’ was originally given by Gierz and Lawson [5] as the open sets are the Scott closed sets of $P$. Two recent results on the lower topology are due to Wen and Xu [21], who proved that for any bounded complete dcpo $P$,

(i) the lower topology on $P$ is sober;

(ii) the Scott closed sets of $P$ are exactly the compact saturated subsets of $P$ with the lower topology.

In this section, we prove some more properties of the lower topology.

**Lemma 3.1.** Let $X$ be a $T_0$ space and $Q \subseteq X$. Then $Q$ is compact saturated if and only if $Q = \uparrow \text{Min}(Q)$ and $\text{Min}(Q)$ is compact, where $\text{Min}(Q)$ is the set of minimal elements of $Q$ with respect to the specialization order.

**Proof.** Note that for any $A \subseteq X$, $A$ is compact iff $\uparrow A$ is compact (see [6] after Definition O-5.7]). Thus the Sufficiency is trivial.

Suppose now that $Q$ is a compact saturated set. To prove $Q = \uparrow \text{Min}(Q)$, it suffices to prove $Q \subseteq \uparrow \text{Min}(Q)$. Let $x \in Q$. Then there is a maximal chain $C \subseteq Q$ (with respect to the specialization order) that contains $x$, here we use the Hausdorff Maximality Theorem. As $Q$ is compact and $\downarrow y \cap Q \neq \emptyset$ for all $y \in C$, we have that $\bigcap_{y \in C} \downarrow y \cap Q \neq \emptyset$. Let $x_0 \in \bigcap_{y \in C} \downarrow y \cap Q$. Since $x_0$ is a lower bound of $C$ and $x_0 \in Q$, $C \cup \{x_0\}$ is also a chain in $Q$. By the maximality of $C$, we deduce that $x_0 \in C$ and $x_0 = \bigwedge C$. It follows that $x_0 \in \text{Min}(Q)$ and $x_0 \leq x$, implying that $x \in \uparrow \text{Min}(Q)$. Thus $Q \subseteq \uparrow \text{Min}(Q)$. It is straightforward to check that $\text{Min}(Q)$ is compact. Thus the necessity follows. □

**Lemma 3.2.** Let $P$ be a bounded complete dcpo. Then both spaces $\Omega P$ and $\text{Max}_{\omega}(P)$ are compact.

**Proof.** We first show that $\Omega P$ is compact. Let $\{\uparrow x_i : i \in \Delta\}$ be a family of subbasic closed subsets of $\Omega P$ such that for any $J \in \Delta^{(\omega_0)}$, $\bigcap_{i \in J} \uparrow x_i \neq \emptyset$. Then $\{\uparrow x_j : j \in J\}$ has an upper bound, and since $P$ is bounded complete, $x_J := \bigvee \{x_i : i \in J\}$ exists. Note that $\{x_J : J \in \Delta^{(\omega_0)}\}$ is a directed subset of the dcpo $P$, so $x := \bigvee \{x_J : J \in \Delta^{(\omega_0)}\}$ exists. It follows that $x = \bigvee \{x_i : i \in \Delta\}$, which implies that $x \in \bigcap_{i \in \Delta} \uparrow x_i$. Thus $\bigcap_{i \in \Delta} \uparrow x_i \neq \emptyset$. Using Alexander Subbase Lemma, we obtain that $\Omega P$ is compact.

By Remark 2.1 and Lemma 3.1, $\text{Max}_{\omega}(P)$ is compact. □

**Theorem 3.3.** For any bounded complete poset $P$, the following statements are equivalent.

(1) $P$ is a dcpo.

(2) $\Omega P$ is compact.
exists a subfamily $\{\uparrow d : d \in D\}$ is a filtered family of closed sets in $\Omega P$. Since $\Omega P$ is compact, we have that $\bigcap_{d \in D} \uparrow d \neq \emptyset$, showing that $D$ has an upper bound, Thus $\bigvee D$ exists because $P$ is bounded complete. Therefore, $P$ is a dcpo. $\square$ 

**Definition 3.4.** A subset $B$ of a poset $P$ is called join-dense (resp., directed-join-dense) if for each $x \in P$, (resp., $\downarrow x \cap B$ is directed and) $x = \bigvee (\downarrow x \cap B)$, and $\bot \in B$ whenever the least element $\bot$ of $P$ exists.

**Remark 3.5.** Let $P$ be a poset and $B$ a join-dense subset of $P$. Then for any $x \in P$, $\downarrow x \cap B \neq \emptyset$. It is trivial if $x \neq \bot$. For the case $x = \bot$, we have $\bot \in \downarrow \bot \cap B$.

**Proposition 3.6.** Let $P$ be a poset. If $B$ is a join-dense subset of $P$, then $B$ is dense in $\Omega P$, that is, $\text{cl}_{\Omega P}(B) = P$.

**Proof.** It suffices to verify that every nonempty basic open set meets $B$. Let $F$ be a finite subset of $P$ such that $P \setminus \uparrow F \neq \emptyset$. We need to show $(P \setminus \uparrow F) \cap B \neq \emptyset$. Take $x \in P \setminus \uparrow F$. By Remark 3.5, we have that $\downarrow x \cap B \neq \emptyset$. Let $b \in \downarrow x \cap B$. Note that $P \setminus \uparrow F$ is a lower set that contains $x$, so $b \in (P \setminus \uparrow F)$. Thus $b \in (P \setminus \uparrow F) \cap B \neq \emptyset$. Therefore, $B$ is dense in $\Omega P$. $\square$

The converse conclusion of Proposition 3.6 is not true in general.

**Example 3.7.** Let $P$ be an infinite set with the discrete order (i.e., $\forall x, y \in P, x \not\leq y$ and $y \not\leq x$). Fix an element $x_0 \in P$, and let $S := P \setminus \{x_0\}$. Then $S$ is a dense subset of $\Omega P$, but $\downarrow x_0 \cap S = \emptyset$, whose supremum does not exist in $P$. Thus $S$ is not join-dense in $P$.

**Proposition 3.8.** Let $P$ be a poset and $B \subseteq P$. Then the following statements are equivalent:

1. $B$ is join-dense.
2. $(\uparrow x : x \in B)$ is a subbase for the closed sets in $\Omega P$.
3. $(\uparrow F : F \in B^{<\omega})$ is a base for the closed sets in $\Omega P$.

**Proof.** (2) $\Leftrightarrow$ (3) is trivial.

(1) $\Rightarrow$ (3) For the sake of convenience, we denote $\mathcal{B} = \{\uparrow F : F \in B^{<\omega}\}$. Suppose $F_0 = \{x_1, x_2, \ldots, x_n\}$ is a finite subset of $P$. Without loss of generality, we assume the least element of $P$ (when it exists) is not in $F_0$. Since $B$ is join-dense, we have that $x_k = \bigvee (\downarrow x_k \cap B)$ for all $1 \leq k \leq n$. It follows that

$$
\uparrow F_0 = \bigcup_{1 \leq k \leq n} \uparrow x_k = \bigcup_{1 \leq k \leq n} \bigvee (\downarrow x_k \cap B) = \bigcup_{1 \leq k \leq n} \bigcap_{x \in \downarrow x_k \cap B} \uparrow x = \bigcap_{x \in \downarrow x_1} \uparrow x \cap \cdots \bigcup_{1 \leq k \leq n} \uparrow x_k = \mathcal{B},
$$

where $\Lambda = \prod_{1 \leq k \leq n} \downarrow x_k \cap B$. Since for any $\varphi \in \Lambda$ and $1 \leq k \leq n, \varphi(k) \in B$, we have that $\bigcup_{1 \leq k \leq n} \uparrow \varphi(k) = \{\uparrow \varphi(k) : 1 \leq k \leq n\} \subseteq \mathcal{B}$. So $\uparrow F_0$ can be expressed as the intersection of some subfamily of $\mathcal{B}$. Since $\{\uparrow F : F \in P^{<\omega}\}$ is a base for the closed sets in $\Omega P$, we conclude that $\mathcal{B}$ is a base.

