Mathematical Problem Solving in Singapore Schools

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Kaur (1995) investigated the problem-solving strategies used by Singapore students in Primary 5, 6, Secondary 1 and 2 to solve non-routine problems and attempted to establish the possibility of a "gap", or time interval, between a problem solver's ability to carry out particular mathematical calculations and operations, and the ability to solve problems employing the same mathematics content knowledge. She also investigated problem solvers' difficulties when solving non-routine problems, given that they possess the necessary mathematical knowledge and skills, and the appropriate behaviours and strategies adopted by good novice problem solvers.

A total of 626 subjects participated in the study with 156 from Primary 5, 157 from Primary 6, 157 from Secondary 1, and 156 from Secondary 2. For all the year levels, the sample was representative of the general student population in Singapore schools. These students had not been exposed to any explicit mathematics problem-solving instructional programme in school. Three paper-and-pencil instruments were developed and used in the study:

- A Problems Test (Appendix A)
- A Computation Test (Appendix B)
- More Problems to Solve (Appendix C)

The data collected were analysed using appropriate techniques and the ten specific questions which guided this investigation, and research among Singapore students from Primary 5, 6, Secondary 1 and 2 were investigated.

KEY FINDINGS

Question 1 *Is there a time interval between a problem solver's ability to carry out particular mathematical calculations and operations and the ability to solve problems employing the same mathematics content knowledge?*

It was found that many of the subjects who were able to perform well on the Computation Test items were not able to solve the corresponding Problems Test items. This finding suggests that there may be a "gap" or time interval between a problem solver's ability to carry out particular mathematical calculations and operations and the ability to solve problems employing the same mathematics content knowledge. This finding lends support to the claims made by Burkhardt (1988) and Bastow *et al.* (1984) that such a "gap" exists for many students.

Question 2 *If this time interval exists, what are some of its characteristics?*

It was found that, in general, the difference between each of the means for a Computation Test item and the corresponding Problem item narrowed with increasing year levels for all the five problems. It was also noted that increasing proportions of subjects through increasing year levels were successful on the Problem items, given that they were also successful on the corresponding Computation Test items. This finding suggests that the "gap", or time interval, narrowed through increasing year levels for all the five problems. This finding lends support to the observation by Burkhardt (1988), Bastow *et al.* (1984) and Resnick (1989) that most students are only comfortable solving problems which employ the mathematics they have learnt some time ago. This finding is further strengthened by the observation that although all the Secondary 2 subjects had been taught simultaneous equations 6 months prior to their participating in the study, only 17.95 per cent of them attempted to solve the Cows and Chickens problem, using simultaneous equations.

As there were subjects from all year levels who were able to solve almost all the Computation Test items and the corresponding Problem items, it appears that the measurement of this "gap" or time interval is not generalizable across cohorts of students in terms of number of years, as suggested by Burkhardt (1988) and Bastow *et al.* (1984). However, it can be suggested that the time interval is different for each individual problem solver. The varied performance of the subjects on the five Problem items also suggests that this time interval may also be different for different problems and for each individual problem solver.

Question 3 *What are the difficulties experienced by unsuccessful problem solvers who possess the necessary mathematical knowledge and skills to solve particular problems ?*

It was noted during the interviews that students almost always attempted to work out the solution for a problem using only one strategy. They did not demonstrate any flexibility by first trying one strategy and if it did not work, try another. Students who worked their solutions using an inappropriate strategy were often not aware that the solution was incorrect. Furthermore, students made no attempt to check the correctness of their solutions or whether it was a solution which satisfied the conditions in the problem.

It was found that some of the difficulties experienced by problem solvers (who were hindered from obtaining a solution), although they possessed the necessary mathematical knowledge and skills to solve particular problems, were:

- lack of comprehension of the problem posed;
- lack of strategy knowledge; and
- inability to translate the problem into a mathematical form.

Students who obtained incorrect solutions although they possessed the necessary mathematical knowledge and skills did so because of one or more of the following:

- an inappropriate strategy used;
- incorrect formulation of the mathematical form;
- computational errors;
- imperfect mathematical knowledge;
- misinterpretation of the conditions of the problem;
- misinterpretation of mathematical concepts; and
- interference by extraneous knowledge.

Question 4 *Are there particular mathematical problem-solving strategies used by most novice problem solvers in Primary 5, 6, Secondary 1 and 2?*

It was found that some strategies such as "guess and check" and "systematic investigation" were problem specific while others such as "modelling" and "number manipulation" were common to all the problems. The strategies used by the sample of novice problem solvers to solve the problems were "guess and check", "systematic investigation", "modelling", "logical reasoning", "use of algebra", "use of formula", "look for a pattern", and "number manipulation".

