TRYING TO BE NEGATIVE: TEACHING THE RULES FOR SIGNS

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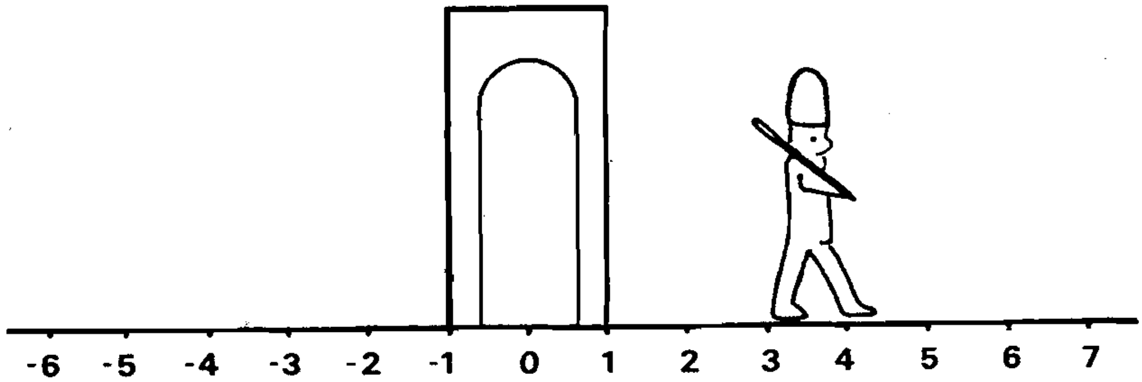
Ever since my own school days I have felt that many of the methods of introducing the rules for operating with negative numbers leave much to be desired. I was introduced to the subject by a schoolmaster who used the "money in the bank" method. You have \$10 in the bank and write a cheque for \$16. Now you are overdrawn by \$6 which we can write as -6 dollars. Fair enough – so far, so good. But now he tries to convince me that if some kind benefactor pays off my debt – takes away my debt – then somehow I can represent my wealth as $-(-6) = +6$ dollars. Now whilst I could accept that I was clearly \$6 better off, it seemed to me that my account must stand at zero dollars and not \$6. So I did not think too much of this "explanation". Perhaps I misunderstood the explanation but this is an inherent defect of such methods.

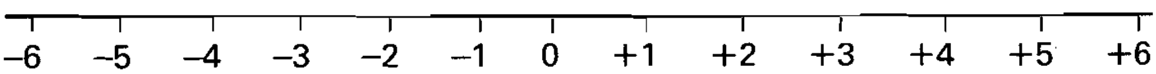
As a teacher, like most others I suppose, I tended to use whatever method came to hand in whatever textbook I was using, always with a slight feeling of unease. The day came, however, when I was forced into a complete review of the topic. One of my students came to me complaining bitterly that she was required to teach this topic from an unfamiliar textbook and she was just not able to understand the method put forward, let alone teach it. We looked at the text and after some considerable time unravelled the method. It was based on the following idea. There is a soldier on sentry duty who takes ten steps to the right of his sentry box and ten steps to the left. He steps at the rate of one step per second. The diagram will give the reader some idea of the basis for the type of explanation that is to come. It will involve positive motion to the right and negative motion to the left; positive direction to the right and negative direction to the left. The writer was looking for combinations of motion and direction to give the combinations of positive and negative numbers.



I have a fundamental objection to methods like this. Unless they are presented with great skill, they obscure the issue. They employ a difficult concept (in this case associated with motion and direction) to explain an easier concept. It is not surprising that they fail.

To return to my story. In assisting this student I evolved a method of teaching the topic which she and subsequent students have used with some success. It is an entirely numeric approach and does not rely upon any real life situation. The method draws only upon the child's experience of number; so it is uncluttered by any appeal to other concepts which might themselves be imperfectly understood and so impede the acquisition of the particular skills in question. As it is entirely numeric, some readers may see the method as a straightforward adaptation of the axiomatic method to the classroom, and indeed that is how I regard this method. It relies on the ordering of the integers. If you like, the order of the integers is taken as axiomatic. In doing this I feel I am in rather good company for I believe that Peano used a similar postulate.