(3) $\Rightarrow$ (1) Let $x \in P$. As $\uparrow x$ is closed in $\Omega P$ and $\mathcal{B} = \{\uparrow F : F \in B^{<\omega}\}$ is a base for the closed sets in $\Omega P$, there exists a subfamily $\{F_i : i \in \Lambda\} \subseteq \mathcal{B}$ such that $\uparrow x = \bigcap_{i \in \Lambda} \uparrow F_i$. Then for each $i \in \Lambda, x \in \uparrow F_i$, and thus there exists $a_i \in F_i$ such that $x \in \uparrow a_i$. It follows that $\uparrow x \subseteq \bigcap_{i \in \Lambda} \uparrow a_i \subseteq \bigcap_{i \in \Lambda} \uparrow F_i = \uparrow x$, so $\uparrow x = \bigcap_{i \in \Lambda} \uparrow a_i$. Thus $x = \bigvee_{i \in \Lambda} a_i$, implying that $\{a_i : i \in \Lambda\} \subseteq \downarrow x \cap B$. Thus $x = \bigvee_{i \in \Lambda} a_i \leq \bigvee (\downarrow x \cap B) \leq x$, so $x = \bigvee (\downarrow x \cap B)$. Therefore, $B$ is a join-dense subset of $P$. $\square$

**Lemma 3.9.** Let $P$ be a bounded complete poset. If $B$ is a join-dense subset of $P$, then

$$
B_0 := \bigvee \{ F : F \text{ is a finite subset of } B \text{ such that } \bigvee F \text{ exists} \}
$$

is a directed-join-dense subset of $P$. 

Proof. We prove this in two steps.
Step 1. For each \( x \in P, \downarrow x \cap B_0 \) is directed. First, from Remark 3.5 and the fact \( B \subseteq B_0 \), it follows that \( \downarrow x \cap B_0 \neq \emptyset \). Let \( x_1, x_2 \in \downarrow x \cap B_0 \). Then \( x \) is an upper bound of \( x_1, x_2 \), which implies that \( x_1 \lor x_2 \) exists because \( P \) is bounded complete. Since \( x_1, x_2 \in B_0 \), there exist two finite subsets \( F_1, F_2 \subseteq B \) such that \( x_1 = \lor F_1 \) and \( x_2 = \lor F_2 \). Now let \( F_3 = F_1 \cup F_2 \). Then \( F_3 \) is a finite subset of \( B \) and \( x_1 \lor x_2 = \lor F_1 \lor \lor F_2 = \lor (F_1 \cup F_2) = \lor F_3 \), showing that \( x_1 \lor x_2 \in \downarrow x \cap B_0 \). Therefore, \( \downarrow x \cap B_0 \) is directed.
Step 2. \( B_0 \) is join-dense. For each \( x \in P \), since \( B \subseteq B_0 \) and \( B \) is a join-dense subset of \( P \), we have that \( x = \lor \downarrow x \cap B \leq \lor \downarrow x \cap B_0 \leq x \), so \( x = \downarrow x \cap B_0 \). Hence, \( B_0 \) is a join-dense subset of \( P \).
All these show that \( B_0 \) is a directed-join-dense subset of \( P \). \( \square \)

It is important to note that the set \( B_0 \) in the proceeding lemma is countable whenever \( B \) is countable.

Lemma 3.10. ([2] Theorem 1.1.15]) If the minimal cardinality of the bases for a topological space \( X \) is \( \leq m \), then for every base \( B \) for \( X \), there exists a base \( B_0 \subseteq B \) such that \( |B_0| \leq m \).

Recall that a topological space is said to be second-countable if it has a countable base.

Theorem 3.11. Let \( P \) be a poset. The following statements are equivalent:
(1) \( \Omega P \) is second-countable.
(2) \( P \) has a countable join-dense subset.
If \( P \) is bounded complete, these are equivalent to
(3) \( P \) has a countable directed-join-dense subset.

Proof. (1) \( \Rightarrow \) (2) Assume \( \Omega P \) is second-countable. Since \( \{ \uparrow F : F \in P^{<\omega} \} \) is a base for the closed sets in \( \Omega P \) and by Lemma 3.10 there exists a countable subfamily \( B \subseteq \{ \uparrow F : F \in P^{<\omega} \} \) which is a base for the closed sets in \( \Omega P \). We may assume \( B = \{ \uparrow F_n : n < \omega \} \). Let \( B = \bigcup_{n<\omega} F_n \), which is countable. Since \( B \) is a base for the closed sets in \( \Omega P \) and \( B \subseteq \{ \uparrow F : F \in B^{<\omega} \} \), we conclude that \( \{ \uparrow F : F \in B^{<\omega} \} \) is also a base for the closed sets in \( \Omega P \). Thus by Proposition 3.8, \( B \) is a join-dense subset of \( P \).
(2) \( \Rightarrow \) (1) Assume \( B \) is a countable join-dense subset of \( P \). By Proposition 3.8, the family \( B = \{ \uparrow F : F \in B^{<\omega} \} \) is a countable base for \( \Omega P \), completing the proof.

If \( P \) is bounded complete, then (2) \( \Rightarrow \) (3) is a direct consequence of Lemma 3.9 and (3) \( \Rightarrow \) (2) is trivial. \( \square \)

For a topological space \( X \), denote by \( C^*(X) \) the set of all nonempty closed subsets of \( X \). Consider the poset \( (C^*(X), \supseteq) \).

1. The poset \( (C^*(X), \supseteq) \) is bounded complete: for each nonempty subset \( \mathcal{A} \subseteq C^*(X) \),
   \[ \bigwedge \mathcal{A} = \text{cl} \left( \bigcup \mathcal{A} \right). \]

2. If \( \mathcal{B} \) is a base for the closed sets in \( X \), then \( \mathcal{B} \) is a join-dense subset of \( (C^*(X), \supseteq) \). This is because for each \( C \in C^*(X) \),
   \[ C = \bigcap \{ B \in \mathcal{B} : C \subseteq B \} = \bigvee_{C \subseteq X} C \cap \mathcal{B}. \]

By the above arguments and Theorem 3.11, we deduce the following.

Corollary 3.12. For any topological space \( X \), the following statements are equivalent:
(1) \( \Omega \) is second-countable.
(2) \( \Omega C^*(X), \supseteq \) is second-countable.
(3) \((C^*(X), \supseteq)\) has a countable join-dense subset.

(4) \((C^*(X), \supseteq)\) has a countable directed-join-dense subset.

Note that a topological space \(X\) is compact if and only if the intersection of each filtered (under the set inclusion order) subfamily of \(C^*(X)\) is nonempty, which is equivalent to that \((C^*(X), \supseteq)\) is a (bounded complete) dcpo. Thus by Theorem 3.3, we obtain the following result.

**Corollary 3.13.** Let \(X\) be a topological space. The following statements are equivalent:

1. \(X\) is compact.
2. \(\Omega(C^*(X), \supseteq)\) is compact.
3. \((C^*(X), \supseteq)\) is a bounded complete dcpo.

### 4. Bounded Complete Poset LT-Models of \(T_1\) Spaces

In this section, we prove that \(T_1\) spaces are exactly the set of maximal points of posets with the relative lower topology. Furthermore, we show that some topological properties can be characterized via lower topology poset models.

**Definition 4.1.** A lower topology poset model (poset LT-model) of a topological space \(X\) is a poset \(P\) with a homeomorphism \(\phi: X \rightarrow \text{Max}_\omega(P)\), where \(\text{Max}_\omega(P)\) is the set of maximal points of \(P\) with the relative lower topology.

