Students’ Challenges to Reason Quantitatively When They Solve Mathematical Word Problems

Luis Tirtasanjaya LIOE  
Yan LIU  
Yanping FANG  
Centre for Research in Pedagogy and Practice, National Institute of Education, Nanyang Technological University, Singapore  
Email: luis.lioe@nie.edu.sg

Abstract

Students’ limited view of mathematics and their strong orientation on numbers have been major hindrances in developing students’ capacity in problem solving. Our study aims to understand students’ challenges in reasoning quantitatively and gain insights on how they overcome these challenges. To this end, seven pairs of Grade 5 students and eight pairs Grade 7 were observed and videotaped when they solved one routine task and one non-routine task. Their processes in solving these two tasks were analyzed using the method of conceptual analysis (Glasersfeld, 1995). Findings suggest that there were two major areas of difficulties that they experienced in reasoning quantitatively. Some of the major challenges behind those difficulties were the lack of quantitative understanding in whole numbers and fractions, the stronger focus on the magnitudes of quantities than the quantities themselves, and the stronger reliance on visually-perceived relationships in the model representation of situation than the quantitative relationships. A case of a Grade 7 pair is presented to discuss a context of pair collaboration that fosters the overcoming challenges in reasoning quantitatively that suggests a practical implication of using collaborative learning to support students’ development of quantitative reasoning capacity.

Overview and Objective

The central focus of Singapore mathematics curriculum framework is to develop students’ capacity in mathematical problem solving (MPS) in terms of dealing with mathematical problems that cover a wide range of problem situations and making use of the relevant mathematical knowledge and thinking processes (Curriculum Planning and Development Division, 2000). The biggest hindrances are students’ limited view of mathematics and their strong orientation on numbers when they solve mathematical problems. Yeap & Kaur (1996) found that students tended to view mathematics as a collection of tools and algorithms to be used to obtain definite numerical answers to given word problems. Foong & Koay (1997) found that students also tended to disregard the problem situation and directly inferred the needed arithmetic operation to combine numerical data presented in the problems. Evidence from Hedberg, Wong, Ho, Lioe, & Tiong’s (2005) and Lioe, Ho, & Hedberg (2006) confirmed such stereotypical thinking, and also showed that students experienced difficulties in dealing with mathematical problems that are beyond routine classroom type of problems and those that require higher level thinking and reasoning. In order to inform ways that might help them understand and overcome these challenges, it is essential to investigate types of challenges that they experience.
Research on students’ challenges in solving word problems in Singapore have shed light on students’ behaviors of successful and unsuccessful problem solvers (Foong, 1994; Wong, 2007; Lioe, Ho, & Hedberg, 2006; Yeap & Menon, 1996) and on developing a problem-solving model for unsuccessful solvers to adopt (Teong, 2003, Lioe, Ho, & Hedberg, 2005). However, little research done that focuses on understanding students’ thinking that might have led them to behave in ways that they demonstrate as they solve problems. Therefore, the objective of this study is to investigate the types of challenges that Singapore students experience when they try to reason quantitatively as they solve word problems and gain insights on ways of overcoming them from ways they struggle with these challenges.

**Theoretical Framework**

One of the most important aspects of the students’ mathematical ability is to see the problem situation as a network of quantities and their internal relationships, and to reason with them by forming a series of quantitative operations in the establishment of the solution. Such reasoning is called quantitative reasoning. A quantitative operation refers to a mental operation of combining two or more quantities to produce a new quantity (Thompson, 1993). The actual evaluation of this new quantity by substituting each quantity by its magnitude is called numerical operation. The result of such numerical operation is the magnitude of the new quantity. For example, if one’s income is $3,000 a month and his monthly expenses is $2,000, conceiving “savings = income – expenses” is a quantitative operation of combining two quantities, “income” and “expenses”, to produce a new quantity “savings”. The evaluation of the amount of “savings” by substituting both “income” and “expenses” by their magnitudes, i.e. $3,000 and $2,000, is a numerical operation.

From the above definitions and examples, it could be seen that in reasoning quantitatively, numbers are of secondary importance and most of the time reasoning takes place independent of the numbers. In fact, quantitative reasoning contextualizes how those numbers, if any, are applied. Its theoretical model has been well developed by Thompson (Thompson, 1988, 1993, & 1995), and further theorized to be a base in the development of algebraic reasoning (Smith and Thompson, 2007).