Now for a beginning. The notion of a negative number is readily accepted these days. Surely we all begin with a discussion of the thermometer and degrees below zero. From this idea we establish the extension of the number line to include the negative numbers.



Having established this ordering of the integers, I shall emphasise to the children that it is "sacred". There is good reason for this. Without this postulate, mathematics would have a different face from the one we know and, indeed, it might well be impossible to make it logically consistent.

The Operation of Addition

Having established the order of the numbers I now emphasise to the children the nature of addition. It is simply the process of moving from one number to its successor on the number line. In the language of Primary School Mathematics "addition is just counting on". You see I am making addition axiomatic – what else is there to do?

Now a word about notation. It is commonplace these days to emphasise that we must distinguish between the notion of positive and negative number and the operations of addition and subtraction. This is central to the subject. There are two notations in use. The subtraction of "minus three" can be represented by $-(-3)$ or in a more recently evolved notation the negative sign is written smaller and at the top of the number symbol like this: $^{-}3$ –, and hence we have $- {}^{-}3$.

For my purposes, I prefer this latter notation for the reason that the method involves doing a lot of examples, many more than in the traditional treatment. If we are to teach this subject to the average child, then in my view we must give him the opportunity to do a lot of examples. He is not going to absorb the topic in one or two quick lessons. Thus I prefer the modern notation. It avoids writing a myriad of brackets.

Armed with these two basic ideas, the order of the integers and the inviolable nature of addition and a convenient notation, I take my class back to their earliest primary school days and ask them to write down an addition table like this.

$^{+}5 + {}^{+}1 = {}^{+}6$
$^{+}5 + {}^{+}2 = {}^{+}7$
$^{+}5 + {}^{+}3 = {}^{+}8$
$^{+}5 + {}^{+}4 = {}^{+}9$
$^{+}5 + {}^{+}5 = {}^{+}10$
$^{+}5 + {}^{+}6 = {}^{+}11$
$^{+}5 + {}^{+}7 = {}^{+}12$

I suggest the teacher select an addition table for 4, 5, 6 or 7, whichever is preferred. Here I use the addition table for $^{+}5$.
We notice that the number line is evolving in the answers.

No, I am not wasting the children's time! With such a beginning I achieve several objectives. The class becomes accustomed to the new notation. The class notes that the number line is being evolved in the answers, which is a reinforcement of the nature of addition. Finally they are being assured that they are dealing with a very easy subject so that we can watch their confidence grow.

The next step is to ask them to write down the same table but this time – just for fun – we will do it backwards.

$^+5 + {}^+6 = {}^+11$ $^+5 + {}^+5 = {}^+10$ $^+5 + {}^+4 = {}^+9$ $^+5 + {}^+3 = {}^+8$ $^+5 + {}^+2 = {}^+7$ $^+5 + {}^+1 = {}^+6$ ↓	*The number line* is being evolved again...
$^+5 + 0 = {}^+5$ ↓	... and this line surely gives no difficulty...
$^+5 + {}^-1 = ?$ $^+5 + {}^-2 = ?$ $^+5 + {}^-3 = ?$ $^+5 + {}^-4 = ?$ $^+5 + {}^-5 = ?$	...but what answers *must* the child write to this next part of the table? I do use the word "must" advisedly.
$^+5 + {}^-1 = {}^+4$ $^+5 + {}^-2 = {}^+3$ $^+5 + {}^-3 = {}^+2$ $^+5 + {}^-4 = {}^+1$ $^+5 + {}^-5 = 0$ ↓	We have noted that the answers always follow the number line and unless the nature of the addition process is to change, then the table must be completed in this fashion...
$^+5 + {}^-6 = {}^-1$ $^+5 + {}^-7 = {}^-2$ $^+5 + {}^-8 = {}^-3$ $^+5 + {}^-9 = {}^-4$ $^+5 + {}^-10 = {}^-5$ ↓	...and there is no difficulty in proceeding still further. At this point we must pause for almost without noticing a vital position has been achieved.

