Top-Down Approach to Teaching Problem Solving Heuristics in Mathematics

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Abstract
Singapore mathematics syllabuses identified eleven heuristics which are applicable to problem solving at the upper primary level (MOE, 2001a), and thirteen heuristics at lower secondary level (MOE, 2001b). According to the different characteristics of these thirteen heuristics, how and when they can be used in the process of mathematical problem solving, they can be classified into four categories: “representation heuristics”, “simplification heuristics”, “pathway heuristics”, and “generic heuristics”. Together they form a model of problem solving in mathematics (Tiong, Hedberg, & Lioe, 2005). Using this model as a structure for teaching heuristics, this paper proposes a top-down approach to teaching heuristics.

Background

The Singapore mathematics syllabuses, developed by Curriculum Planning and Developing Division (CPDD), Ministry of Education Singapore (MOE), have identified thirteen heuristics that are applicable to mathematical problem solving.

1. Act it out
2. Use a diagram/model
3. Use guess-and-check
4. Make a systematic list
5. Look for patterns
6. Work backwards
7. Use before-after concept
8. Make suppositions
9. Restate the problem in another way
10. Simplify the problem
11. Solve part of the problem
12. Think of a related problem
13. Use equations

    (Heuristics 12 and 13 are not in the primary syllabus.)

Though these heuristics are listed in the syllabus, the use of these heuristics are not fully reflected in Singapore published textbooks (Fan & Zhu, 2000), and “it is by no means clear how these heuristics should be incorporated into teaching and when” (Lee & Fan, 2002 p. 5). This paper from a theoretical point of view proposes a top-down approach to teaching heuristics.

What are heuristics?

Heuristic methods, heuristic strategies, or simply heuristics, are rules of thumb for making progress on difficult problems (Polya, 1973). They are general suggestions on strategy that are designed to help when we solve problems (Schoenfeld, 1985). For Bruner (1960) they are methods and strategies that can be helpful in problem solving. In sum, they can be explained as non-rigorous methods of achieving solutions to problems, ideas that have been useful in previous problem solving that we might want to apply when we solve our current problems. Heuristics have been generally recognized as a crucial component for problem solving (Polya 1973; Schoenfeld, 1985; Rubinstein, 1986; Mayer, 2003). In fact, according to Schoenfeld (1985), "heuristics have now become nearly synonymous with mathematical problem solving" (p. 23).

After analyzing the roles and functions of the thirteen heuristics in the Singapore syllabus, we can summarize them into four ideas: "representation", "simplification", "pathway", and "bring in solution" (Tiong, Hedberg, & Lioe 2005). Putting them together, we have a model for problem solving in mathematics, as shown in figure 1. We can also treat these four ideas as four general heuristics, "use different representations", "simplify your problem", "approach your problem from different directions", and "bring in solutions".

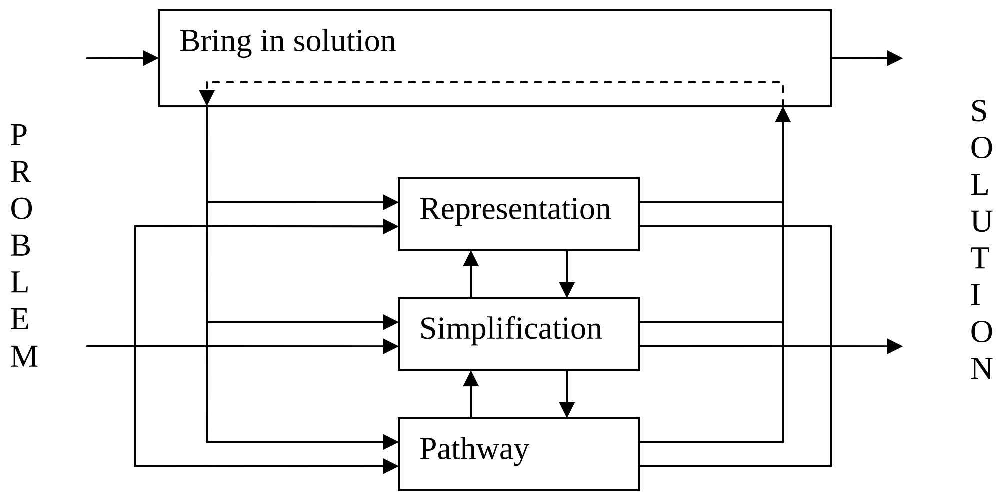

Figure 1: Model for problem solving in mathematics (Tiong, Hedberg, & Lioe, 2005)

By definition, all heuristics have the following two characteristics:

1. Heuristics do not guarantee a solution. All heuristics do is pointing us towards possible ways in which we might be able to find our solution.
2. Heuristics do not come with specific procedures. When we use heuristics, we are required to make some judgments of our own regarding what we should do.