The strategies of "guess and check (unsystematic)", "guess and check (systematic)", "systematic investigation", "modelling (diagrams)", "modelling (systematic counting)", "logical reasoning", and "number manipulation" were those used by problem solvers from Primary 5 and 6. In addition to these strategies used by problem solvers from Primary 5 and 6, the Secondary 1 problem solvers also used algebra (linear equations), algebra (simultaneous equations), and the strategy "use of formula". The Secondary 2 problem solvers used all of the strategies used by the problem solvers from Primary 5, 6 and Secondary 1 and in addition, the strategy "look for a pattern".

It can be concluded that there are particular mathematical problem-solving strategies used by most novice problem solvers in Primary 5, 6, Secondary 1 and 2 to solve the five problems: Stamps, Rectangular Shape, Loans, Handshakes and Cows and Chickens. These strategies were "guess and check (unsystematic)", "guess and check (systematic)", "systematic investigation", "modelling (diagrams)", "modelling (systematic counting)", "logical reasoning", and "number manipulation".

Question 5 *Does the range of problem-solving strategies used by novice problem solvers expand with increasing age and mathematical maturity?*

It appears that the range of problem-solving strategies used by novice problem solvers does expand with increasing age and mathematical maturity. The inclusion of some of these strategies may be a consequence of mathematics instruction in the classroom related to the year level of study of the problem solver.

Question 6 *Do good novice problem solvers differ from poor novice problem solvers in their selection and use of strategies when attempting to solve problems?*

In this study, it appears that good novice problem solvers did differ in their selection and use of strategies from poor novice problem solvers when attempting to solve the five problems: Stamps, Rectangular Shape, Loans, Handshakes and Cows and Chickens. Generally, the good novice problem solvers used a wide repertoire of strategies compared to the poor novice problem solvers. The former tended mainly to use algebra, formulae and the strategies of "guess and check", "systematic investigation", "modelling", and "logical reasoning", while the latter tended to use the strategy of "number manipulation". Also, poor novice problem solvers often presented solutions which were classified under "unable to detect method used" or did not attempt the problem.

Question 7 *Which of the problem-solving strategies commonly prescribed in mathematical syllabus documents are good novice problem solvers able to use?*

The majority of the good novice problem solvers from all the four year levels who attempted to solve the four problems (Frog and Well, Birds on Trees, Mathematics Competition and Football Tournament) using the specified strategy(s) in the question were successful in obtaining the solutions to the respective problems, except for the Primary 5 problem solvers who attempted to solve the Mathematics Competition problem using the strategy "make a table". Only 30 per cent of those Primary 5 problem solvers who attempted to find a solution using "draw a diagram or picture" to the Frog and Well problem arrived at the correct solution. However, a study of their written solutions revealed that those who obtained an incorrect answer did so because they allowed the frog to rest on the rim of the well instead of it climbing out on the sixth day, rather than their inability to use the strategy "draw a diagram or picture" correctly.

It appears that the common strategies "draw a diagram or picture", "work backwards" and "make a list" belong to the problem solving "tool kit" of the good novice problem solvers from Primary 5, 6, Secondary 1 and 2, but the strategy "make a table" only belongs to the "tool kit" of these good problem solvers from Primary 6, Secondary 1 and 2.

Question 8 *Are good novice problem solvers able to solve problems in more than one way?*

It was found that good novice problem solvers from all the four year levels were able to only suggest or use one way of solving the Sets of Coins problem, and only two types of strategies were suggested or used. The strategies were "guess and check" and "make a systematic list". For all four year levels, a larger proportion of problem solvers suggested or used the strategy "guess and check" compared to "make a systematic list".

For the Cows and Chickens problem, the problem solvers from all four year levels suggested or used one or two ways of solving the problem. The percentage of problem solvers who suggested two ways increased through progressively from Primary 5 to Secondary 2. The strategy "guess and check" was suggested or used by the majority of the problem solvers from all the four year levels. Algebra was also suggested or used by the majority of the Secondary 2 problem solvers and also a sizeable number of Secondary 1 problem solvers. From the above findings, it appears that it is not possible to conclude that good novice problem solvers are able to solve problems in more than one way. This conclusion is weakened because only two problems were used, one of which was similar to the one the subjects had attempted before during an earlier phase of the study.