The theoretical assumption behind our method of analysis is that students have their own mathematics (Confrey, 1991). By that, we mean that in our communication with students about mathematical ideas or when they are solving mathematical problems, they can have their own interpretations that we are not aware of, or we are aware of, but we simply dismiss them as being wrong, even though they might have made sense to students themselves. In order to unravel these challenges, we employ a conceptual analysis approach (Glasersfeld, 1995). The base for this analysis is to view any mental content (percepts, images, concepts, thoughts, words, etc) as a result of mental operations (Thompson & Saldanha, 2003). In principle, the main objective of conceptual analysis is to answer the question: “What mental operations must be carried out to see the presented situation in the particular way one is seeing it?” (Glasersfeld, 1995, p. 78). Such mental operations are called conceptual operations. In other words, we use conceptual analysis to answer the question “what can this student be thinking so that his/her actions make sense from his perspective?”
In order to observe mental operations that are carried out by students, it is essential to have students articulate their mental activities which are usually hidden. An effective way to make students externalize their internal mental activities is through thinking-aloud procedure where students are asked to constantly verbalize their thoughts aloud. Although such procedure is proven effective, there is a risk of students being uncomfortable with such unnatural setting and eventually will hinder their mental activities. To tackle this issue, Teong (2003) combined thinking-aloud procedure with Schoenfeld’s (1985) pair-work setting to make it natural for students to verbalize their thoughts aloud. The discussions with peer, such as clarifying ideas and assumptions, discussing interpretations, and decision making, have higher chance to stimulate the externalization of their mental activities in a natural way and at the same time prevent the hindrance of their mental activities. Teong’s adaptation of pair work and thinking-aloud procedure becomes a base for data collection method in this study.

**Methods and Data**

This study drew its data from a funded 3-year project titled “Developing the Repertoire of Heuristics for Mathematical Problem Solving” at the Centre for Research in Pedagogy and Practice, Singapore. The project observed two batches of students when they solved five word problems in pairs from February to October 2004. The first batch consists of 14 Grade 5 students from two primary schools and 16 Grade 7 students from two secondary schools and the second batch consists of 24 Grade 5 students from three primary schools and 20 Grade 7 students from three secondary schools. Each session lasted from 30 to 60 minutes.

Among all pairs of students in the first batch of Grade 5 and Grade 7 students, seven pairs of Grade 5 students and eight pairs of Grade 7 students were selected to be used in this study. Table 1 lists all the fifteen pairs with pseudonyms starting from the first pair from Grade 5 to the eighth pair from Grade 7. The abbreviations PP and SP refer to “Grade 5 pair” and “Grade 7 pair” respectively by adopting from the term “primary school” for elementary school and “secondary school” for middle school in the Singapore context.

<table>
<thead>
<tr>
<th>No</th>
<th>Grade 5 (PP)</th>
<th>No</th>
<th>Grade 7 (SP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PP1</td>
<td>Francisca and Zoe</td>
<td>SP1</td>
<td>Jerry and Richard</td>
</tr>
<tr>
<td>PP2</td>
<td>Gerry and Kelvin</td>
<td>SP2</td>
<td>Christine and Mary</td>
</tr>
<tr>
<td>PP3</td>
<td>Jack and Ester</td>
<td>SP3</td>
<td>David and Ray</td>
</tr>
<tr>
<td>PP4</td>
<td>Abraham and Robin</td>
<td>SP4</td>
<td>George and Julie</td>
</tr>
<tr>
<td>PP5</td>
<td>Joanne and Yvonne</td>
<td>SP5</td>
<td>Nelly and Susan</td>
</tr>
<tr>
<td>PP6</td>
<td>Billy and Leo</td>
<td>SP6</td>
<td>James and William</td>
</tr>
<tr>
<td>PP7</td>
<td>Johnson and Chad</td>
<td>SP7</td>
<td>Ron and Neville</td>
</tr>
<tr>
<td></td>
<td></td>
<td>SP8</td>
<td>Alvin and Samuel</td>
</tr>
</tbody>
</table>

Two of the five word problem items that deal directly with quantitative reasoning were used, see below:
Thompson (1993) used the term *quantitative complexities* to refer to the kind of complexities that demand different level of quantitative reasoning in solving them. With regard to such complexities, the second task is more complex than the first task. There were at least four differences that constitute higher level of quantitative complexities in Q2 than Q1. The four differences are listed in Table 2.