**Remark 4.2.** For any poset \(P\), the space \(\text{Max}_\omega(P)\) is always \(T_1\) because for every \(x \in \text{Max}(P)\), \(\uparrow x \cap \text{Max}(P) = \{x\}\). Thus topological spaces having a poset LT-model must be \(T_1\).

Given a \(T_1\) space \(X\), let \(C^*(X)\) be the set of all nonempty closed subsets of \(X\). The poset \((C^*(X), \supseteq)\) is bounded complete by the argument before Corollary 3.12. The set of maximal points of \((C^*(X), \supseteq)\) are the singleton sets:

\[
\text{Max}(C^*(X), \supseteq) = \{\{x\} : x \in X\}.
\]

The following result shows that the space \(\text{Max}_\omega(C^*(X), \supseteq)\) is homeomorphic to \(X\).

**Lemma 4.3.** Let \(X\) be a \(T_1\) space. The mapping \(\phi: X \rightarrow \text{Max}_\omega(C^*(X), \supseteq)\) defined by

\[
\phi(x) = \{x\}, \ \forall x \in X
\]

is a homeomorphism.

**Proof.** Clearly, \(\phi\) is a bijection. For any closed set \(C \subseteq X\), we have that

\[
\phi(C) = \{\{x\} : x \in C\} = \uparrow_{C^*(X)} C \cap \text{Max}_\omega(C^*(X), \supseteq),
\]

which is closed in \(\text{Max}_\omega(C^*(X), \supseteq)\). So \(\phi\) is a closed mapping. It is also a continuous mapping since for any \(C \in C^*(X)\),

\[
\phi^{-1}(\uparrow_{C^*(X)} C \cap \text{Max}_\omega(C^*(X), \supseteq)) = \{x : x \in C\} = C,
\]

which is closed in \(X\). Thus \(\phi\) is a homeomorphism.

As a consequence, we obtain the following result.

**Theorem 4.4.** Every \(T_1\) space has a bounded complete poset LT-model.
The above theorem and Remark 4.2 show that the $T_1$ spaces are precisely the spaces that have a poset LT-model.

Recall that a subset $I$ of a poset $P$ is an ideal if $I$ is directed and $I = \downarrow I$. In the following, we use $\text{Id}(P)$ to denote the set of all ideals on the poset $P$. For a subset $B \subseteq P$, let $\text{Id}_\lor(B) = \{ I \in \text{Id}(B) : \downarrow I \text{ exists in } P \}$. We should note that the sets $\text{Id}(B)$ and $\text{Id}_\lor(B)$ coincide whenever $P$ is a dcpo.

**Lemma 4.5.** Let $P$ be a poset and $B$ a directed-join-dense subset of $P$. Then $\text{Max}(\text{Id}_\lor(B), \subseteq) = \{ \downarrow a \cap B : a \in \text{Max}(P) \}$.

**Proof.** First, as $B$ is a directed-join-dense subset of $P$, we have that for each $x \in P$, $\downarrow x \cap B$ is directed and $x = \bigvee (\downarrow x \cap B)$. Since $\downarrow x \cap B$ is a lower subset of $B$, we have that $\downarrow x \cap B \subseteq \text{Id}_\lor(B)$.

Let $I \in \text{Max}(\text{Id}_\lor(B), \subseteq)$. Then $\downarrow I$ exists and $I \subseteq (\downarrow \bigvee I) \cap B$. Since $(\downarrow \bigvee I) \cap B \subseteq \text{Id}_\lor(B)$ and $I$ is maximal, we have that $I = (\downarrow \bigvee I) \cap B$. We now show that $\downarrow I \in \text{Max}(P)$. Let $x \in P$ with $\downarrow I \leq x$. Then $I \subseteq \downarrow x \cap B$. Since $\downarrow x \cap B \subseteq \text{Id}_\lor(B)$ and $I$ is maximal, we have that $I = \downarrow x \cap B$. This implies that $\downarrow I = \bigvee (\downarrow x \cap B) = x$. Therefore, $\downarrow I \in \text{Max}(P)$.

Conversely, assume that $a \in \text{Max}(P)$. Let $I \in \text{Id}_\lor(B)$ such that $\downarrow a \cap B \subseteq I$. Then $a = \bigvee (\downarrow a \cap B) \leq \downarrow I$. As $a$ is maximal in $P$, it follows that $a = \downarrow I$. Thus we have that $\downarrow a \cap B \subseteq I \subseteq (\downarrow \bigvee I) \cap B = \downarrow a \cap B$, implying that $\downarrow a \cap B = I$. Hence $\downarrow a \cap B \in \text{Max}(\text{Id}_\lor(B), \subseteq)$. □

**Lemma 4.6.** Let $P$ be a poset and $B$ a directed-join-dense subset of $P$. Then $\text{Max}_ω(P)$ and $\text{Max}_ω(\text{Id}_\lor(B), \subseteq)$ are homeomorphic.

**Proof.** Define $\phi : \text{Max}_ω(P) \longrightarrow \text{Max}_ω(\text{Id}_\lor(B), \subseteq)$ by

$$\phi(a) = \downarrow a \cap B, \forall a \in \text{Max}(P).$$

By Lemma 4.5, $\phi$ is a bijection. Since $B$ is a directed-join-dense subset of $P$, we have that for any $x, y \in P$, $x \leq y$ iff $\downarrow x \cap B \subseteq \downarrow y \cap B$. Thus for any $x \in P$, we have

\[
\phi(\uparrow x \cap \text{Max}(P)) = \{ \downarrow a \cap B : x \leq a \text{ and } a \in \text{Max}(P) \} = \{ \downarrow a \cap B : \downarrow x \cap B \subseteq \downarrow a \cap B \text{ and } a \in \text{Max}(P) \} = \{ I \in \text{Max}(\text{Id}_\lor(B), \subseteq) : \downarrow x \cap B \subseteq I \} = \uparrow_{\text{Id}_\lor(B)}(\downarrow x \cap B) \cap \text{Max}(\text{Id}_\lor(B), \subseteq),
\]

which is closed in $\text{Max}_ω(\text{Id}_\lor(B))$. Thus $\phi$ is a closed mapping. For any $I \in \text{Id}_\lor(B)$, we have

\[
\phi^{-1}(\uparrow_{\text{Id}_\lor(B)}(I \cap \text{Max}(\text{Id}_\lor(B), \subseteq))) = \{ a \in \text{Max}(P) : I \subseteq \downarrow a \cap B \} = \{ a \in \text{Max}(P) : \bigvee I \leq a \} = \uparrow(\bigvee I) \cap \text{Max}(P),
\]

which is closed in $\text{Max}_ω(P)$. Thus $\phi$ is a continuous mapping. Therefore, $\phi$ is a homeomorphism. □

Let $B$ be a directed-join-dense subset of a poset $P$. Then $(\text{Id}_\lor(B), \subseteq)$ is an algebraic poset whose compact elements are $\downarrow b, b \in B$. In particular, $P$ is a directed-join-dense subset of itself. Thus we obtain the following result.

**Corollary 4.7.** Let $P$ be a poset. Then $\text{Max}_ω(P)$ and $\text{Max}_ω(\text{Id}_\lor(P), \subseteq)$ are homeomorphic.

**Lemma 4.8.** Let $P$ be a bounded complete poset and $B$ a join-dense subset of $P$. Define

$$B_0 = \{ \bigvee F : F \text{ is a finite subset of } B \text{ such that } \bigvee F \text{ exists} \}.$$

Then $(\text{Id}_\lor(B_0), \subseteq)$ is a bounded complete algebraic poset.
Proof. Let \( \{I_\alpha : \alpha \in \Delta \} \subseteq \text{Id}_\forall(B_0) \) be a nonempty family. We prove the following.

(c1) \( B_0 \cap \bigcap_{\alpha \in \Delta} I_\alpha \neq \emptyset \).