Question 9 *Do good novice problem solvers rely on individual problem-solving frameworks to guide them when solving problems?*

Of the 63 problem solvers from Primary 5, 6, Secondary 1 and 2, 52 had frameworks comprising the four Phases: Understand/Represent the Problem, Find a Way to Solve the Problem, Solve the Problem, and Check the Solution. Eleven problem solvers, of whom five were from Primary 5, three were from Primary 6 and another three were from Secondary 1, had frameworks comprising only the three Phases: Understand/Represent the Problem, Find a Way to Solve the Problem and Solve the Problem. It seemed that the good novice problem solvers in the sample attempted to solve non-routine problems in a structured manner. Most tend to go systematically through the 4 phases when solving non-routine problems but manifesting different behaviours. It can be concluded that good novice problem solvers from all year levels were found to rely on individual problem-solving frameworks to guide them when solving non-routine problems.

Question 10 *If such frameworks exist, do they increase in degree of sophistication relative to problem solvers' ages and year levels?*

From the frameworks, it seems that the number of problem-solving behaviours demonstrated by the subjects during the different phases varied, ranging from a minimum of one to a maximum of three for Primary 5 problem solvers, a minimum of one to a maximum of four for Primary 6 problem solvers, a minimum of one to a maximum of three for Secondary 1 problem solvers, and a minimum of one to a maximum of five for Secondary 2 problem solvers.

It appears that these frameworks increased very gradually in degrees of sophistication for problem solvers from Primary 5, 6 and Secondary 1 relative to their year levels, but the frameworks of the problem solvers from Secondary 2 were very detailed and comprehensive in scope compared to those of Primary 5, 6 and Secondary 1. In particular, the behaviours listed under the Check the Solution Phase by the Secondary 2 subjects suggest that they were making an attempt to reflect on their solution to the problem. From the above findings, it appears that the frameworks of the Secondary 2 problem solvers were significantly more sophisticated when compared to those of Primary 5, 6 and Secondary 1 problem solvers.

IMPLICATIONS FOR THE CURRICULUM

This project has, during its course, documented significant findings on:

- the existence of a "gap" or time interval;
- problem solvers' difficulties;
- problem solution strategies used by novice problem solvers from Primary 5, 6, Secondary 1 and 2; and
- characteristic strategies used by and behaviours of good novice problem solvers.

These findings have implications for the curriculum.

The evidence of the existence of a "gap" or time interval between a problem solver's ability to carry out particular mathematical calculations and operations, and the ability to solve problems employing the same mathematics content knowledge has implications for curriculum developers, assessment boards and classroom teachers. In particular, it is important for classroom teachers to have an awareness of this "gap" when teaching and assessing mathematical problem solving.

The difficulties experienced by unsuccessful problem solvers who possess the required mathematical knowledge and skills documented in this study has important implications for classroom teachers, who often assume that when an incorrect answer is given to a mathematical task, the error had occurred because the student lacked the necessary mathematical knowledge or skill. The simple interview format used in this study is easy to use and could be adapted and used by classroom teachers to diagnose their students' difficulties and hence, remediate their difficulties.

The particular problem-solving strategies used by most novice problem solvers at Primary 5, 6, Secondary 1 and 2 found in the study, and the evidence that the range of problem-solving strategies used by novice problem solvers does expand with increasing age and mathematical maturity are significant for curriculum developers, assessment boards, and classroom teachers. Often, mathematics syllabuses contain lists of strategies for problem solving, and classroom teachers are expected to decide when and what to teach with little, if any, guidance. The findings documented in this study about the nature of the strategies used by novice problem solvers at the different year levels can help to provide suitable insights into which strategies are most appropriate and should be developed at different stages of mathematical development and assist in answering the question "When should particular strategies be taught ?" The differences noted in the selection and use of problem solution strategies by good and poor novice problem solvers are significant for classroom teachers. The reliance on coping strategies such as "number manipulation" by novice problem solvers signals to the teacher that they are failing and need help.

In this study, good novice problem solvers were found to be able to use some strategies commonly prescribed in mathematical syllabus documents but not others at certain year levels and this again, has implications for curriculum developers, assessment boards and classroom teachers. The good novice problem solvers were generally not able to solve problems in more than one way and this appears to be a consequence of classroom practice and hence, has implications for teacher trainers and classroom teachers. Good novice problem solvers were found to rely on individual problem-solving frameworks when solving problems. These frameworks showed that only problem solvers at Secondary 2 were beginning to reflect during the "Check the Solution" Phase. This finding has implications for curriculum developers and classroom teachers as it may caution against asking very young children to reflect fruitlessly.

