Problem Posing To Promote Mathematical Thinking

Yeap Ban Har &
Berinderjeet Kaur

INTRODUCTION

Problem posing has, in recent years, gained increasing attention from Mathematics educators around the world (Silver 1994). Problem posing in mathematics education is an unexplored phenomenon in Singapore. It is one of the skills identified for the Thinking Programme in Mathematics (Ministry of Education 1997).

Problem posing is the generation of new problems and the reformulation of given ones (Silver 1994). Some examples of problem posing activities include:

- the generation of a problem (Ellerton 1986);
- the generation of a problem based on a given context (Silver & Cai 1993);
- the generation of a problem based on a given calculation (Green & McCaan 1991);
- the generation of a problem given the solution;
- the generation of sub-problems in the solution of a larger problem;
- the generation of questions to problems with unstated questions (Kruestskii 1976) and
- the generation of "what-if" questions (Brown & Walter 1985).

Problem-posing activities are suitable vehicles to promote mathematical thinking amongst students. According to Silver (1994), mathematical problem posing can be seen as a feature of mathematical thinking as well as a feature of creativity, among other perspectives. Kilpatrick (1987) argued that problem posing should be both the goals and means of instruction in mathematics. Formulating questions is one of the thinking skills in the Dimensions of Learning framework (Marzano 1992).

This article describes several problem posing activities for the upper secondary classroom and how these have been used to promote mathematical thinking, in particular, and thinking, in general. These activities are selected to illustrate how existing teaching materials can be adapted into problem posing activities to promote thinking.

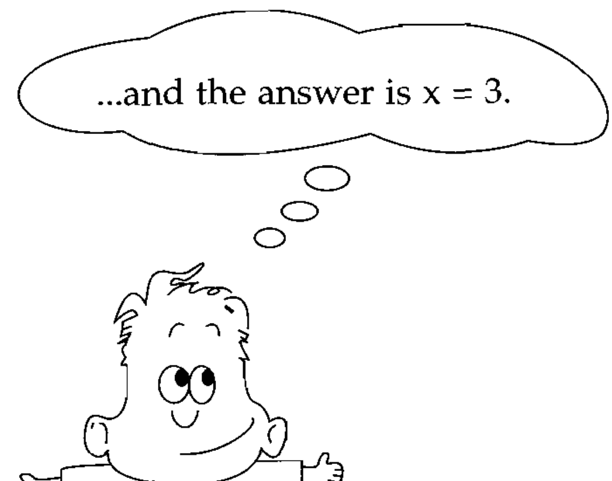
CLASSROOM ACTIVITIES

ACTIVITY ONE: WHAT'S THE QUESTION?

Generating a problem given the solution

This activity involves students writing possible questions to a given solution.

What's The Question?



- What could the question be?

Such activities are created by simply selecting an answer such as "the area is 154 cm^2" or "the derivative is $3x^2 - 4$", which could be the answer to a host of questions, and requiring students to generate possible questions to the answer.

In this activity, the questions generated could have ranged from a simple $x - 3 = 0$ to $x^2 - x = 6$ to $x^3 - x^2 - 9x + 9 = 0$ to $\sqrt{(x + 1)} = 2$ to a fairly complex $\lg(x + 7) = 1$. Students get the opportunity to take stock of and use their existing knowledge. In a way, the activity provides a stimulus for students to summarize all they know about solving equations. In the same way, "the area is 154 cm^2" serves as a stimulus for students to summarize all they know about area of figures.

ACTIVITY TWO: WHAT'S THE PLAN?

Generating sub-problems in solving a larger problem

This activity requires students to identify the main goal in a problem as well as to identify sub-questions that must be answered before the main goal can be achieved. Students are also required to sequence the sub-questions so that the answer to the main question may be derived.

What's The Plan?

The points P and Q are on the curve $y = x^2$. The value of x at P is -2 and the value of x at Q is 3. The tangent at P meets the y-axis at A and the normal at Q meets the y-axis at B. Find the distance AB.

Adapted from GCE Ordinary Level Additional Mathematics November 1996 Paper 1 Question 3

Before you solve the main problem, you need to first solve other sub-problems.

Ask yourself:
- What is the main problem?
- What are the sub-problems?

Organize your plan using a graphic organizer.

Such activities could be created using questions that teachers typically use in a mathematics class. The identification of the main problem and the sub-problems as well as the planning stage, however, have to be made explicit.