As \( I_\alpha \neq \emptyset \) and \( P \) is bounded complete, \( \bigwedge I_\alpha \) exists. Similarly, \( x_0 := \bigwedge \{I_\alpha : \alpha \in \Delta \} \) exists. Since \( B_0 \) is a directed-join-dense subset of \( P \), by Remark 3.5, we have \( \emptyset \neq B_0 \cap \downarrow x_0 \subseteq B_0 \cap \bigcap_{\alpha \in \Delta} I_\alpha \), so \( B_0 \cap \bigcap_{\alpha \in \Delta} I_\alpha \neq \emptyset \).

(c2) \( \bigcap_{\alpha \in \Delta} I_\alpha \) is an ideal of \( B_0 \).

Let \( x, y \in \bigcap_{\alpha \in \Delta} I_\alpha \). Fix an arbitrary \( \beta \in \Delta \). Since \( x, y \in I_\beta \subseteq B_0 \) and \( I_\beta \) is an ideal, there exists \( z \in I_\beta \) which is an upper bound of \( \{x, y\} \), thus \( x \lor y \) exists because \( P \) is bounded complete. From the definition of \( B_0 \), it follows that \( x \lor y \in B_0 \). Note that \( x \lor y \leq z \in I_\beta \) and \( I_\beta \) is a lower subset of \( B_0 \), so \( x \lor y \in I_\beta \). By the arbitrariness of \( \beta \), we have that \( x \lor y \in \bigcap_{\alpha \in \Delta} I_\alpha \).

(c3) \( \bigvee \bigcap_{\alpha \in \Delta} I_\alpha \) exists.

For a fixed \( \beta \in \Delta \), since \( I_\beta \in \text{Id}_\forall(B_0) \), we have that \( \bigvee I_\beta \) exists and \( \bigcap_{\alpha \in \Delta} I_\alpha \subseteq I_\beta \subseteq \bigvee I_\beta \), implying that \( \bigvee I_\beta \) is an upper bound of \( \bigcap_{\alpha \in \Delta} I_\alpha \), so \( \bigvee \bigcap_{\alpha \in \Delta} I_\alpha \) exists because \( P \) is bounded complete.

All these show that \( \bigcap_{\alpha \in \Delta} I_\alpha \in \text{Id}_\forall(B_0) \), and thus \( \bigwedge \text{Id}_\forall(B_0) \subseteq \Delta \) exists. Therefore, \( \text{Id}_\forall(B_0) \subseteq X \) is bounded complete. It is algebraic by the argument before Corollary 4.7.

The following corollary is an immediate consequence of the above lemma.

Corollary 4.9. If \( P \) is a bounded complete poset, then \( \text{Id}_\forall(P) \) is a bounded complete algebraic poset.

By Lemma 4.3, Corollary 4.7 and Corollary 4.9 we deduce the following.

Corollary 4.10. Every \( T_1 \) space has a bounded complete algebraic poset LT-model.

Remark 4.11. Compared with proof for the result in [14], the above construction provides a more straightforward method to the bounded complete algebraic poset LT-models.

Theorem 4.12. Let \( X \) be a \( T_1 \) space. The following statements are equivalent:

1. \( X \) is second-countable.
2. \( X \) has a bounded complete poset LT-model that has a countable directed-join-dense subset.
3. \( X \) has a bounded complete \( \omega \)-algebraic poset LT-model.

Proof. (1) \( \Rightarrow \) (2) is immediate by Corollary 3.12 and Lemma 4.3.

(2) \( \Rightarrow \) (3) Let \( P \) be the bounded complete poset LT-model and \( B \) a countable directed-join-dense subset of \( P \). Then the set \( B_0 \) constructed from \( B \) in Lemma 3.9 is a countable directed-join-dense subset of \( P \), and by Lemma 4.8, \( \text{Id}_\forall(B_0) \subseteq \) is a bounded complete \( \omega \)-algebraic poset with the compact elements \( \downarrow b, b \in B_0 \). Then by Lemma 4.6, \( \text{Id}_\forall(B_0) \subseteq \) is a bounded complete \( \omega \)-algebraic poset LT-model of \( X \).

(3) \( \Rightarrow \) (1) Suppose \( P \) is a bounded complete \( \omega \)-algebraic poset LT-model of \( X \). Then the set \( K(P) \) of all compact elements of \( P \) is a countable directed-join-dense subset of \( P \). From Theorem 3.11, \( P \) has a countable base, so is \( \text{Max}_\omega(P) \). As \( X \) is homeomorphic to \( \text{Max}_\omega(P) \), \( X \) is second-countable.

The two families \( \text{Id}(P) \) and \( \text{Id}_\forall(P) \) coincide whenever \( P \) is a dcpo. Then by Corollary 4.7, we deduce the following.

Corollary 4.13. For any dcpo \( P \), \( \text{Max}_\omega(P) \) and \( \text{Max}_\omega(\text{Id}(P) \subseteq \) are homeomorphic.

The above corollary need not be true for a general poset.

Example 4.14. Let \( N \) be the set of natural numbers with the usual order. Define \( P = N \cup \{a\} \) such that \( a \) is incomparable with any element of \( N \). Then \( \text{Max}(P) = \{a\} \), while \( \text{Max}(\text{Id}(P) \subseteq \) = \( \{N, \{a\}\} \). Hence, they are not homeomorphic.

Lemma 4.15. If \( P \) is a (resp., bounded complete) dcpo, then \( (\text{Id}(P) \subseteq \) is an (resp., bounded complete) algebraic dcpo.
Proof. First, note that for any poset $P$, $(\text{Id}(P), \subseteq)$ is an algebraic dcpo with the compact elements $\downarrow x, x \in P$. Now assume $P$ is a bounded complete dcpo. Then $\text{Id}(P) = \text{Id}_0(P)$. Since $P$ is a directed-join-dense subset of itself and $P_0$ constructed from $P$ in Lemma 4.8 is exactly $P$, we have that $(\text{Id}(P), \subseteq)$ is a bounded complete algebraic dcpo.

The following is a corollary of Lemma 4.3, Lemma 4.15 and Corollary 4.13.

**Corollary 4.16.** If a topological space has a (resp., bounded complete) dcpo LT-model, then it also has an (resp., bounded complete) algebraic dcpo LT-model.

Xi and Zhao proved that spaces that have a bounded complete dcpo model must be well-filtered and coherent [22]. Later, they showed that every Hausdorff $k$-space has a bounded complete dcpo model [26]. However, there is still no characterization for the spaces that have a bounded complete dcpo model, while there is a fully description for the case of LT-model shown as follows.

**Theorem 4.17.** Let $X$ be a $T_1$ space. The following statements are equivalent:

1. $X$ is compact.
2. $X$ has a bounded complete dcpo LT-model.
3. $X$ has a bounded complete algebraic dcpo LT-model.

**Proof.** (1) $\Rightarrow$ (2) follows from Corollary 5.13 and Lemma 4.3.
(2) $\Rightarrow$ (3) is an immediate consequence of Corollary 4.16.
(3) $\Rightarrow$ (1) is an immediate consequence of Lemma 3.2.

The following result, originally proved by Wen and Xu (2018), will be used later. Here, we give a simpler proof.

**Lemma 4.18.** [21] Let $P$ be a poset. Then the following statements are equivalent:

1. $\Omega P$ is sober.
2. For each irreducible closed subset $A$ of $\Omega P$, $\bigwedge A$ exists.