We have established that "addition of a negative number is equivalent to subtraction". This point must be drawn to the attention of the children. A line of our table reads as follows:

$^+5 + {}^-3 = {}^+2$ ...but it may be expanded thus...

$^+5 + {}^-3 = 5 - 3 = 2$ ...a brief return to good old-fashioned numbers.

${}^{+}5 + {}^{-}9 = 5 - 9 = {}^{-}4$

Notice that we have also established this result which will be new to the class – for the first time they can subtract a larger number from a smaller one.

At this point in the development of the subject, it is probably wise to pause and set the class a variety of examples to reinforce the achievements so far.

The next step is to remind the children that addition is *commutative.*

$${}^{-}3 + {}^{+}5 = {}^{+}2$$
$${}^{-}3 + {}^{+}5 = -3 + 5 = 2$$
$${}^{-}9 + {}^{+}5 = {}^{-}4$$
$${}^{-}9 + {}^{+}5 = -9 + 5 = {}^{-}4$$

This is all new to the children and the great variety of patterns of positive and negative numbers must be developed and reinforced with examples.

$${}^{-}5 + {}^{+}10 = {}^{+}5$$
$${}^{-}5 + {}^{+}9 = {}^{+}4$$
$${}^{-}5 + {}^{+}8 = {}^{+}3$$
$${}^{-}5 + {}^{+}7 = {}^{+}2$$
$${}^{-}5 + {}^{+}6 = {}^{+}1$$
$${}^{-}5 + {}^{+}5 = 0$$

As much for completeness as anything else, the next step is for the class to set up the addition table for ${}^{-}5$.

$${}^{-}5 + {}^{+}4 = {}^{-}1$$
$${}^{-}5 + {}^{+}3 = {}^{-}2$$
$${}^{-}5 + {}^{+}2 = {}^{-}3$$
$${}^{-}5 + {}^{+}1 = {}^{-}4$$
$${}^{-}5 + 0 = {}^{-}5$$

We note that the *number line* turns up again ...

$${}^{-}5 + {}^{-}1 = ?$$
$${}^{-}5 + {}^{-}2 = ?$$
$${}^{-}5 + {}^{-}3 = ?$$
$${}^{-}5 + {}^{-}4 = ?$$

...but what answers should be written here?

$${}^{-}5 + {}^{-}1 = {}^{-}6$$
$${}^{-}5 + {}^{-}2 = {}^{-}7$$
$${}^{-}5 + {}^{-}3 = {}^{-}8$$
$${}^{-}5 + {}^{-}4 = {}^{-}9$$

Why the number line must be *continued,* of course...

This last portion of the table I consider to be of great importance for we can write

$${}^{-}5 + {}^{-}2 = -5 - 2 = -7$$

In my experience, the child having difficulty with his mathematics is very inclined to make mistakes at this point. Thus faced with $-5 - 2$ he will recall the famous words "minus a minus is a plus" or even "two minuses is a plus" and write $-5 - 2 = +7$ or even $-5 -2 = +14$. I have seen this error on a number of occasions, and it is very difficult to correct for usually upon investigation one finds that the child has no proper concept of the operation that is involved. He does not know whether he is adding, subtracting or multiplying. An advantage of this approach is that it puts emphasis on the operation involved.

Again, before taking the next step the experienced teacher will provide his class with many examples for them to do.