This activity helps students develop skills to solve complex problems. Students need to identify the main problem (the skill of focusing) as well as problems that have to be solved to provide information crucial to the solution of the main problem (the skills of analyzing and gathering information). The problems generated must be sequenced in a way to create a plan that works (the skill of organizing). A piece of work done by a student for *Activity 2* is shown in Figure 1.

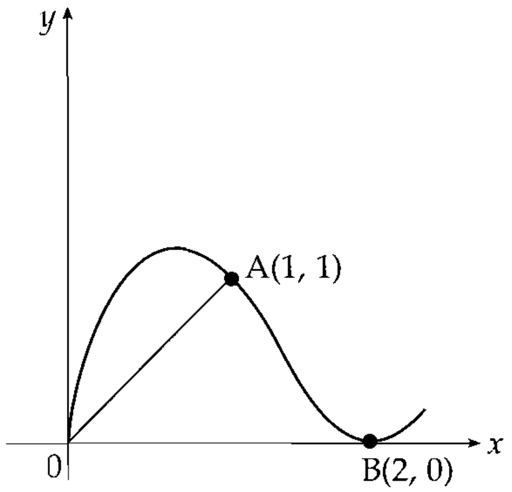
ACTIVITY THREE: THE MISSING QUESTION

Generating questions to problems with unstated questions

This activity requires students to write questions for a given problem situation.

The Missing Question?

The diagram shows part of the curve $y = x(x - 2)^2$ which passes through A (1, 1) and touches the x-axis at B (2, 0).



Adapted from GCE Ordinary Level Additional Mathematics November 1990 Paper 1 Question 16

- Write a question for the problem.
- Answer the question you wrote.

This type of activities are created by deleting questions from typical mathematics problems. The problem scenario should ideally be rich in information to allow the generation of questions that range from simple ones to complex types. In this activity, the questions generated could vary from simple ones such as "Find the gradient of the curve at A." to complex ones such as "Find the intersection point between the tangent at A and normal at B." to "Find the ratio of area P to area Q.".

Students are required to examine all the information present in the given problem and to pose a question that would, hopefully, acknowledge the richness of the information. It is in effect an act of summarizing key information contained in a problem.

ACTIVITY FOUR: WHAT IF?

Generating "what-if" questions

This activity involves students generating "what-if" questions after they have solved a given question.

What If?

Find $\frac{dy}{dx}$ if $y = \sin 3x$.

What if

- $y = 3 \sin 3x$?
- $y = \sin^3 3x$?
- $y = \sin 3x^3$?
- $y = \sin\left(\frac{x}{3}\right)$?
- $y = \sin\left(\frac{3}{x}\right)$?
- $x = \sin 3y$?

Generate another ten "what-if" questions and answer them. Look out for any interesting observation / patterns.

The "what-if" questions you have generated must allow you to show your ability to use chain rule, product rule and implicit differentiation.

Such activities are merely an extension to any questions that students have solved. This activity provides students the opportunity to vary a problem condition systematically and to observe the effects of doing so. It allows students to integrate their knowledge on a particular aspect of mathematics. By allowing the students to make alterations to problem conditions, the activity enables them to realize how the solution process is affected each time a certain change is made. Students would also be provided with the opportunity to select suitable techniques and reject others in solving each problem.

ACTIVITY FIVE: WHAT'S THE PROBLEM?

Generating a problem based on a given context

This activity involves students writing questions for a given context.

What's The Problem?

Create a calculus problem that involves maximum/minimum values. Solve the problem.

Marks are given for:
- how realistic the problem is [3 marks]
- how interesting/original the problem is [3 marks]
- how difficult the problem is [3 marks]
- how well the problem is solved [6 marks]

Such activities are created by providing certain information upon which students can create their own problems. For example, in this activity, the problem created has to be a calculus problem and one that involves maximum or minimum concept. The teacher's expectation is also clearly indicated to students.

While the boundaries of the activity are defined, the activity allows sufficient opportunities for students to exercise their creative thinking. Metacognitive processes are often evoked in the process of formulating the question. In the event that the solution does not make sense, students need to analyse the problem to rectify the situation. This problem allows each student to show his ability. Weaker students could pose a relatively simple problem while the better ones are not restricted in any way.