**Proof.** (1) $\Rightarrow$ (2) Let $A$ be an irreducible closed set. As $\Omega P$ is sober, there exists $x \in P$ such that $A = \uparrow x$ with respect to the partial order on $P$. Hence, $\bigwedge A = x$.
(2) $\Rightarrow$ (1) Let $A$ be an irreducible closed set. By assumption, $x := \bigwedge A$ exists. To prove $A = \uparrow x$, it suffices to prove $x \in A$. Otherwise, $x \in P \setminus A$. Since $P \setminus A$ is open in $\Omega P$, there exists a finite subset $F$ of $P$ such that $x \in P \setminus \uparrow F \subseteq P \setminus A$. It follows that $A \subseteq \uparrow F = \bigcup_{y \in F} \uparrow y$. Since $A$ is irreducible, there exists $y_0 \in F$ such that $A \subseteq \uparrow y_0$, implying that $x = \bigwedge A \in \uparrow y_0 \subseteq \uparrow F$, a contradiction. Thus $x \in A$ and $A = \uparrow x$. Therefore, $\Omega P$ is a sober space.

**Corollary 4.19.** Every bounded complete poset is sober with respect to the lower topology.

The following example shows that neither the sobriety nor the well-filteredness of $P$ endowed with the lower topology is inherited by its maximal point space $\text{Max}_\omega(P)$.

**Example 4.20.** Let $X$ be an infinite set equipped with the co-finite topology (the proper closed sets are finite subsets). It is easy to verify that $X$ is not a well-filtered space, hence not sober. Let $P = (C(X), \geq)$. By Theorem 4.4, the $T_1$ space $X$ is homeomorphic to $\text{Max}_\omega(P)$, implying that $\text{Max}_\omega(P)$ is not well-filtered. However, since $P$ is a bounded complete poset, by Corollary 4.19, $\Omega P$ is a sober space.
5. Algebraic dcpo LT-Models of $T_1$ Spaces

It is well-known that spaces that have a domain model must be Baire [17]. In this section, we show that every $T_1$ space has a domain LT-model. This means that spaces having a domain LT-model need not be Baire.

In [24], Zhao (2009) proved that every $T_1$ space has a bounded complete algebraic poset model. Subsequently, from a bounded complete algebraic poset $(P, \leq P)$, Zhao and Xi (2016) constructed a dcpo $\hat{P}$ as follows:

$$\hat{P} = \{(x, a) : x \in P, a \in \text{Max}(P) \text{ and } x \leq_P a\}$$

ordered by

$$(x, a) \leq (y, b) \text{ iff either } a = b \text{ and } x \leq_P y, \text{ or } y = b \text{ and } x \leq_P b.$$  

It was proved that a bounded complete algebraic poset $P$ and the dcpo $\hat{P}$ has the homeomorphic maximal point space relative to the Scott topology, therefore every $T_1$ space has a dcpo model [25].

We elaborate the construction of $\hat{P}$ by the following simple example.

Example 5.1. (8) Let $P = \{a_1, a_2, \ldots, a_n, \ldots\} \cup \{d_1, d_2, \ldots, d_n, \ldots\}$ with the partial order $\leq_P$ on $P$ defined by

$$a_i \leq_P d_i \text{ and } a_i \leq_P a_{i+1}$$

for any $i = 1, 2, \ldots$. Then $(P, \leq_P)$ is a bounded complete algebraic poset, shown in Figure 1. The dcpo $\hat{P}$ constructed from $P$ is shown in Figure 2.

It needs to be reminded that $\hat{P}$ need not be a dcpo if $P$ is only a poset, which can be seen from Proposition 5.5.

Remark 5.2. The following facts on the poset $\hat{P}$ constructed from a poset $P$ will be used later. They can be proved by using a similar approach to [25, Lemma 1].
(i) The directed subset $D$ of $\hat{P}$ has two cases: either $\bigvee D \in D$ or $D = \{ (x_i, a) : i \in I \}$ for some $a \in \text{Max}(P)$ and some directed subset $\{ x_i : i \in I \}$ of $P$.

(ii) The set of maximal points of $\hat{P}$ equals $\{ (a, a) : a \in \text{Max}(P) \}$.

We call a poset $P$ conditionally directed complete if for every directed subset $D$, $\bigvee D$ exists whenever $D$ has an upper bound. Clearly, every bounded complete poset is conditionally directed complete.

**Proposition 5.3.** If $P$ is a conditionally directed complete poset, then $\hat{P}$ is a dcpo.

**Proof.** Let $D$ be a directed subset of $\hat{P}$. By Remark 5.2, we may assume $D = \{ (x_i, a) : i \in I \}$, where $\{ x_i : i \in I \}$ is a directed subset of $P$ and $a \in \text{Max}(P)$. As $P$ is conditionally directed complete, $\bigvee_{i \in I} x_i$ exists, and then $\bigvee D = (\bigvee_{i \in I} x_i, a)$. Hence $\hat{P}$ is a dcpo. $\square$

**Lemma 5.4.** Let $(P, \leq_p)$ be a poset and $\{ (x_i, a) : i \in I \}$ a directed subset of $\hat{P}$. If $\bigvee_{i \in I} (x_i, a)$ exists, then $\bigvee_{i \in I} x_i$ exists and $\bigvee_{i \in I} (x_i, a) = (\bigvee_{i \in I} x_i, a)$.

**Proof.** For the sake of convenience, let $\bigvee_{i \in I} (x_i, a) = (y, b)$. We prove this in two steps.

Step 1. $\bigvee_{i \in I} x_i = y$. Since $(y, b)$ is an upper bound of $\{ (x_i, a) : i \in I \}$, $y$ is an upper bound of $\{ x_i : i \in I \}$. If $z \in P$ is another upper bound of $\{ x_i : i \in I \}$, then $(z, a)$ is an upper bound of $\{ (x_i, a) : i \in I \}$, which implies that $(y, b) \leq (z, a)$, hence $y \leq z$. Therefore, $\bigvee_{i \in I} x_i = y$.

Step 2. $a = b$. Suppose, on the contrary, that $a \neq b$. Since $x_i \leq a \leq b$, we have that $y = b$. Note that $a$ is an upper bound of $\{ x_i : i \in I \}$, so $y = b = \bigvee_{i \in I} x_i \leq_p a$. Thus $b \leq_p a$, contradicting that $b$ is maximal. Therefore, $a = b$ holds. $\square$

**Proposition 5.5.** Let $(P, \leq_p)$ be a poset such that $P = \downarrow \text{Max}(P)$. Then $P$ is a conditionally directed complete poset if and only if $P$ is a dcpo.

**Proof.** Assume $\hat{P}$ is a dcpo. Let $D$ be a directed subset of $P$ with an upper bound $y$. As $y \in P = \downarrow \text{Max}(P)$, there exists $a \in \text{Max}(P)$ such that $y \leq_p a$. It follows that $\{ (x, a) : x \in D \}$ is a directed subset of $\hat{P}$. Thus by Lemma 5.4, $\bigvee D$ exists in $\hat{P}$. Therefore, $P$ is a conditionally directed complete poset. The converse is trivial by Proposition 5.3. $\square$

The following lemma shows that for a poset $P$, the maximal point spaces of $P$ and $\hat{P}$ are homeomorphic when each equipped with the relative lower topology.

**Lemma 5.6.** Let $(P, \leq_p)$ be a poset. Then $\text{Max}_u(P)$ and $\text{Max}_u(\hat{P})$ are homeomorphic.

**Proof.** Define $f : \text{Max}_u(P) \rightarrow \text{Max}_u(\hat{P})$ by $f(a) = (a, a)$ for each $a \in \text{Max}(P)$. Then $f$ is a bijection. Let $x \in P$ such that $\uparrow x \cap \text{Max}(P) \neq \emptyset$, and fix an element $a_0 \in \uparrow x \cap \text{Max}(P)$. Then $f(\uparrow x \cap \text{Max}(P)) = \{ (a, a) : a \in \text{Max}(P) \}$ and $x \leq_p a$.

