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The Measurement of Mathematics Pedagogical Content Knowledge

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Abstract: An important determinant of student learning is the quality of teaching which is, in turn, affected by teachers' knowledge of discipline-specific pedagogy. Thus a clearer understanding of what constitutes Mathematics Pedagogical Content Knowledge (MPCK) and its development in teachers is crucial for improving teacher education courses. This symposium reports through three papers the results from a project which seeks to study how primary teachers develop MPCK. This first paper describes an operational definition of MPCK in terms of four constructs and the development of an instrument to measure MPCK. The items of the instrument were designed specifically to measure these constructs in the context of Singapore's primary mathematics curriculum.
Keywords: mathematics pedagogical content knowledge, pedagogical content knowledge.

Introduction

While mathematics education research has a long history in the twentieth century, research in mathematics teacher education has a substantially shorter history. It should be a fundamental principle that as mathematics teaching is based on mathematics education research on pupil learning, the design and delivery of curriculum for mathematics teacher preparation and teacher professional development should likewise be informed by corresponding research findings on student teachers and practicing teachers. One area of concern in teacher education curriculum is the determination of the knowledge base which teachers are expected to have in order to practise their profession effectively. While researchers in the late eighties have made different categorizations of teacher knowledge, the four general areas delineated by Grossman (1990) provide a useful way of discussing the various aspects of teacher knowledge. These are: general pedagogical knowledge, subject matter knowledge, pedagogical content knowledge and knowledge of context. Shulman (1987) defines Pedagogical Content Knowledge (PCK) as "that special amalgam of content and pedagogy which is uniquely the province of the teacher". While general pedagogical knowledge can be generically applied to all teaching subjects, much of PCK is specific to individual teaching subjects. An emerging consensus is that teachers' knowledge of discipline-specific pedagogy is critical (see for example, Darling-Hammond, 2000).

The National Institute of Education in Singapore (NIE) is an institute of the Nanyang Technological University, one of three universities in Singapore. It is the sole provider of pre-service teacher education in Singapore and all primary school teachers receive their teacher preparation through programmes conducted by the Institute. Much of Singapore's teacher professional development is also provided by the NIE. With this background, the research project entitled Knowledge for Teaching Primary Mathematics was initiated in 2003 with the objective of studying the development of beginning primary school teachers' Mathematics Pedagogical Content Knowledge (MPCK).

While parts of a teacher's MPCK could be innate, most of this knowledge is acquired through the teacher's own education comprising general education, mathematics education, professional teacher preparation and development courses, interacting with experiential learning on the job. Part of the research would be to ascertain the state of teachers' MPCK at different points of the teachers' development in the beginning years and thus there was a need of an instrument to measure teachers' MPCK.

One of the reasons for focussing on primary teachers is that in Singapore, almost all primary teachers who teach mathematics are generalist teachers who teach the main subjects of English Language, Mathematics, Science and Social Studies. Very few of these teachers would have taken any substantial amount of mathematics content courses at post-secondary levels. The assignment of teaching subjects is done by school management based on the assumption that all have been "trained" through the methodology courses and does not take into account of the teachers' particular aptitude for or inclination towards any specific subject but is rather based on needs of the schools.

Defining and measuring teachers' MPCK is not a simple task as MPCK involves complex interactions between knowledge of generic pedagogy, a strong understanding of the discipline of mathematics and a sound grasp of the principles of mathematics-specific pedagogy. Such measurement is thus not achieved through measuring the level of competence in mathematical processes or depth of understanding of mathematical concepts. It was felt necessary to develop a specially designed instrument with scoring system which could measure MPCK. It was also intended to analyse teachers' understanding (or lack) of mathematics pedagogy through their answers as this can inform the design and delivery of more effective teacher professional development courses.

This paper describes process of developing the instrument and provides details of the instrument and describes the essential aspects of the scoring system with some illustrations.

