Mathematical Thinking Strategies for Solving Challenging Problems

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INTRODUCTION

Problem solving has been, and still is, the main emphasis in school mathematics (NCTM 1980). Many children, however, face difficulties in solving mathematical problems for various reasons; one of which being the inability to or inadequacy in using mathematical thinking strategies. In view of the importance placed on problem solving in mathematics, two questions need to be addressed: What are some types of mathematical thinking that pose difficulty for children, thereby preventing them from solving "difficult" problems? What strategies can best be used to enhance children's skill in solving challenging problems, based on the type of mathematical thinking identified?

This article addresses these questions by discussing three mathematical thinking strategies that teachers can effectively use to help students solve specific mathematical problems. They are: conjecturing, specializing and checking, hypothesis-testing, and modelling.

MATHEMATICAL THINKING STRATEGIES

1. CONJECTURING, SPECIALIZING AND CHECKING

One problem-solving strategy which may be used to solve mathematical problems is to make a guess of the answer and then to check the constraints that are given in the question or that are implied from some question statements. An example of the thinking involved in the use of this strategy is shown as follows:

A full tin of oil weighs 7 kg.
When the tin is half full, it
weighs 4 kg.
What is the weight of the tin when it is empty?

Several methods can be used to solve this problem. One method is concerned with conjecturing. There are two possibilities which conjecturing on the answer can lead to:

(a) contradiction of one or more constraints given in the question and

(b) supporting the explicit or implicit statements from the question.

The following paragraphs illiustrate these two possibilities in the process of using this strategy.

First, a problem solver may make a wrong guess of the weight of the tin as 2 kg. When a full tin of oil weighs 7 kg and the empty tin is 2 kg, a full tin of oil minus the empty tin will weigh 7 kg - 2 kg = 5 kg. However, when a half tin of oil weighs 4 kg and the empty tin is 2 kg, the half tin of oil minus the empty tin will weigh 4 kg - 2 kg = 2 kg. The latter (2 kg) is not half of 5 kg. This result contradicts the fact that the weight of a half tin of oil should be half of the weight of a full tin of oil. Thus, the assumption that the weight of the empty tin is 2 kg is not valid.

Next, you make a correct guess of the weight of the empty tin to be 1 kg. Now, when a full tin of oil weighs 7 kg and the empty tin is 1 kg, the full tin of oil minus the empty tin will weigh 7 kg - 1 kg = 6 kg. However, when a half tin of oil weighs 4 kg and the empty tin is 1 kg, the half tin of oil minus the empty tin is 4 kg - 1 kg = 3 kg. The weight 3 kg is half of 6 kg. Now, the result supports the fact that the weight of half a tin of oil should be half of the weight of a full tin of oil. Thus, the assumption that the weight of the empty tin is 1 kg is valid.

Another problem which illustrates the use of this mathematical procedure is as follows:

Linda, John and Devi shared a sum of money.
Linda's share is half of what John and Devi received.
John's share is $\frac{4}{9}$ of the sum of money.
What fraction of the sum of money is Linda's share?

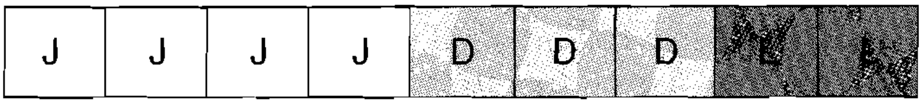
One possible way to solve this problem is described as follows:

First guess: Let L (Linda) have 1 share, then the balance is for D (Devi) as shown in the diagram below:

Linda received 1 share and Devi received 4 shares.
Sum of J's (John's) share and D's (Devi's) share $= 4 + 4 = 8$.
But $\frac{8}{2} = 4$ whereas L's share is 1.

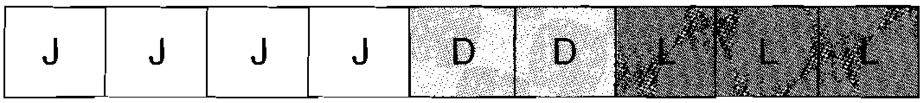
From the question statement, L's share should be half of J's and D's shares. Thus, the hypothesis that L's share = 1 unit is not valid.

Second guess: Let L have 2 shares then the number of D's share is 3 as shown in the following diagram:



Since J has 4 shares and D received 3 shares, the sum of J's share and D's share $= 4 + 3 = 7$.
But $\frac{7}{2} = 3\frac{1}{2}$ whereas L's share is 2.