Heuristics help us to deal with difficult problems or problems that we are not familiar with. Other than that, heuristics usually enables us to find solutions with less time and effort as compared to when we use algorithms to find solutions.

Not all heuristics are the same in term of specificity. There are ones that give very general and ambiguous instruction, while the others have more specific procedures that we as problem solvers would like to follow. For example, heuristic "represent you problem differently" only gives us a very general direction to what we should do, where as heuristic "draw a diagram" tells how we can represent our problem, visually. Heuristic "draw a histogram" on the other hand gives a much more specific instruction than the previous two.

To follow this heuristic "draw a histogram", we will need to know first the procedures in which we can draw a histogram, whereas to follow the heuristic "draw a diagram" we can invent our own diagrams and the rules or procedures in which we can manipulate them; here we have the freedom to choose and be creative. Of course, we can still end using the histogram, since it might be the best representation for our problems.

Here we can construct hierarchy of heuristics according to their specificity. We propose to put the four "general heuristics" from the model above on top of the hierarchy. Using the example in the previous paragraph, we have Figure 2. We should note that Figure 2 is not an exhaustive hierarchy for "representations", we can still add in more representation into the hierarchy, such as "manipulative" into the second level, "pie chart" into the third level, and so on. Here we are just trying to get a rough idea of how a hierarchy of heuristics might look like. Since the number of heuristics are only limited by our creativity and imagination, it is not possible to construct a hierarchy that contain all possible heuristics.

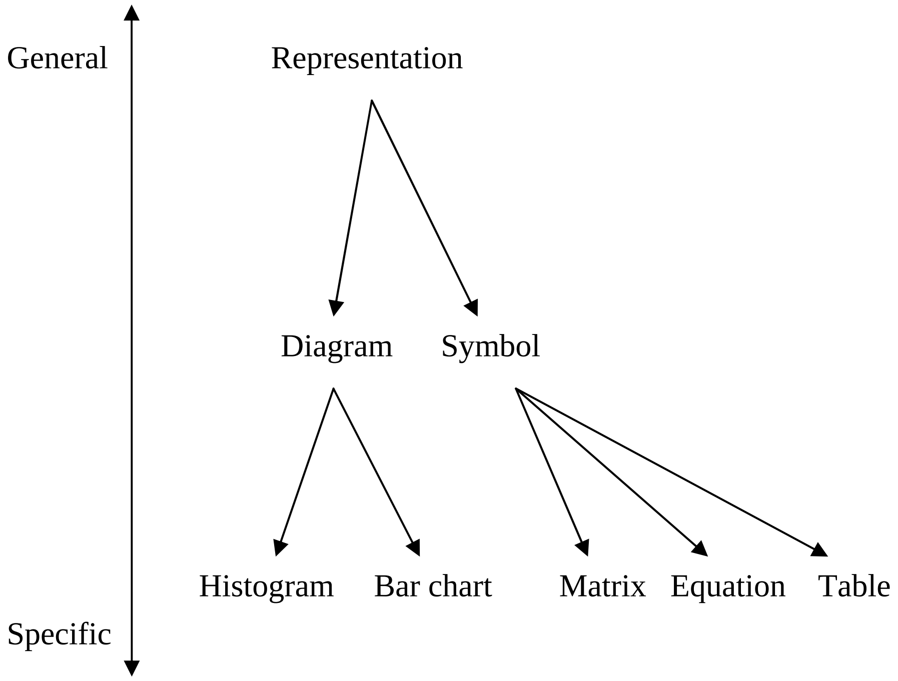

Figure 2: An example of hierarchy of heuristics

Here we need to clarify first that diagram, symbol or histogram, matrix themselves are not heuristics, but "use diagram", "use equation" and "draw table" are heuristics. Representations themselves are not heuristics, however suggestions to use representations are. Of course Figure 2 is only part of the whole hierarchy of heuristics for representations, and we can have similar hierarchy for "simplification", "pathway", and "bring in solution".