The Operation of Subtraction

For secondary school children $5 - 3 = 2$ is a long established fact. Using our new notation, we express the result as follows:

$$5 - 3 = {}^{+}5 - {}^{+}3 = {}^{+}2.$$

The notation emphasises that the operation here is subtraction – subtraction of positive numbers. Usually there is no problem here, so we can proceed straightaway to set the class to work constructing a subtraction table. I shall use the subtraction table for ${}^{+}5$.

$$\begin{aligned}
{}^{+}5 - {}^{+}5 &= 0 \\
{}^{+}5 - {}^{+}4 &= {}^{+}1 \\
{}^{+}5 - {}^{+}3 &= {}^{+}2 \\
{}^{+}5 - {}^{+}2 &= {}^{+}3 \\
{}^{+}5 - {}^{+}1 &= {}^{+}4 \\
{}^{+}5 - 0 &= {}^{+}5 \\
{}^{+}5 - {}^{-}1 &= ? \\
{}^{+}5 - {}^{-}2 &= ? \\
{}^{+}5 - {}^{-}3 &= ? \\
{}^{+}5 - {}^{-}4 &= ?
\end{aligned}$$

Here is the number line appearing yet again ...

...but now what answers must be written here?...

$$\begin{aligned}
{}^{+}5 - {}^{-}1 &= {}^{+}6 \\
{}^{+}5 - {}^{-}2 &= {}^{+}7 \\
{}^{+}5 - {}^{-}3 &= {}^{+}8 \\
{}^{+}5 - {}^{-}4 &= {}^{+}9 \\
{}^{+}5 - {}^{-}5 &= {}^{+}10 \\
{}^{+}5 - {}^{-}6 &= {}^{+}11 \\
{}^{+}5 - {}^{-}7 &= {}^{+}12
\end{aligned}$$

...can subtraction have possibly changed its nature just because we are subtracting the next few numbers in sequence beyond zero? Certainly not! And experience to date indicates that the children will have no difficulty in obtaining the next set of answers.

And we have established, painlessly if not effortlessly, the key result

$${}^{+}5 - {}^{-}4 = 5 + 4 = {}^{+}9$$

We have demonstrated that the subtraction of negative numbers is equivalent to addition. Again, of course, there must be a pause for a multiplicity of examples providing the opportunity to practise the skills associated with addition and subtraction of the integers. Algebraic notation can and, indeed, should be introduced at this point.

The Operation of Multiplication

To establish the required results, I introduce the class to the easiest of multiplication tables: *the one times table.* This is usually a popular move.

$$\begin{aligned} {}^{+}1 \times {}^{+}5 &= {}^{+}5 \\ {}^{+}1 \times {}^{+}4 &= {}^{+}4 \\ {}^{+}1 \times {}^{+}3 &= {}^{+}3 \\ {}^{+}1 \times {}^{+}2 &= {}^{+}2 \\ {}^{+}1 \times {}^{+}1 &= {}^{+}1 \end{aligned}$$

The class will have no difficulty obtaining these results. Our old friend the *number line* appears yet again ...

$$\begin{aligned} {}^{+}1 \times 0 &= 0 \\ {}^{+}1 \times {}^{-}1 &= ? \\ {}^{+}1 \times {}^{-}2 &= ? \\ {}^{+}1 \times {}^{-}3 &= ? \\ {}^{+}1 \times {}^{-}4 &= ? \end{aligned}$$

...but what answers are to be written here?...

...and yet again we must reply that unless multiplication has changed its spots, then the mind is forced to accept that the *number line must continue.*

$$\begin{aligned} {}^{+}1 \times {}^{-}1 &= {}^{-}1 \\ {}^{+}1 \times {}^{-}2 &= {}^{-}2 \\ {}^{+}1 \times {}^{-}3 &= {}^{-}3 \\ {}^{+}1 \times {}^{-}4 &= {}^{-}4 \end{aligned}$$

And an important result has been established: multiplication of a *positive* and a *negative* gives a *negative result.*