This contradicts the fact that L's share should be half of J's and D's shares. Again, the hypothesis that L's share = 2 units is not valid.



Third guess: Let L have 3 shares, then the balance is for D as shown in the following diagram:

Since J has 4 shares and D received 2 shares, the sum of J's share and D's share $= 4 + 2 = 6$.
But $\frac{6}{2} = 3$ whereas L's share is 3.

Since L's share is half of J's and and D's shares, the hypothesis that L's share = 3 units is valid. Therefore, Linda receives $\frac{3}{9}$ of the sum.

In the above question, there are two constraints: (1) John has 4 shares and (2) the sum of John's shares and Devi's shares is twice that of Linda's. In the solution, a problem solver needs to make a guess of Linda's share. Using the guessed figure, the result should not lead to any contradiction to the given constraints. The above example shows the application of this form of mathematical thinking. Specifically, a few tests were carried out until the constraint that the sum of John's share and Devi's share doubles that of Linda's share is satisfied.

2. HYPOTHESIS-TESTING

Another type of mathematical-thinking strategy for solving mathematical problems is hypothesis-testing. The stages involved in using this strategy are:

(a) collecting data;

(b) looking for some pattern from an organized set of data;

(c) forming a hypothesis;

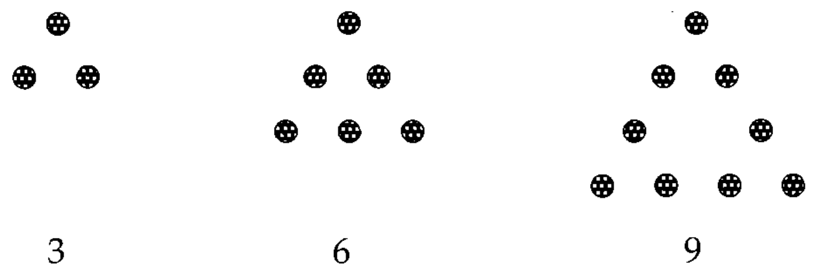
(d) testing the hypothesis with some specific examples;

(e) forming generalizations and

(f) using the generalized information to solve the problem.

The following investigation problem is used to illustrate this process of thinking:

> *Dots are used to form triangles as follows. The dot number refers to the number of dots of the triangle.*



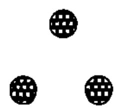
> *What triangle dot number has (i) 25 dots and (ii) 999 dots on each side of the triangle?*

There are many ways of solving this problem. The paragraphs on the next page illustrate the hypothesis-testing strategy for solving this investigation problem.

(a) Collection of Data

The first stage of the solution is to collect a set of data which can be organized so that a pattern can be determined. For each triangle of a given number of sides, a corresponding dot number is determined. For example, in the first diagram on the left (see above), the number of dots on each side of the triangle is 2 and the dot number is given as 3. The number of dots of the diagram can be found in the following way:

Since there are 3 sides and each side has 2 dots, the total number of dots is 6 minus 3. The number 3 is subtracted because the dot at each corner is repeated when the number of dots is counted. The diagram below shows the repeated corner dots.



Using a similar procedure, the dot number can be determined given that the number of dots on each side of a triangle is 3. The dot number is given by:

3 sides x 3 dots on each side - 3 = 6.

When the actual number of dots is counted, it is also 6.

Similarly, when the number of dots on each side of a triangle is 4, the dot number is given by:

3 sides x 4 dots on each side - 3 = 9.

When the actual number of dots is counted, it is also equal to 9.

A set of triangles is used and an organized set of data is shown in (b).

(b) Looking for Pattern

A set of data on the number of dots on each side of the triangle corresponding to the dot number and the algorithm of obtaining the dot number is summarized in the table below:

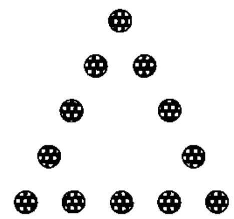
Number of Dots on Each Side of Triangle	*Dot Number (Number of Dots)*	*Algorithm for Obtaining the Dot Number*
2	3	$2 \times 3 - 3 = 3$
3	6	$3 \times 3 - 3 = 6$
4	9	$4 \times 3 - 3 = 9$
5	12	$5 \times 3 - 3 = 12$
6	15	$6 \times 3 - 3 = 15$

Notice that in the last column, the number of dots on each side of the triangle varies whereas the number of sides of the triangle and the number 3 representing the number of dots at the corner are unchanged. Here, you can derive a pattern to obtain the next set of data.