<table>
<thead>
<tr>
<th></th>
<th>Q1</th>
<th>Q2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Resembles <em>routine</em> types of word problems</td>
<td>1. Resembles <em>non-routine</em> types of word problems.</td>
</tr>
<tr>
<td>2</td>
<td>The magnitudes of all quantities are made explicit.</td>
<td>2. The magnitudes of all quantities are all hidden.</td>
</tr>
<tr>
<td>3</td>
<td>All the quantitative relationships are made explicit.</td>
<td>3. There is a hidden quantitative relationship.</td>
</tr>
<tr>
<td>4</td>
<td>Deals with <em>whole numbers</em>, requires students to reason additively.</td>
<td>4. Deals with <em>fractions</em>, requires students to reason multiplicatively</td>
</tr>
</tbody>
</table>

From Table 2, the second task resembles *non-routine* task while the first task resembles *routine* task based on Foong (2002)’s task classification. One point that makes the second task non routine is that it requires students’ familiarity with the real-life context such as marriage practices adopted in society. Hence, not all quantitative relationships are readily translated into arithmetical expressions since some assumption made about real-life situation is required. An example of such an assumption is the relationship drawn from the monogamous marriage practice adopted which is close to marriage practices in Singapore. Secondly, the magnitudes of quantities in Q2 are all hidden and hence demand students to reason based on quantities and quantitative relationships instead of numbers and numerical relationships. Thirdly, there is a hidden quantitative relationship in Q2, which is the relationship between the number of men and women in the town. Hence, it requires students to unpack this hidden quantitative relationship to see the whole network of quantities in Q2. Lastly, Q2 deals with fractions that have higher demand on students’ multiplicative reasoning instead of additive reasoning as in Q1 (Thompson & Saldanha, 2003). All these four differences suffice to guarantee that the second task requires higher level of quantitative reasoning than the first task. Hence, the use of two tasks with different complexities are hoped to understand possible challenges that students might experience in solving a wide range of problems, and to inform ways of helping students overcome these challenges.

**Results**

The conceptual analysis on the seven Grade 5 pairs and eight Grade 7 pairs when they solved Q1 and Q2 suggest that students experienced challenges in reasoning quantitatively at least in the following two areas: 1) In unpacking/understanding

**Q1.** A group of tourists paid $200 for admission to a theme park. Adults paid $8 each and children $4 each. If there were 7 more adults than children, how many adults and children were there in the group?

**Q2.** In a certain town, two-thirds of the adult men are married to three-fifths of the adult women. What fraction of the adults in the town are married?
quantitative relationships in the problem situation and 2) in differentiating quantitative operations from numerical operations. The former refers to how students conceive quantitative relationships from their understanding of the problem situation, represent the network of quantities if necessary, and use the quantitative relationships that they have conceived in their reasoning. The latter specifically refers to the performed calculations or procedures, of which the students based the calculations on numbers instead of quantities. Within each area, there are specific challenges that only surfaced in the non-routine task. The following discusses cases of students’ difficulties with regard to solving Q1 (routine) and Q2 (non-routine) tasks.

**Students’ challenges in reasoning quantitatively when they solved Q1 (routine task)**

Two difficulties were noted in each area of students’ difficulties when they solved the routine task. The two difficulties under students’ difficulties in unpacking/understanding quantitative relationships (category A) were A1) conceiving inconsistent quantitative relationships and A2) ignoring crucial quantitative relationships. The two difficulties under students’ difficulties in differentiating quantitative operations and numerical operations (category B) were B1) failing to identify a quantity evaluated by a calculation and B2) conceiving inconsistent quantitative relationships. In the following discussions, each of these four difficulties is illustrated by cases of the subjects who experienced it.

**A1. Conceiving inconsistent quantitative relationships**

PP2 illustrate the case when students conceived inconsistent quantitative relationship (A1). In solving Q1, they conceptualized a new quantity, which is a “pair”, by combining the given quantities, “children” and “adults” in the group of tourists. In conceptualizing a “pair”, they conceived a quantitative relationship “1 pair = 1 adult + 1 child”. However throughout their reasoning, such quantitative relationship was conceived differently as shown in the following excerpt taken from Turn 53 – 57 in the transcript.