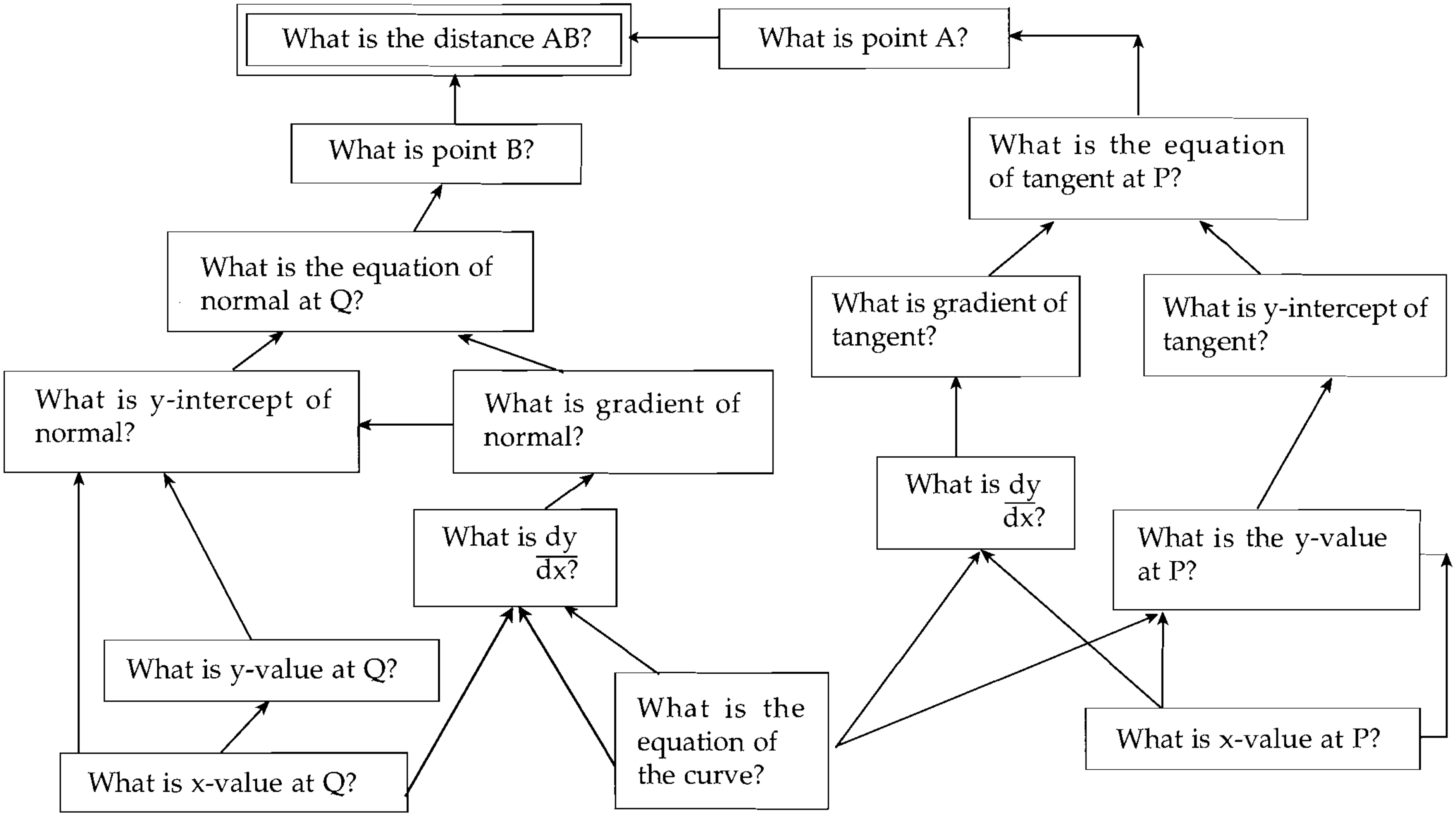


Figure 1 : Student's Work

CONCLUSION

The activities described are analysed to identify core thinking skills from the Dimensions of Thinking and Learning frameworks (Marzano 1988, 1992) that are included.

An Analysis of Several Classroom Activities in terms of Skills Specified in the Dimensions of Learning Framework

Skills	Activity 1	Activity 2	Activity 3	Activity 4	Activity 5
Focusing					
defining the problem		√			√
Information gathering					
formulating question		√			
goal setting					
Remembering					
encoding				√	
recalling					
Organizing					
ordering		√			
representing					
comparing					
classifying					
Analyzing					
identifying components		√			
identifying patterns					
identifying relationship		√		√	
identifying errors					
identifying main ideas			√		
Generating					
inferring					
inducting					
predicting				√	
deducting					
Integrating					
summarizing	√		√		
restructuring					
Evaluating					
verifying					√
establishing criteria					

**only main skills are identified*

The activities described are intended to show how typical classroom activities can be modified or extended to provide opportunities for students to engage in problem-posing situations. Problem-posing activities, we believe, are one powerful means to encourage thinking in the mathematics classroom.

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