Heuristics on the top of the hierarchy are just very basic and general ideas that can be applied to most problems. These ideas can be further broken downward to make them more specific, making them easier to follow and apply to problems. However by doing so, we have restricted the applicability of the heuristics to only a specific few types of problems. Heuristics with specific procedures are usually less applicable than those with fewer procedures. Besides that, specific heuristics require less of problem solvers' interpretation and intuition or creativity. As we can see in Figure 2, the "heuristics" at the bottom of the hierarchy are pointing us to topics that we learn in mathematics lessons that come with a lot of procedures and rules governing how to create and manipulate them. After all, we can see mathematics as a big collection of tools or "representations" in which we use to solve problems.

Learning and teaching heuristics

Before we start talking about the teaching of heuristics, it is important for us to understand what it means to have learned heuristics, and how they are taught. "The aim of heuristic (here heuristic refers the branch of study, not a heuristic strategy) is to study the methods and rules of discovery and invention" (Polya 1973, p. 112). Heuristics help us discover and invent, and in case of problem solving, discover and invent solutions to problems or procedures to solve problems. Heuristics are not the discoveries and inventions themselves, but methods and rules of discovery and invention, so when we learn heuristics, we are learning ideas or strategies, not the actual discovery and invention.

Of course, as mentioned above, this does not mean that heuristics guarantee us discovery and invention, but they give us a sense of direction and rules of thumb for making sense, especially when we run into difficult and unfamiliar problems. When we have learned heuristics, we have grasp ideas and apply them to help us solve problems. It can not be emphasized enough that heuristics are about ideas not procedures.

Not all heuristics that can be used to solve problems are taught explicitly by teachers. In fact, sometimes the heuristics that we use are discovered from our own problem solving experience, or are identified when we observe other people solving problems. We can also learn them through examining and studying worked examples in textbooks. Schoenfeld (1985) summarizes these processes as:

> Occasionally the person solves a problem using a technique that was successful earlier, and some thing clicks. … If that method succeeds twice, the individual may use it when faced with another similar problem. In that way a method becomes a strategy. Over a period of years each individual problem solver comes to rely – quite possibly unconsciously – upon those methods that have proven useful for himself or herself. That is the individual develops a personal and idiosyncratic collection of problem solving strategies. (p. 70-71)
>
> By means of introspection (Polya's method) or by making systematic observations of experts solving large number of problems, it might be possible to identify and characterize the heuristic strategies that are used by expert problem solvers. (p. 71)

The idea of teaching heuristics explicitly is so that we can expedite these processes of discovering and identifying heuristics, and apply them in problem solving. Whether it is possible to simply expect the discovery and identification of processes, rather than explicitly teaching the heuristics, is questionable, it definitely requires more time and effort and is a less efficient approach to learning heuristics. However, the heuristics learned through discovery and identification are usually ones that we use most often, since we have a much better understanding and appreciation of them.

The common way of teaching heuristics explicitly is to teach them as if they were mathematical concepts or skills. We teach one heuristic at a time and give students problems that can be solved by that particular heuristic at the end of the lesson. The problem with such an approach is that we have isolated each heuristic from the others, since often the relationships between heuristics and the ideas for these heuristics are not mentioned. Students might treat these heuristics as algorithms; procedures they need to follow when they solve problems without really understanding the ideas behind the heuristics, why and when they should them.

Schoenfeld (1985) has taken a different approach to teaching heuristics. He argued that “most “general heuristic strategies” are so broadly defined that their definitions are far too vague to serve as a guide to their implementation” (p. 95). He proposed breaking heuristics down to what he called “more precise and usable descriptions of heuristic strategies” or substrategies. Students not only need to learn these substrategies, they also need to learn how to break down “general strategies” to substrategies.