We claim that $f(\uparrow x \cap \text{Max}(P)) = \uparrow (x, a_0) \cap \text{Max}(\hat{P})$. Suppose $(a, a) \in f(\uparrow x \cap \text{Max}(P))$. Then $a \in \text{Max}(P)$ and $x \leq_p a$. By the definition of the order on $\hat{P}$, we have that $(x, a_0) \leq (a, a)$, implying that $(a, a) \in \uparrow (x, a_0) \cap \text{Max}(\hat{P})$. Conversely, suppose $(b, b) \in \uparrow (x, a_0) \cap \text{Max}(\hat{P})$. Then $b \in \text{Max}(P)$ and $(x, a_0) \leq (b, b)$, implying that $x \leq_p b$, so $(b, b) \in f(\uparrow x \cap \text{Max}(P))$. All these show that $f(\uparrow x \cap \text{Max}(P)) = \uparrow (x, a_0) \cap \text{Max}(\hat{P})$, which is closed in $\text{Max}_u(\hat{P})$. So $f$ is a closed mapping.

For any $(x, a) \in \hat{P}$, we have $f^{-1}(\uparrow (x, a) \cap \text{Max}(\hat{P})) = \{ b \in \text{Max}(P) : (x, a) \leq (b, b) \} = \uparrow x \cap \text{Max}(P)$, which is closed in $\text{Max}_u(P)$. So $f$ is a continuous mapping. Therefore, $f$ is a homeomorphism. $\square$

**Remark 5.7.** The above lemma has been proved by H. Li and Q. Li [14] when $(P, \leq_p)$ is assumed to be a conditionally complete dcpo. Here, we generalize the result to any poset.
Given a $T_1$ space $X$, $P = (C^*(X), \supseteq)$ is a bounded complete poset such that $P = \downarrow \text{Max}(P)$. By Proposition 5.5, $\widehat{P}$ is a dcpo. Further, by Lemma 4.3 and Lemma 5.6, it follows that

$$X \equiv \text{Max}_w(P) \equiv \text{Max}_w(\widehat{P}).$$

As a consequence, we obtain the following result.

**Theorem 5.8.** Every $T_1$ space has a dcpo LT-model.

**Remark 5.9.** As pointed out in the introduction, the above result is an immediate consequence of the results in [14], though the authors did not state it.

In the following, the dcpo LT-model constructed above for a $T_1$ space $X$ will be denoted by $D(X)$, that is, $D(X) = (C^*(X), \supseteq)$.

By Corollary 4.16 and Theorem 5.8, we deduce the following result.

**Corollary 5.10.** Every $T_1$ space has an algebraic dcpo LT-model.

**Remark 5.11.** The referee pointed out that the above result was also obtained by H. Li and Q. Li in [15], which was published after our submission to this journal.

In the next part, we study some properties of the dcpo LT-model $D(X)$ of a $T_1$ space $X$.

For each $a \in \text{Max}(P)$ and $Q \subseteq \widehat{P}$, denote

$$Q_a = \{(x, a) \in Q : x \leq_P a\}.$$

**Lemma 5.12.** Let $P$ be a bounded complete poset and $Q \subseteq \widehat{P} \setminus \text{Max}(\widehat{P})$. The following statements are equivalent:

1. $Q$ is compact saturated in $\Omega(\widehat{P})$.
2. For each $a \in \text{Max}(P)$, $Q_a$ is compact saturated in $\Omega(\widehat{P})$.
3. For each $a \in \text{Max}(P)$, $Q_a$ is Scott closed.

**Proof.** We first prove a useful result:

(F) If $\{\Uparrow(x_i, a_i) : i \in \Delta\}$ satisfies that for each $J \in \Delta^{(<\omega)}$, $\bigcap_{i \in J} \Uparrow(x_i, a_i) \cap Q \neq \emptyset$, then there exists $a \in \text{Max}(P)$ such that $a_i = a$ for all $i \in \Delta$.

Let $t_i \in \Delta$. By assumption that $\Uparrow(x_i, a_i) \cap Q \neq \emptyset$, there exists $(x, a) \in Q$ such that $(x, a) \leq (x, a)$. Note that $Q \cap \text{Max}(\widehat{P}) = \emptyset$, so $x \neq a$. From the definition of the order on $\widehat{P}$, it follows that $a \uparrow = a$. Now take an arbitrary $i \in \Delta$. Then

$$\Uparrow(x_i, a_i) \cap \Uparrow(x_i, a_i) \cap Q = \Uparrow(x_i, a_i) \cap \Uparrow(x_i, a) \cap Q \neq \emptyset,$$

so there exists $(y, b) \in Q$ such that $(x_i, a_i), (x_i, a) \leq (y, b)$. Since $Q \cap \text{Max}(\widehat{P}) = \emptyset$, we have that $y \neq b$. By the definition of the order on $\widehat{P}$, it holds that $a_i = b = a$. Therefore, $a_i = a$ for all $i \in \Delta$.

We now prove the lemma.

(1) $\Rightarrow$ (2) Suppose $\{\Uparrow(x_i, a_i) : i \in \Delta\}$ satisfies that for each $J \in \Delta^{(<\omega)}$, $\bigcap_{i \in J} \Uparrow(x_i, a_i) \cap Q_a \neq \emptyset$. By the proceeding argument, we deduce that $a_i = a$ for all $i \in \Delta$. Since $Q$ is compact such that $\bigcap_{i \in J} \Uparrow(x_i, a) \cap Q \neq \emptyset$ for all $J \in \Delta^{(<\omega)}$, we have that $\bigcap_{i \in \Delta} \Uparrow(x_i, a) \cap Q \neq \emptyset$, so let $(y, b) \in \bigcap_{i \in \Delta} \Uparrow(x_i, a) \cap Q$. Note that $Q \cap \text{Max}(\widehat{P}) = \emptyset$, hence $y \neq b$. By the definition of the order on $\widehat{P}$, it holds that $b = a$, so $(y, b) = (y, a) \in \bigcap_{i \in \Delta} \Uparrow(x_i, a) \cap Q_a \neq \emptyset$. By Alexander Subbase Lemma, $Q_a$ is compact.

(2) $\Rightarrow$ (3) By Remark 2.11, $Q_a$ is a lower set. Now let $D$ be a directed subset of $Q_a$. We need to prove $\bigvee D \in Q_a$. It is trivial when $\bigvee D \in D$. Otherwise, by Remark 5.2, $D = \{(x_i, a) : i \in \Delta\}$, where $\{x_i : i \in \Delta\}$ is a directed subset of $P$ and $a \in \text{Max}(P)$. Then for each $J \in \Delta^{(<\omega)}$, there exists $i_{J} \in \Delta$ such that $x_{i_{J}}$ is an upper bound of $\{x_i : i \in J\}$, implying that $(x_{i_{J}}, a) \in \bigcap_{i \in J} \Uparrow(x_i, a) \cap Q_a \neq \emptyset$. Since $Q_a$ is compact, we have that
Proposition 6.1. For all adjoint of following are equivalent:

1. There is still no analog result established for poset models (using the Scott topology).
2. Bounded complete posets. This result shows some advantages of considering lower topology models, as there is no analog result established for general spaces.

Problem 5.15. Is it true that for any co-sober $T_0$ space $X$, the set $D(X)$ equipped with the lower topology is co-sober?

6. A Functor from the Category of $T_1$ Spaces to a Category of Bounded Complete Posets

In section 4, we constructed a bounded complete poset from each $T_1$ space. In this section, we show that this construction can be extended to a functor from the category $TOP_1$ of $T_1$ spaces to a category of bounded complete posets. This result shows some advantages of considering lower topology models, as there is still no analog result established for poset models (using the Scott topology).

For monotone maps $f : P \rightarrow Q$ and $g : Q \rightarrow P$ between posets, $f$ is a left adjoint of $g$ and $g$ is a right adjoint of $f$ if

$$f(p) \leq q \iff p \leq g(q)$$

for all $p \in P, q \in Q$.