<table>
<thead>
<tr>
<th>Turns</th>
<th>Speaker</th>
<th>Utterances</th>
</tr>
</thead>
<tbody>
<tr>
<td>53.</td>
<td>Gerry</td>
<td>There are 12 pairs of</td>
</tr>
<tr>
<td>54.</td>
<td>Kelvin</td>
<td>No, not pair, yea, yea, yea, eh?</td>
</tr>
<tr>
<td>55.</td>
<td>Gerry</td>
<td>Pair. Because its tenses are sets.</td>
</tr>
<tr>
<td>56.</td>
<td>Kelvin</td>
<td>There are 12 pairs of adults and children. OK, then we will take 12 divided by 2, and 6, this is, there are 6 children.</td>
</tr>
<tr>
<td>57.</td>
<td>Gerry</td>
<td>And adults.</td>
</tr>
</tbody>
</table>

This excerpt shows that in the later reasoning, they conceived “12 pairs = 6 adults + 6 children” which was not consistent with the initial conceived quantitative relationship “1 pair = 1 adult + 1 child”. It was noted that students’ cognitive challenge behind experiencing this difficulty was the lack of quantitative understanding in differentiating the number of “pairs” and the number of “people in pairs”.

**A2. Ignoring crucial quantitative relationships**

An example of students ignored some crucial quantitative relationship (A2) was shown by PP3. The students ignored the quantitative relationship “# of adults = # of children + 7” that came from the fact that there were 7 more adults, and conceived an operation of dividing the total amount paid by the “whole tourists”, which is $200, by the amount paid
by 1 pair of adult and child, which is $12. In this way, they ignored the existence of seven more adults and treated the number of adults as the same as the number of children. In this case, the students’ cognitive challenge behind experiencing this difficulty was driven by their objective of finding the right combination between “adults” and “children”, which suggests the stronger focus on the magnitudes of quantities than the quantities themselves.

**B1. Failing to identify a quantity evaluated by a calculation**

SP3 gives an example of students failed to identify a quantity evaluated by a calculation (B1). Figure 1 shows the numerical operation that SP3 performed and the quantity of which they failed to identify is circled.

![Figure 1: “something” = amount paid by all pairs ÷ # of units](image)

From Figure 1, prior to drawing the model\(^1\), SP3 had calculated the “amount paid by all pairs of adults and children”, i.e. 144. The model represents the amount paid by one adult and one child, of which the amount paid by one child ($4) was taken as “1 unit”\(^2\). They then conceived the operation of dividing the amount $144 by the number of units (3) but failed to identify what quantity does 1 unit associate with. The fact that this pair conceived a division: “amount paid by all pairs of adults and children ÷ amount paid by 1 pair of adult and child” suggests that they have actually conceptualized a quantitative operation to get the number of pairs of adults and children. However, this conceptualization was interfered by their perception on the model that they used. They were unable to re-connect the operations of dividing by the number of units to the quantitative operation of finding the number of pairs.

**B2. Conceiving inconsistent identification of a quantity**

PP4’s reasoning is used to illustrate students’ inconsistent identification of a quantity. Figure 2 shows the numerical operations that PP4 performed and the numbers, of which the students inconsistently identified with a quantity, are circled.

![Figure 2](image)

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\(^1\) A “model” in Singapore context refers to a diagram, normally made up of rectangles, to represent the network of quantities in the problem situation.

\(^2\) “Units” is the term used in Singapore context that refers to the rectangles in the model. In using model method, the units that represent the quantities of known magnitudes are used as a base of reasoning to evaluate the value of the units that represent the quantities of unknown magnitudes.
Figure 2: Inconsistent identification of “134” as the amount paid by all pairs and the number of pairs

Figure 2 shows that PP4 calculated “200 – 56 = 134” in the second step of their calculation. Their intention in performing this step was to find “the amount paid by all pairs of adults and children”. However in the following step, they identified “134” as “the number of all pairs”. Hence, they divided this number by two to get the number of children in the group. It was noted that the conceptualization of divisor “two” was based on the model representation that they drew, which was the two boxes that represented the quantities with “unknown” values. In this case, the students’ reasoning in deriving an operation is based on their visually-perceived relationships in the model representation that they drew. The stronger reliance on visually-perceived relationships than on quantitative relationships was the challenge behind this inconsistent identification.