The classroom practice of mathematics lessons in most Singapore schools was exhibited in the ways the subjects solved problems. It was noticed during the interviews and also from the paper-and-pencil solutions that subjects were in the habit of trying to work the solution of the current problem using only one strategy. They did not demonstrate any flexibility by trying a strategy and if it did not work, trying an alternative. The taxonomy of problem-solving behaviours of good novice problem solvers showed that behaviours such as "underline the answer", "check the answer against an answer sheet", and "check the answer with my friends" were common among them. These kinds of classroom practices have serious implications for curriculum designers and teacher trainers. If the "problem-solving curricula" is to be successfully brought into classrooms in Singapore schools, these practices may have to be reconsidered and classroom teachers be made aware of how they can successfully implement mathematical problem solving in the classroom.

This study also offers some recommendations for future teaching of mathematics in the classroom. It recommends that teachers should model mathematical problem solving in the classroom by:

- demonstrating flexibility in the use of strategies by solving problems in several ways;
- showing that methods such as "guess and check" and "modelling" are as mathematical as "logical reasoning" and "use of formulae";
- encouraging students to solve problems in more than one way;
- providing students with experience in finding several solutions to a problem; and
- encouraging students to reflect while solving problems.

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APPENDIX A

PROBLEMS TEST

Answer all questions. Show all your workings and also explain how you work out the problems.

1. **Stamps**

   Wei Min has to post a parcel costing $23. He has plenty of $5 and $2 stamps but no others. How many of each kind of stamps would he use so that not more than 8 stamps are used altogether? Show all your working.

2. **Rectangular Shape**

Alice has 20 cm of wire. She makes a rectangular shape with the wire. The shape has the largest possible area. What is the length and width of the shape Alice made? Explain how you worked it out.

3. **Loans**

Meena owes Siti \$7. Siti had already borrowed \$15 from Rani and \$32 from Alice. Alice owes \$3 to Rani and \$7 to Meena. One day the girls got together at school to sort out their accounts. Which girl left with \$18 more than she arrived with ? Show all your working.

4. **Handshakes**

At a party there are 10 people. If everyone at the party shakes hands with everyone else, how many handshakes would there be? Explain how you worked it out.

5. **Cows and Chickens**

On a farm there are cows and chickens. Altogether they have 41 heads and 100 legs. How many chickens are there? Explain how you worked it out.

APPENDIX B

COMPUTATION TEST

Answer all questions. Show all your workings.

1 (a) $23 = ? \times 2 + 1 \times 5$ (b) $23 = 4 \times 2 + ? \times 5$
? = ______ ? = ______

2 (a) Figure A is a square of side 5 cm.
The perimeter of figure A = ______ cm.
The area of figure A = ______ cm^2.

(b) Figure B is a rectangle of sides 7cm and 3 cm.
The perimeter of figure B = ______ cm.
The area of figure B = ______ cm^2.

(c) Is a square a rectangle? Yes / No

3 (a) \$15 + \$3 = ______ (b) \$32 – \$3 – \$7 = ______
(c) \$7 – \$7 = ______ (d) \$32 + \$15 – \$7 = ______

4 (a) (10 × 9) ÷ 2 = ______
(b) 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = ______

5 (a) * + 20 = 41
* = ______
* × 4 + 20 × 2 = ______

(b) * + 10 = 41
* = ______
* × 4 + 10 × 2 = ______

APPENDIX C

MORE PROBLEMS TO SOLVE

1. **Frog and Well**

A frog is at the bottom of a well 13 m deep. Every day it climbs up 3 m, but at night it slips back 1 m.

Draw a diagram or a picture to show how many days it takes the frog to climb out of the well.

2. **Birds on Trees**

16 birds are sitting in 2 trees, a palm tree and a cherry tree. 2 birds fly away from the palm tree, and then 5 birds fly from the cherry tree to the palm tree. Now there are the same number of birds in each tree.

Draw a diagram or a picture and work backwards to show how many birds were in each tree to start with.

3. **Mathematics Competition**

In a mathematics competition, 20 problems were given. 5 marks were given for each correct answer, and 2 marks were deducted for each incorrect answer. Mutu did all 20 problems and his score was 72.

Make a table to show how you would find the number of correct answers he had.

4. **Football Tournament**

   6 football teams play in a tournament, so that each team plays each of the other teams only once.

   Make a list to show all the different games which will be played.

5. **Sets of Coins**

   A sum of $2.60 is made up of 12 coins of denominations: 10 cents, 20 cents and 50 cents.

   Suggest some ways to find all the possible sets of coins.

6. **Cows and Chickens**

   When he was visiting his grandfather's farm, Prem noticed that he raised only cows and chickens. He counted 38 heads and 100 legs belonging to the cows and chickens.

   Suggest some ways for Prem to find how many cows and how many chickens his grandfather has.

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