Proposition 6.1. Let $P$ and $Q$ be two bounded complete posets, $g : Q \rightarrow P$ a monotone mapping. Then the following are equivalent:
(1) \( g \) has a left adjoint.

(2) For each \( p \in P \), \( g^{-1}(\uparrow p) = \uparrow q \) for some \( q \in Q \).

(3) For each \( B \subseteq Q \), if \( \bigwedge B \) exists, then \( g(\bigwedge B) = \bigwedge g(B) \).

We call a monotone mapping \( g : P \rightarrow Q \) lower continuous if it has a left adjoint such that \( g(\text{Max}(P)) \subseteq \text{Max}(Q) \).

Denote by \( \text{BCPOSET} \) the category of bounded complete posets and lower continuous mappings.

**Theorem 6.2.** The assignment \( C^* \) defines a functor from \( \text{TOP}_1 \) to \( \text{BCPOSET} \). On morphisms \( g : X \rightarrow Y \) in \( \text{TOP}_1 \), \( C^*(g) : (C^*(X), \supseteq) \rightarrow (C^*(Y), \supseteq) \) is defined by \( C^*(g)(A) = \text{cl}_Y(g(A)) \) for each \( A \in C^*(X) \), as shown below:

\[
\begin{array}{ccc}
X & \rightarrow & (C^*(X), \supseteq) \\
g & \downarrow & \downarrow \quad C^*(g) \\
Y & \rightarrow & (C^*(Y), \supseteq)
\end{array}
\]

**Proof.** We first show that \( C^*(g) \) is a lower continuous mapping. On one hand, we have

\[
C^*(g)(\text{Max}(C^*(X), \supseteq)) = \{ \text{cl}_Y(g(x)) : x \in X \} \\
= \{ \{g(x)\} : x \in X \} \\
\subseteq \text{Max}(C^*(Y), \supseteq).
\]

On the other hand, for any \( B \in C^*(Y) \), it follows that

\[
C^*(g)^{-1}(\uparrow_{C^*(Y)} B) = \{ A \in C^*(X) : g(A) \subseteq B \} \\
= \{ A \in C^*(X) : g(A) \subseteq B \} \\
= \{ A \in C^*(X) : A \subseteq g^{-1}(B) \} \\
= \uparrow_{C^*(X)} g^{-1}(B).
\]

By Proposition 6.1, \( C^*(g) \) has a left adjoint. Thus \( C^*(g) \) is a morphism in \( \text{BCPOSET} \). It is straightforward to check that \( C^* \) preserves identities and composition. Therefore, \( C^* \) is a functor. \( \square \)

**Lemma 6.3.** The assignment \( \text{Max}_\omega \) defines a functor from \( \text{BCPOSET} \) to \( \text{TOP}_1 \). On morphisms \( g : Q \rightarrow P \) in \( \text{BCPOSET} \), \( \text{Max}_\omega(g) : \text{Max}_\omega(Q) \rightarrow \text{Max}_\omega(P) \) is defined by \( \text{Max}_\omega(g)(q) = g(q) \) for each \( q \in \text{Max}(Q) \), as shown below:

\[
\begin{array}{ccc}
Q & \rightarrow & \text{Max}_\omega(Q) \\
g & \downarrow & \downarrow \quad \text{Max}_\omega(g) \\
P & \rightarrow & \text{Max}_\omega(P)
\end{array}
\]

**Proof.** It is trivial by the definition of the lower continuity. \( \square \)

**Proposition 6.4.** There is a natural isomorphism \( \phi : I_{\text{TOP}_1} \rightarrow \text{Max}_\omega \circ C^* \) defined as follows: for each \( T_1 \) space \( X \),

\[
\phi_X : X \rightarrow \text{Max}_\omega(C^*(X), \supseteq), \phi_X(x) = \{ x \}, \forall x \in X.
\]

**Proof.** First, by Lemma 4.3, each \( \phi_X \) is an order-isomorphism. In addition, since for each \( x \in X \), \( \text{Max}_\omega \circ C^*(f)(\phi_X(x)) = \text{Max}_\omega(f)(\{x\}) = \{f(x)\} = \phi_Y(f(x)) \). Thus the following diagram commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{\phi_X} & \text{Max}_\omega(C^*(X), \supseteq) \\
\downarrow f & & \downarrow \text{Max}_\omega \circ C^*(f) \\
Y & \xrightarrow{\phi_Y} & \text{Max}_\omega(C^*(Y), \supseteq)
\end{array}
\]

Therefore, \( \phi \) is a natural isomorphism. \( \square \)
Theorem 6.5. The functor $C^*$ is a left adjoint of $\text{Max}_{\omega}$. 

Proof. Let $X$ be a $T_1$ space. Define $\phi_X : X \to \text{Max}_{\omega}(C^*(X), \supseteq)$ by $\phi_X(x) = \{ x \}$ for each $x \in X$. Let $P$ be a bounded complete poset and $f : X \to \text{Max}_{\omega}(P)$ a continuous mapping.

Define $g : (C^*(X), \supseteq) \to P$ by $g(A) = \bigwedge f(A)$ for each $A \in C^*(X)$. Since $A \neq \emptyset$, it follows that $f(A) \neq \emptyset$, and since $P$ is bounded complete, $\bigwedge f(A)$ exists, so $g$ is well-defined (refer to the following diagram).

$$
\begin{array}{c}
X \xrightarrow{\phi_X} \text{Max}_{\omega}(C^*(X), \supseteq) \\
\downarrow f \\
\text{Max}_{\omega}(P) \xleftarrow{g} P
\end{array}
$$

We now prove this result in three steps.

Step 1. $g$ is a lower continuous mapping.

First, we have that

$$g(\text{Max}(C^*(X), \supseteq)) = \{ g([x]) : x \in X \} = \{ f(x) : x \in X \} \subseteq \text{Max}(P).$$

Additionally, for each $p \in P$, it follows that

$$g^{-1}(\uparrow p) = \{ A \in C^*(X) : g(A) = \bigwedge f(A) \in \uparrow p \} = \{ A \in C^*(X) : f(A) \subseteq \uparrow p \cap \text{Max}(P) \} = \uparrow_c f^{-1}(\uparrow p \cap \text{Max}(P)).$$

Since $\uparrow p \cap \text{Max}(P)$ is closed in $\text{Max}_{\omega}(P)$ and $f$ is continuous, we obtain that $f^{-1}(\uparrow p \cap \text{Max}(P)) \subseteq C^*(X)$. Thus by Proposition 6.7, $g$ has a left adjoint. Therefore, $g$ is lower continuous.

Step 2. $\text{Max}_{\omega}(g) \circ \phi_X = f$.

This is easy since for each $x \in X$, we have

$$\text{Max}_{\omega}(g) \circ \phi_X(x) = \text{Max}_{\omega}(g)([x]) = g([x]) = \bigwedge \{ f(x) \} = f(x).$$

Step 3. $g$ is the unique lower continuous mapping such that $\text{Max}_{\omega}(g) \circ \phi_X = f$.

Suppose $h : (C^*(X), \supseteq) \to P$ is a lower continuous mapping such that $\text{Max}_{\omega}(h) \circ \phi_X = f$. Let $A \in C^*(X)$. Since $h$ is monotone, it follows that $h(A) \subseteq h([x]) = f(x)$ for all $x \in A$, so $h(A) \subseteq \bigwedge f(A) = g(A)$. Additionally, since $h$ has a left adjoint, by Proposition 6.7, there exists $B \in C^*(X)$ such that $h^{-1}(\uparrow g(A)) = \uparrow_{C^*(X)} B$. Then for each $x \in A$, we have that $h([x]) = f(x) \geq \bigwedge f(A) = g(A)$, showing that $h([x]) \in \uparrow g(A)$, so

$$[x] \in h^{-1}(\uparrow g([A])) = \uparrow_{C^*(X)} B,$$

implying that $[x] \in \uparrow_{C^*(X)} B$, i.e., $\{ x \} \subseteq B$. It follows that $A \subseteq B$, and thus $A \in \uparrow_{C^*(X)} B = h^{-1}(\uparrow g([A]))$. This shows that $h(A) \in \uparrow g(A)$, that is, $g(A) \subseteq h(A)$. Therefore, $g(A) = h(A)$.