We wish to note that not all inconsistent identification of a quantity was driven by students’ visually-perceived relationships. The findings also show that in some cases, students did not bear in mind the quantities that they had conceived in their quantitative operations in performing a numerical operation. Hence they mistook it for another quantity in the next calculation step.

Students’ challenges in reasoning quantitatively when they solved Q2 (non-routine task)

Similar to students’ experiences in solving routine task, the students also experienced all the difficulties that we have discussed in the previous section, which were A1, A2, B1, and B2. However in solving this non-routine task, there were two additional difficulties in unpacking/understanding quantitative relationships and one additional difficulty in differentiating quantitative operations from numerical operations.

The two additional difficulties under category A were A3) failing to identify hidden quantitative relationships and A4) embedding unintended quantitative relationships in model representation of a situation. Under category B, the additional difficulty was B3) basing the operations on numerical relationships. In the following, we shall illustrate the additional difficulties that were unique to this non-routine task, which were A3, A4 and B3, and the challenges behind experiencing these difficulties.

A3. Failing to identify hidden quantitative relationship

PP7 shows an example of students’ failing to identify hidden quantitative relationship. In this non-routine task, the hidden quantitative relationship was “# of men ≠ # of women” that was not mentioned in the problem statement. When PP7 drew the model to represent the situation, they made an assumption that the number of men and women were equal. By assuming this relationship, they drew the model as shown in Figure 3.

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3 Computational mistake was made by the students.
The existence of hidden quantitative relationship itself appeared to be the challenges that led them to make of such assumption. Firstly, this suggests that such situation was indeed unfamiliar to students. Secondly, this suggests students’ tendency to infer the hidden quantitative relationship by considering the simplest relationship (equal number of men and women) rather than trying to identify it by other means.

A4. Embedding unintended quantitative relationships in model representation of a situation

SP6 shows an example of students’ embedding an unintended quantitative relationship in the model that they drew. Figure 4 shows the model that they drew.

Students’ intention in drawing this model was to represent the “women” in five equal size fractional parts. However due to drawing the model in sketch, the unit size of “single women” was drawn close to the unit size of the “single men” which were circled. The students then visually perceived the equal unit size between “single men” and “single women”. Hence, the quantitative relationship: “1 fractional part of single men = 1 fractional part of single women” was the unintended relationship to the students, but it was conceived by the students and used in their reasoning. Hence, this relationship became “unintended relationship” that was embedded in the drawn model. Therefore, the challenge behind this difficulty was students’ reliance on visually-perceived relationships in their reasoning.

B3. Basing the operations on numerical relationships

Calculating “2/3 + 3/5” based on a numerical relationship: (Frac) married adults = (frac) married men + (frac) married women, is an example that was commonly found in both Grade 5 and Grade 7 pairs. Figure 5 shows a diagram of students’ drawn models prior to calculate “2/3 + 3/5”. From here, we could see that this calculation was disconnected from the model that they drew. By that, we mean that the conceived operation was not based on any quantitative relationships that they conceived earlier that were represented in their models or from the problem situation.
Overcoming challenges in reasoning quantitatively

Having discussed types of challenges in reasoning quantitatively, the issue of how students overcome such challenges raised. Specifically, the question raised is “does interacting with peer help the students to overcome their challenges in reasoning quantitatively? If so, what aspects of peer interaction that contributes to the overcoming challenges?” To address this issue, we will present one case of a Grade 7 pair, SP3, who was successful in overcoming the challenges that they experienced in Q1 (routine task). The difficulty in reasoning quantitatively that they experienced was failing to identify a quantity evaluated by a calculation (B1) and it had been discussed in the illustration of B1. The insight of ways this pair overcome their challenges will lead us to the discussion in the next section.

As earlier indicated, SP3’s conceived quantitative operation of which a quantity failed to identify was “something” = amount paid by all pairs ÷ # of units. The summary of their numerical operation had been displayed in Figure 1, and it is re-displayed in Figure 6 for convenience sake.

As discussed in previous section, the major challenge behind this difficulty was that their conceptualization of a quantitative operation in getting the number of pairs of adults and children was interfered by their visually-based relationships perceived from the model. They were unable to re-connect the operations of dividing by the number of units to the operation of finding the number of pairs. Recall that the model represented the amount paid by 1 pair of child and adult while model 1 represented the amount paid
by the whole group of tourists. The interaction that led them to overcoming the challenges is extracted from Turn 88 to 98.