We expect that multiplication will retain its commutative property. Thus

$${}^{+}1 \times {}^{-}3 = {}^{-}3 \times {}^{+}1 = {}^{-}3$$

and $\quad {}^{+}1 \times {}^{-}1 = {}^{-}1 \times {}^{+}1 = {}^{-}1.$

Such avid readers as those who are still with me will anticipate the last stage of our drama. It begins by persuading the children to write down *the minus one times table.*

$$\begin{aligned}
{}^{-}1 \times {}^{+}5 &= {}^{-}5\\
{}^{-}1 \times {}^{+}4 &= {}^{-}4\\
{}^{-}1 \times {}^{+}3 &= {}^{-}3\\
{}^{-}1 \times {}^{+}2 &= {}^{-}2\\
{}^{-}1 \times {}^{+}1 &= {}^{-}1\\
{}^{-}1 \times 0 &= 0\\
{}^{-}1 \times {}^{-}1 &= ?\\
{}^{-}1 \times {}^{-}2 &= ?\\
{}^{-}1 \times {}^{-}3 &= ?
\end{aligned}$$

These first few results follow directly from what has been established, and we see once again that *the number line* is emerging ...

...and what answers should we write here?

$$\begin{aligned}
{}^{-}1 \times {}^{-}1 &= {}^{+}1\\
{}^{-}1 \times {}^{-}2 &= {}^{+}2\\
{}^{-}1 \times {}^{-}3 &= {}^{+}3\\
{}^{-}1 \times {}^{-}4 &= {}^{+}4\\
{}^{-}1 \times {}^{-}5 &= {}^{+}5
\end{aligned}$$

...what else can we write? We must follow *the number line.*

And so the desired result is established that negative times negative gives a positive result. Much practice must now follow this rule in its arithmetic and algebraic context. The task will not be complete until there has also been a great deal of practice using examples that require the application of the rules for addition, subtraction and multiplication simultaneously, so to speak.

There may be some complaint that in establishing these last results I have only considered the special cases of multiplication by ${}^{+}1$ and ${}^{-}1$. Shouldn't the approach be more general? The answer is that this result is really very general. Consider, for example, ${}^{-}3 \times {}^{+}4$.

$$\begin{aligned}
{}^{-}3 \times {}^{+}4 &= ({}^{-}1 \times {}^{+}3) \times ({}^{+}1 \times {}^{+}4)\\
&= {}^{-}1 \times {}^{+}3 \times {}^{+}1 \times {}^{+}4\\
&= {}^{-}1 \times {}^{+}1 \times {}^{+}3 \times {}^{+}4\\
&= {}^{-}1 \times {}^{+}12
\end{aligned}$$

However, if it is thought desirable, there is no reason why the ${}^{-}2$ or ${}^{-}3$ times tables should not be developed for they reveal the same utterly compelling pattern.

Though I have made a great fuss about the mathematical purity of this approach, and it is an aspect that pleases me a great deal, its principal strength lies in its value as a teaching device. It is psychologically sound. It allows the child to proceed through the material step by step acquiring a full understanding and as complete a conceptual background as can be obtained at this stage of development. The initial knowledge required by the demands of this approach is minimal: ie, knowledge of number bonds and multiplication facts.

I have not given the development of the rules for division. There is in fact a little more to be done. I prefer to remark that division is merely multiplication by a multiplicative inverse, ie,

$$\div (-2) \rightarrow \times (-\tfrac{1}{2})$$

so that the rules for division are going to be the same as those for multiplication. This needs to be expanded in the classroom and followed by a great deal of practice, although it is a minor extension of what has been outlined.

In demonstrating this approach, I have myself found the overhead projector extremely valuable. Writing out all those tables without the answers, then filling the answers in as a check on the children's work can be done very effectively and in colour on the OHP.

My method, it may be complained, is long-winded – it takes up a great deal of time. Unashamedly I agree that this is the case. I do not believe in a quick run through the rule in the first ten minutes of the lesson, and we are on to some meaty algebraic examples by the time the bell goes. That may be all right for the clever kids (though it is only going to give them parrot knowledge) but it is no good for the average child. Algebra is a great stumbling block for the average child. Its abstraction and complexity defeat him. The rules for signs are one of the most important features of this complexity. I am advancing this method as one that may help some of those students who are struggling with their mathematics, help them with their understanding. I firmly believe that it will help even the clever ones. They really obtain very little benefit from the quick, slick "learn it off by heart" approach.