All these show that $C^*$ is a left adjoint of $\text{Max}_{\omega}$. □

Lemma 6.6. Let $P$ be a bounded complete poset and $B \subseteq \text{Max}(P)$. Then $\bigwedge B = \bigwedge \text{cl}_{\text{Max}_{\omega}(P)}(B)$.

Proof. Since $B \subseteq \text{cl}_{\text{Max}_{\omega}(P)}(B)$, it follows that $\bigwedge \text{cl}_{\text{Max}_{\omega}(P)}(B) \subseteq \bigwedge B$. Additionally, since $B \subseteq \uparrow \bigwedge B$ and $\uparrow \bigwedge B$ is closed in $\Omega P$, it follows that $\text{cl}_{\Omega P}(B) \subseteq \uparrow \bigwedge B$, hence $\text{cl}_{\text{Max}_{\omega}(P)}(B) = \text{cl}_{\Omega P}(B) \cap \text{Max}(P) \subseteq \uparrow \bigwedge B$, implying that $\bigwedge B \subseteq \bigwedge \text{cl}_{\text{Max}_{\omega}(P)}(B)$. Therefore, $\bigwedge B = \bigwedge \text{cl}_{\text{Max}_{\omega}(P)}(B)$. □

Proposition 6.7. There is a natural transformation $\psi : C^* \circ \text{Max}_{\omega} \to \mathcal{I}_{\text{BPOSET}}$ defined as follows:

$$\psi = (C^*(\text{Max}_{\omega}(P)), \supseteq) \to P, \psi(A) = \bigwedge A, \forall A \in C^*(\text{Max}_{\omega}(P)).$$
Proof. First, note that for each $A \in C^*(\operatorname{Max}_\omega(P))$, $A \neq \emptyset$, and since $P$ is bounded complete, $\land A$ exists, so $\psi_P$ is well-defined. We prove the result in two steps.

Step 1. $\psi_P$ is lower continuous.

(c1) Let $A, B \in C^*(\operatorname{Max}_\omega(P))$. If $A \supseteq B$, then $\psi_P(A) = \land A \subseteq \land B = \psi_P(B)$, hence $\psi_P$ is monotone.

(c2) Since $\operatorname{Max}_\omega(P)$ is a $T_1$ space, it follows that $\operatorname{Max}(C^*(\operatorname{Max}_\omega(P), \supseteq)) = \{p : p \in \operatorname{Max}(P)\}$. For each $p \in \operatorname{Max}(P)$, $\psi_P[p] = \land \{p\} = p \in \operatorname{Max}(P)$. Thus $\psi_P(\operatorname{Max}(C^*(\operatorname{Max}_\omega(P), \supseteq)) \subseteq \operatorname{Max}(P)$.

(c3) For each $p \in P$, we have that

$$
\psi_P^{-1}(\uparrow p) = \{A \in C^*(\operatorname{Max}_\omega(P), \supseteq) : A \in \uparrow p\}
= \{A \in C^*(\operatorname{Max}_\omega(P), \supseteq) : \land A \subseteq \uparrow p\}
= \uparrow(\operatorname{Max}(C^*(\operatorname{Max}_\omega(P), \supseteq))) \cap \operatorname{Max}(P).
$$

Therefore, $\psi_P$ is a lower continuous mapping.

Step 2. Suppose $g : Q \longrightarrow P$ is a morphism in $\text{BCPOSET}$. We need to verify that the following diagram commutes.

$$
\begin{array}{ccc}
C^*(\operatorname{Max}_\omega(Q), \supseteq) & \xrightarrow{\psi_Q} & Q \\
\Downarrow C^* \circ \operatorname{Max}_\omega(g) & & \Downarrow g \\
C^*(\operatorname{Max}_\omega(P), \supseteq) & \xrightarrow{\psi_P} & P
\end{array}
$$

Let $A \in C^*(\operatorname{Max}_\omega(Q))$. Since $g$ has left adjoint, it follows that $\land g(A) = g(\land A)$. Then by Lemma 6.6, we have $\land \operatorname{cl}_{\operatorname{Max}_\omega(P)}(g(A)) = \land g(A)$, thus

$$
\psi_P(C^*(\operatorname{Max}_\omega(g(A)))) = \psi_P(\operatorname{cl}_{\operatorname{Max}_\omega(P)}(g(A))) = \land \operatorname{cl}_{\operatorname{Max}_\omega(P)}(g(A)) = \land g(A) = g(\land A) = g(\psi_Q(A)).
$$

Hence, the above diagram commutes.

All these show that $\psi_P$ is a natural transformation. $\square$

We call a poset $P$ lower topology determined ($\text{LT}$-determined) if (i) for each $x \in P$, $x = \land \uparrow x \cap \operatorname{Max}(P)$; (ii) for each $A \in C^*(\operatorname{Max}_\omega(P))$, $A = \uparrow(\land A) \cap \operatorname{Max}(P)$.

Remark 6.8. For a $T_1$ space $X$, it is trivial to check that $(C^*(X), \supseteq)$ is an $\text{LT}$-determined bounded complete poset. Therefore, $C^*$ is also a functor from $\text{TOP}_1$ to $\text{LTD} - \text{BCPOSET}$.

Proposition 6.9. Let $P$ be a bounded complete poset. Then the following statements are equivalent:

1. $\psi_P$ is an order-isomorphism.
2. $P$ is $\text{LT}$-determined.

Proof. Let $f$ be the left adjoint of $\psi_P$. By Proposition 6.7, for each $x \in P$, $g^{-1}(\uparrow x) = \uparrow(\operatorname{Max}(C^*(\operatorname{Max}_\omega(P), \supseteq))) \cap \operatorname{Max}(P)$, hence $f(x) = \land g^{-1}(\uparrow x) = \uparrow x \cap \operatorname{Max}(P)$.

(1) $\Rightarrow$ (2) Since $\psi_P$ is order-isomorphic, it follows that $f$ is the inverse of $\psi_P$. Then for each $x \in P$, $x = g(f(x)) = \land \uparrow x \cap \operatorname{Max}(P)$, and for each $A \in C^*(\operatorname{Max}_\omega(P))$, $A = f(g(A)) = \uparrow(\land A) \cap \operatorname{Max}(P)$. Hence $P$ is $\text{LT}$-determined.

(2) $\Rightarrow$ (1) By assumption, for $x \in P$, we have $g(f(x)) = \land \uparrow x \cap \operatorname{Max}(P) = x$, and for each $A \in C^*(\operatorname{Max}_\omega(P))$, $f(g(A)) = \uparrow(\land A) \cap \operatorname{Max}(P) = A$. Thus $f$ is the inverse of $\psi_P$, so $\psi_P$ is an order-isomorphism. $\square$

Let $\text{LTD} - \text{BCPOSET}$ be the category of $\text{LT}$-determined bounded complete posets and lower continuous mappings.

By Proposition 6.7, the natural transformation $\psi : C^* \circ \operatorname{Max}_\omega \longrightarrow I_{\text{LTD-BCPOSET}}$ is a natural isomorphism. By Proposition 6.4, $\phi : I_{\text{TOP}_1} \longrightarrow \operatorname{Max}_\omega \circ C^*$ is also a natural isomorphism. As a consequence, we obtain the following result.

Corollary 6.10. The categories $\text{LTD} - \text{BCPOSET}$ and $\text{TOP}_1$ are equivalent.
References