<table>
<thead>
<tr>
<th>Turns</th>
<th>Speaker</th>
<th>Utterances</th>
</tr>
</thead>
<tbody>
<tr>
<td>88.</td>
<td>David</td>
<td>Um … 200, 144 ... 144 divided by 3 units.</td>
</tr>
<tr>
<td>89.</td>
<td>Ray</td>
<td>Can?</td>
</tr>
<tr>
<td>90.</td>
<td>David</td>
<td>144 divided by 3 units.</td>
</tr>
<tr>
<td>91.</td>
<td>Ray</td>
<td>Then after that?</td>
</tr>
<tr>
<td>92.</td>
<td>David</td>
<td>After that … the answer right?</td>
</tr>
<tr>
<td>93.</td>
<td>Ray</td>
<td>Equals to?</td>
</tr>
<tr>
<td>94.</td>
<td>David</td>
<td>Um…</td>
</tr>
<tr>
<td>95.</td>
<td>Ray</td>
<td>The answer equals to … for this one ((pointing at units))</td>
</tr>
<tr>
<td>96.</td>
<td>David</td>
<td>Because first divided by 3, and then the … adults is 2 units, then er… children is 1 unit. ((referring to model 2))</td>
</tr>
<tr>
<td>97.</td>
<td>Ray</td>
<td>Children is 1 unit can right?</td>
</tr>
<tr>
<td>98.</td>
<td>David</td>
<td>Try lor.</td>
</tr>
</tbody>
</table>

In the above conversation, Ray attempted to clarify the quantitative operations that David conceived. For example, the questions like “then after that?” demanded David’s justification on his sequences of quantitative operations and “equals to?” demanded David’s justification on what quantity he evaluated. David’s responses to Ray indicated the base of his reasoning was on the visually-perceived relationship in the model. He could not justify what quantity he evaluated. The following is the continuation of their conversation.

<table>
<thead>
<tr>
<th>Turns</th>
<th>Speaker</th>
<th>Utterances</th>
</tr>
</thead>
<tbody>
<tr>
<td>99.</td>
<td>Ray</td>
<td>Adult is 2 units, this one ((pointed at model 2)) is for money leh, this one is population.</td>
</tr>
<tr>
<td>100.</td>
<td>David</td>
<td>((refers to model 2)) No you see ah, because of the 144 right? 144… then after that children in 1 unit, adults is 2 units. See ah, divided by 3 what! Right? And then the answer times 2 equals to adults, and then answer times 1 equals to children lah.</td>
</tr>
<tr>
<td>101.</td>
<td>Ray</td>
<td>Wait wait total equals to 144.</td>
</tr>
<tr>
<td>102.</td>
<td>David</td>
<td>Then 144 add the 7 adults lah.</td>
</tr>
</tbody>
</table>

From the above conversation, Ray knew that David could not justify the quantity evaluated by his operation due to the use of model 2. Ray knew that the 3 units represented the “money” or the amount paid by pairs. He pointed out to David that the quantity that they want to evaluate was the “population” which refers to the number of adults and children. On the other hand, David did not get what Ray was referring to. Instead, he reiterated his operations that he based on visually-perceived relationships in model 2 in Turn 100. In that sense, Ray’s attempt to clarify the quantity evaluated was not successful. However, instead of re-explaining to David, it could be seen in Turn 101 that he went interpret David’s reasoning again. This led to the following conversation.