(c) Forming and Testing a Hypothesis

From the pattern you have derived, you can form a hypothesis which can be phrased in verbal or generalized form. In this example, you may hypothesize that "the dot number of a given triangle with a fixed number of dots on each side can be determined by multiplying the number of dots on each side by the number of sides of a triangle and the result is used to subtract 3." To make the statement clearer, symbols may be used to represent the verbal statement. For example, let d be the number of dots on each side of the triangle, then the dot number of this triangle is given by 3d - 3.

In order to test the validity of the hypothesis, you need to check by using some specific examples so that the ideas you obtained can lead to the actual answer of a given situation. For example, you first check with the pattern or formula you have derived. If the number of dots on each side is 5, then the dot number or the number of dots of the triangle is 3 x 5 - 3 = 12. Now, you can also obtain the value 12 by counting the actual number of dots of the triangle given as follows:



Notice that there are altogether 12 dots and thus the dot number for the triangle with 5 dots on each side is 12. You can verify the hypothesis again using triangles with 6 dots and 7 dots on each side of the triangles respectively.

(d) Using the Generalized Information to Solve the Problem

Having verified the validity of the hypothesis, you may now use the generalized formula to solve the problem which requires you to find the dot number for triangles with (i) 25 dots and (ii) 999 dots on each side of the triangle. Using the generalized formula:

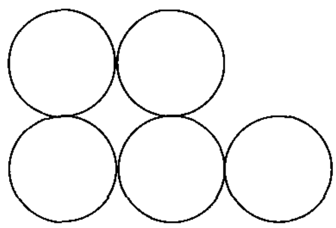
(i) 3d - 3 = 3 x 25 - 3 = 75 - 3 = 72
The dot number for the triangle with 25 dots on each side is 72.

(ii) 3d - 3 = 3 x 999 - 3 = 2997 - 3 = 2994
The dot number for the triangle with 999 dots on each side is 2994.

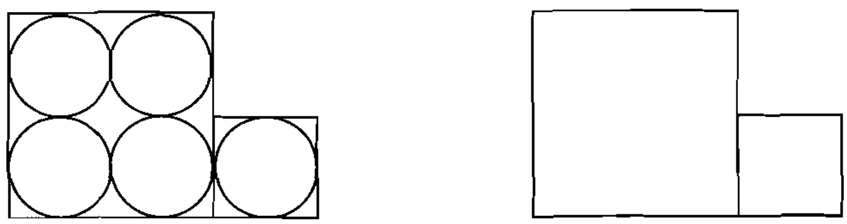
3. MODELLING STRATEGY

Another strategy which is useful for solving mathematical problems is modelling. The modelling strategy requires problem solvers to look at the problem from a familiar but related perspective. Solving a familiar problem helps one to tackle a more difficult problem. In addition, problem solvers are also required to apply some hidden principles which are related to the topic concerned. The following problem and its solution illustrate the processes of the strategy.

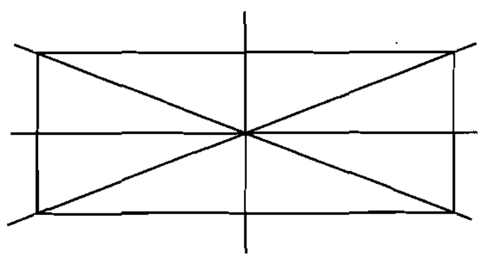
> *A figure is formed by joining 5 identical circles as shown below. Draw a straight line in the figure so that it is cut into two equal parts.*



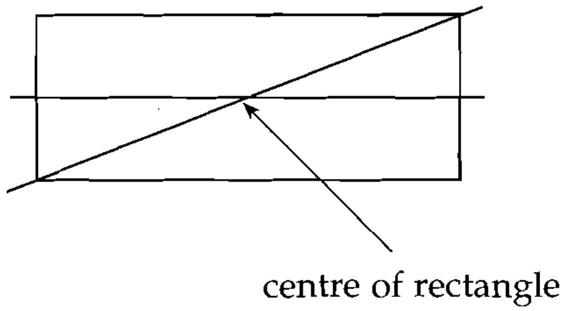
The modelling strategy requires problem solvers to model or transform the problem into a familiar situation. Some children may be able to see that the figure can be modelled by using two squares as in the diagram below:



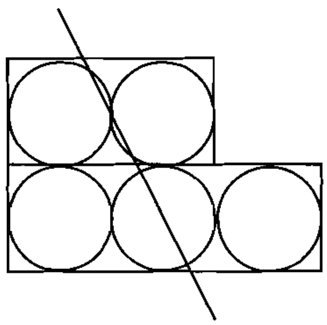
Now, instead of drawing a line to cut the circles into two equal parts, they look at the problem from a different perspective, i.e. to draw a line which cuts the two squares into two equal parts. To many children, drawing a line to cut the two squares is much easier than drawing a line to cut only five circles. However, in order to draw a line to cut the two squares, an underlying principle to cut a rectangle is needed. The following figure shows the principle for cutting a rectangle into two equal parts.

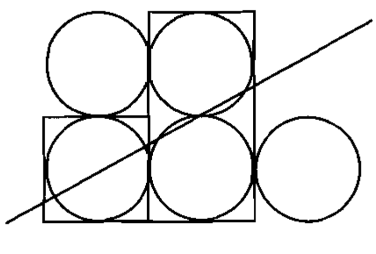


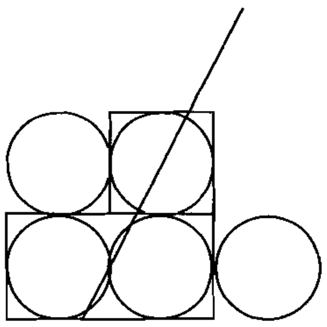
Most children think that there are only four possibilities to cut a rectangle into two equal parts as shown above. In fact, there are infinite number of ways to cut a rectangle into two equal parts as long as the straight line passes through the centre of the rectangle. This is shown in the diagram below.

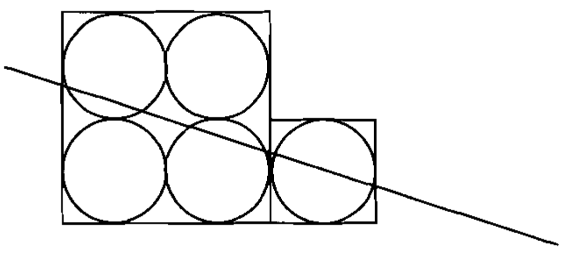


Using this underlying principle, a line is then drawn passing through the centres of the adjoining squares. This line will cut the two squares into two equal parts. Thus, applying this principle, modelling circles into two squares and drawing a line to cut through a rectangle into two equal parts, the following are some solutions to the problem of drawing a straight line to cut five circles into two equal parts.









Conclusion

This article has discussed three different types of uncommon mathematical-thinking strategies which are specifically used to tackle certain types of mathematical problems. They are:

1. the conjecturing, specializing and checking strategy;
2. the hypothesis-testing strategy and
3 the modelling strategy.

Some examples were used to illustrate the features of these mathematical-thinking strategies.

Two questions were found to be suitable for illustrating the features of the conjecturing, specializing and checking strategy. The first question on finding the weight of an empty tin (see p. 54) was solved by guessing the weight of the empty tin, followed by checking the implied constraint that the weight of a half tin of oil should be half of the weight of a full tin of oil. The underlying concept which is used to solve this problem is based on the principle that a wrong assumption made (in this case, the weight of the empty tin) should lead to contradiction whereas a correct assumption will lead to the support of some facts such as the weight of a half tin of oil is half the weight of a full tin of oil. Similarly, in the second question, the correct assumption of Linda's shares would support the fact that the sum of John's shares and Devi's shares is twice that of Linda's.

The hypothesis-testing strategy illustrates a series of mathematical processes involving the collection of data, looking for pattern from the set of collected data in an organized form, forming a hypothesis, using some specific examples to test the hypothesis and using the hypothesis to solve a problem. The example on dot number triangle shows the application of the above features to determine the dot number of a triangle with any dots on each side of the triangle. In this example, an algebraic formula is generalized and used to find the dot number of a triangle with 999 dots on each side of the triangle.

The modelling strategy requires problem solvers to look at the problem from a different perspective which is usually a familiar situation. The problem is solved by modelling or transforming the actual (a more difficult) situation into a simpler situation. An example for determining the straight line which is used to divide five circles into two equal parts illustrates the modelling processes in the solution. In the solution, squares are used to model after the circles. If a solution can be found using squares, it will lead to a solution using circles.

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