> These conditions (for transfer) include breaking down the “general heuristic strategies” into a collection of coherent substrategies, carefully delineating a fairly large sample of these, discussing the conditions under which they seem to be appropriate, and using problems from new domains for practice. (Schoenfeld 1985, p. 95)

We should note that when Schoenfeld proposed this approach, he only meant it for students with reasonable mathematics background, college students or older.

Top-down approach to teaching heuristics

Here we try to propose an alternative approach to teaching heuristics. This approach can be seen as an extension of Schoenfeld’s (1985) approach as it also involves breaking down general heuristics to specific heuristics; however here we start from even more general heuristics, using the problem solving model in Figure 1 above as a structure in which we teach heuristics.

The idea of such an approach is to introduce to students the ideas behind the heuristics, why these heuristics are used, first before we start breaking them down to more specific heuristics and apply them to problem solving. We call this approach the top-down approach to teaching heuristics because we start teaching heuristics from the heuristics on the top of the hierarchy of heuristics we, and slowly work our way down the hierarchy. Again using the example in Figure, we start by first teaching the concepts and ideas of “representations”, then we move down to the heuristic “draw a diagram”, and then move further down to “draw a histogram”.

Also, by deducing heuristics to four simple general heuristics, younger students might have a better chance of understanding and applying heuristics. Besides, with such a structure it will be easier for students to attach meaning to new heuristics that they might encounter, making these heuristics easier to remember and apply. Here we borrow Bruner’s idea of structural teaching, we teach the ideas and structure first before we go into detail.

> Grasping the structure of a subject is understanding it in a way that permits many other things to be related to it meaningfully. To learn structure in short is to learn how things are related. (Bruner 1960, p. 7)

Here we give an example of how the top-down approach can be applied in classrooms.

- First of course we start by giving students a few problems, preferably non-routine or ill-structured problems, problems that can not be solved easily by the students, or else there will be no point talking about heuristics.

- Give a brief introduction of what representations are by showing students a few examples, preferably three examples, one enactive (physical), one iconic (visual), and one symbolic. Discuss with them which representations do they prefer and why.
- Give students some time to come up with their own representations for the problems. Through students' representations, we can understand better how students actually come to understand problems.
- Ask some students to present to the rest of the class their representations and explain why they have represented the problems that way. Ask other students to comment on these representations.
- Now give students time to solve the problems with the representations they have created.
- Again ask students to present their works, if they have managed to solve the problems, and discuss with the class why they have done what they have done.
- Introduce the idea of simplification, again giving a few examples and explain why these heuristics can be useful to solve the problems.
- Also introduce the idea of pathway, explain to students problems have a starting state and ending state, and where we start solving the problems can have different level of difficulties.
- Give more time for students to try to solve the problems, and let them present and discuss their solutions.
- Introduce the concept of "bringing in solutions" as ways to solve problems, and ask them to comment on such methods, and why and when do they think they will be useful.
- Now ask them to solve the problems using as many ways as they possibly find, and comment on which way they think they prefer, and why.

As we can see from the example above, top-down approach to teaching heuristic requires heavy involvements from students. It gives students chance to use their creativity to solve problems, from creating their own representations to discover or invent their own solutions, and creating their own heuristics. Again, here we are emphasizing the ideas not the procedures. The problems we use play a crucial part in the whole process, therefore we have to carefully select the problems to better facilitate the discussions with the students.

As mentioned in the section in the section before, topics in mathematics textbook can be considered as forms of representation. We can introduce them to students as representations to solve the problems and ask them to discuss whether these representations are better than the ones they themselves have come up with. Teacher then will need to explain to students the strength of these "representations". Approaching textbook topics will make the topics more meaningful for students and they can appreciate the topics more also.

Summary

In this paper we have proposed a top-down approach to teaching heuristics, where we put most of the emphasis on the ideas of the heuristics, why these heuristics are useful, not on the heuristics themselves. Also, unlike Schoenfeld's approach, top-down approach is meant for upper primary and lower secondary students, although empirical data will need to be collected to determine the effect of such an approach on students learning of heuristics and problem solving in general. However we do see from the example given that such an approach will be demanding on both teacher and students, especially teacher and students are new to the approach. Here we also try to suggest a more problem-based approach to teaching topics in mathematics textbook.

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