<table>
<thead>
<tr>
<th>Turns</th>
<th>Speaker</th>
<th>Utterances</th>
</tr>
</thead>
<tbody>
<tr>
<td>103.</td>
<td>Ray</td>
<td>But you see ah, if 144 divided by 3 ah, see ah ((mumbles calculating, David calculated “144 divided by 3” on the paper)) equals to … 48 ... 48... 48 plus 7... 48 dollars</td>
</tr>
<tr>
<td>104.</td>
<td>David</td>
<td>48 ((stated the result of his calculation))</td>
</tr>
<tr>
<td>105.</td>
<td>Ray</td>
<td>Oh I know!</td>
</tr>
<tr>
<td>106.</td>
<td>David</td>
<td>48 equals to what er ... 1... 1 unit</td>
</tr>
</tbody>
</table>
Ray’s willingness to interpret David’s reasoning brought to fruition of his attempt to clarify David’s failing to identify a quantity. He let David to calculate his operation first, i.e. \( 144 \div 3 = 48 \), of which David identified it as the amount associated with 1 unit in model 2. Ray then based his clarification from result of David’s calculation, i.e. from the relationship “1 unit \( \rightarrow \$48 \)”. Ray identified it as “the amount that the children have”, which referred to the whole children. Ray then continued with a quantitative operation “dividing the amount that all children have by the amount paid by each adult” to get “the number of children”. As shown in Turn 110, with this David was able to identify the quantity that he earlier failed to identify and hence his difficulty had been overcome.

**Discussion and Implications**

This study attempted to understand the challenges that Singapore students experienced to reason quantitatively as they solve mathematical word problems. Through pair interactions and thinking-aloud instruction, some of the students’ mental activities that constitute the experiences of the challenges were surfaced and analyzed through conceptual operations. We have identified some areas and types of challenges that might have prevented them from reasoning quantitatively, that include unpacking/understanding quantitative relationships in the problem situation and distinguishing quantitative operations from numerical operations. We have also discussed a case of collaborative pair work in overcoming the challenges that the student experienced.

**Students’ challenges and pedagogical interventions**

The use of the routine and non-routine tasks enables us to identify the difficulties that Grade 5 and Grade 7 students experienced to reason quantitatively and the students’ cognitive challenges behind experiencing these difficulties. In particular, the use of non-routine task had also enabled us to identify difficulties and challenges that may not surface when the students deal with routine problems. This insight gives a glimpse of various challenges that the students had to reason quantitatively in a range of mathematical problems and may be helpful for teachers to foresee in which area the student might have struggled in reasoning quantitatively if a certain task type is applied.

In summary, there were four challenges behind the difficulties that the students experienced in unpacking/understanding quantitative relationships and differentiating quantitative operations from numerical operations when they solved both routine and non-routine tasks. The challenges were: 1) students’ stronger focus on the magnitudes of quantities than on the quantities, 2) students’ stronger reliance on visually-perceived relationships in a model than quantitative relationships, 3) not keeping in mind some
quantities in the conceived quantitative operations when performing numerical operations and 4) the lack of quantitative understanding in whole numbers and fractions.

The first three challenges related to students’ orientation in their reasoning. The focus on the magnitudes of quantities, visually-perceived relationships in a model, and not keeping in mind some quantities are the opposite of the orientation in quantitative reasoning. Hence, some pedagogical interventions are needed to re-orient the students back to quantitative orientation. As shown in SP3’s case of overcoming difficulties, to address David’s challenges in reasoning quantitatively, Ray’s questions such as “then after that?” demanded David’s justification on his sequences of quantitative operations and “equals to?” demanded David’s justification on what quantity he evaluated. By responding to these questions, only then the challenges that David experienced was surfaced and offer opportunity for Ray to help David in overcoming his challenges.

The insights from SP3’s case suggest some pedagogical interventions that teachers can use to assist students in understanding and overcoming such challenges. First, by holding conversation in mathematical problem solving discussion with students: expecting students to take responsibility for explaining their reasoning, interpretations of the tasks, assumptions, and decisions that they made in approaching the problems. As shown in David’s justification of his reasoning led to the surfacing of his challenges, teachers can also identify the challenges that the children experience. David’s questions also suggest some ways of asking students to justify their reasoning. In fact, David’s questions are in the same line as the types of questions that Thompson (1993) suggested which are “this is a number of what?”, “what are you trying to find?” and “what did this calculation give you?” (Thompson, 1993). Such questioning could be made a habit for students so that they can constantly orient themselves in reasoning quantitatively when they solve mathematical problems. Especially when they work collaboratively, these questions can be used as a guideline for students to check the other students’ reasoning.

The fourth challenge related to students’ lack of quantitative understanding and appeared to be the major challenge especially in non-routine task. In the routine task, the quantitative understanding related to whole numbers was the understanding in differentiating the number of “pairs” and the number of “people in pairs”. In non-routine task, the quantitative understanding related to fractions was conceiving fraction as an expression of quantitative relationships. Such situation warrants further study to examine how the curriculum and practices supports students’ development of quantitative understanding of some mathematical ideas, especially in how to enhance the instruction to better facilitate students’ conceptual development of mathematical ideas. For example, Steffe (2001, 2003), Saenz-Ludlow (1994), Tzur (1999, 2004), Olive & Steffe (2002), Olive & Vomvoridi (2006) show that children’s construction of fractions and fractional schemes involve series of representational-based activities where children deal with manipulatives or computer program to develop quantitative understanding of fractions and conceptualize schemes to operate fractions such as addition, multiplication, division, and fractional equivalence. Hence, traditional teaching instruction may need to be supported in order to help students develop this understanding. The practice of using “model methods”, which is representation based, as the main tool to approach mathematical tasks among Grade 5 Singapore schools might offer a good opportunity to enhance such instruction. However further study needed to examine how model method is used, whether or not it severs as a problem-solving procedure to apply or as a means to
bridge students’ quantitative reasoning that leads to conceptualization of mathematical ideas.

**Context of pair interaction that fosters students’ overcoming challenges**

The last point of discussion in this paper is how the successful case of SP3 informs a context of pair interaction that fosters students’ overcoming challenges. From the students’ interaction in SP3, it suggests that overcoming the difficulties in reasoning quantitative entails the situation of which at least one student in the pair has quantitative orientation. If both students did not orient themselves in quantities and quantitative relationships, there is a possibility of having a case where they did not experience the difficulties in reasoning quantitatively as a difficulty and the challenges behind as challenges. And hence, to the students there is nothing needs to overcome.

In SP3, Ray’s quantitative orientation played a major role in the negotiation with David that led to overcoming difficulties. It began with the quantitative-oriented student became aware of their experiences of the difficulty in reasoning quantitatively. In the case of SP3, Ray knew that there was a quantity that his peer (David) failed to identify in his conceived quantitative operation. This led to the second factor, i.e. Ray’s willingness to interpret the David conceived quantitative operations. Only after the quantitative-oriented student was able to see his or her peer’s conceived quantitative operations, he or she could clarify the mistake to the peer. Part of quantitative-oriented student’s attempt in understanding his or her peer’s conceived quantitative operation was by asking the peer to clarify his/her quantitative operation, of which the types of questions had been discussed above. The third factor is Ray’s ability to communicate his/her quantitative operation and addressed the challenges behind the difficulty. In the case of SP3, the major challenge behind David’s failing to identify a quantity (B1) was his reliance on the visually-perceived relationship in the model. Hence, Ray helped David to make connection between the unit in the model and the quantity they wanted to evaluate. Once this challenge was addressed, i.e. David saw the connection between the unit in the model and the quantity he wanted to evaluate then his failing to identify a quantity was overcome. The fourth factor which is also important is the willingness from the student who experienced difficulties to listen to the quantitative-oriented student who attempts to communicate his idea.

All the four factors above led the David who experienced difficulties in reasoning quantitatively to re-orient him back to quantities and quantitative relationships. By the fact that David eventually was able to make the connection between his visual-perceived relationships and the quantitative relationships, suggest that he had also developed a quantitative understanding required to solve this task, which is in this case in whole numbers. Recall that the lack of quantitative understanding was the major challenge in the non-routine task. Although it was not mentioned above, some cases show that they had quantitative orientation in solving this non-routine task, but the lack of quantitative understanding of fractions hindered the overcoming difficulties. Therefore, quantitative understanding of mathematical idea becomes the fifth factor in the context of pair interaction that foster students’ quantitative reasoning and the overcoming challenges.

This context may be useful for teachers to consider, especially in employing pair work or group work instruction. By recognizing these five factors, teachers may be able
to select students in forming a group or a pair, where the productive interaction can be expected. For example, pairing up a quantitative-oriented student with a numerical- or visual-oriented student might increase the chance of overcoming any difficulties that they experienced as compared to pairing students who both lack in quantitative orientation. With this, the opportunity of having students’ active learning, as advocated by many problem-solving based curriculums, e.g. National Council of Teachers of Mathematics (NCTM) standards (2000) and the Board of Studies NSW (2002), including Singapore mathematics curriculum, might increase and optimized. In particular, the spirit of teaching via problem solving (Schroeder and Lester, 1989) can be translated into the practices alongside the development of students’ quantitative reasoning capacity.

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