Instrument

To develop an instrument to measure MPCK, the research team comprising experienced teacher educators sought to define important aspects of a primary mathematics teacher which would contribute to effective mathematics teaching and learning. To narrow down to a few manageable aspects, it was decided that broadly, a teacher with strong MPCK would be able to formulate explanations and representations of concepts, to deconstruct mathematical knowledge so as to see it from a learner's perspective, to have a sound grasp of mathematics as applicable to what they will be teaching and to have the ability to make sound choice of action when faced with typical pupil learning difficulties. The research team thus decided on an instrument to explore the following four aspects which are termed MPCK constructs:

(a) a teacher's own understanding of mathematical structure and connections,
(b) a teacher's knowledge of a range of alternative representations of concepts for purpose of explanation,
(c) a teacher's ability to analyse the cognitive demands of mathematical tasks on learners, and
(d) a teacher's ability to understand and take appropriate action for children's learning difficulties and misconceptions.

Since the measurement of the teachers' MPCK would have to take place in the context of primary school mathematics, four broad topic areas of primary school mathematics needed to be covered, namely, *Whole Numbers*, *Fractions & Decimals*, *Geometry* and *Measurement*. Acknowledging that a comprehensive instrument for these topics would be unwieldy, four items were crafted for each topic area, one for each MPCK construct as given above. Quigley, Lim-Teo & Fan (2003) piloted the instrument in early 2003 with earlier cohorts of student teachers and subsequently, there were some minor modifications. The current instrument has 16 items as given in the appendix and the items fall into the matrix table as shown in Table 1. Certainly, the items may test more than one of the MPCK constructs since the constructs are usually not mutually exclusive but it is a combination which comes into play whenever a teacher makes a decision to take a certain line of action. Table 1 thus indicates the primary construct which is being tested.

Table 1: *Distribution of items in the instrument*

MPCK Construct \ Topic Area	Whole Numbers	Fractions & Decimals	Geo-metry	Measure-ment
(a) Teachers' own knowledge of mathematical structure and connections	1	5	9	13
(b) Representations (multiple or alternative) of concepts for the purpose of explanations	2	6	11	14
(c) Cognitive demands of mathematical tasks on learners	3	7	10	15
(d) Pupil difficulties and misconceptions and choice of actions	4	8	12	16

The responses were intentionally open-ended so as to provide richer data than would be available from multiple choice items and participants had to answer the questions on the paper itself.

Scoring on the Instrument

The items are to be scored in two ways; firstly nominal codes are used to assemble a catalogue of responses, and secondly these codes are assessed and translated into ordinal degrees of understanding to enable a quantitative measure of performance. In the pilot study and the tests conducted in 2003, there were many different responses for each item, each with its own nominal code. As many of these responses had an insignificant number of responses, the similar responses were collapsed into larger but meaningful categories of related responses. Each category was given a new nominal code and the number of nominal codes was reduced to at most 12 per item. Some categories of responses were retained in spite of small numbers of responses due to the mathematical importance of those responses.

Also, a maximum of 4 points was used for each question. Although aware that the items may not be of uniform difficulty, the scaling of each item to the same maximum score was done so that we could determine the difficulty of each item relative to each other and to compare the respondent's performance across constructs or across topics.

In the following, we shall discuss four items, one from each construct and topic so as to illustrate the expectations and the relative scoring of the responses.

<u>Item 1</u>

A pupil tells you that when you multiply two numbers together, the product is always larger than either of the two numbers. How do you respond to the pupil?

For this item, this item was intended to test the teachers' own understanding of the concept of multiplication as applied to real numbers and the awareness of conditions ($a > 1$, $b > 1$) when applying general rules such $a \times b$ is greater than both a and b. Mathematical understanding of the use of counter-examples to disprove an assertion and the power of using counter-examples to convince a child of misconception are both operable here.

The full-scoring answers should be that the teacher will ask the pupil to consider the counter-examples of multiplying by zero, one or fractions. Such answers would attain the full score of 4 points. Partially correct answers would be those which state that the child is wrong with a counter-example stated. A yet lower score could be given if the counter-example of negative numbers was given because, while mathematically correct, negative numbers are not in the primary mathematics syllabus.

Item 6

When 23 is divided by 4, three possible answers are given

(a) 5.75

(b) $5\frac{3}{4}$

(c) 5 with remainder 3.

For each of them, write one story problem for which that answer is most appropriate.

The intent of this item was to see if the teacher could draw on different contexts to illustrate the different answers which could result from division. Stories which are considered appropriate are as follows:

(a) The story problem should use the context of money or some other measure where decimals were appropriate;
(b) The story problem should be about the sharing of pizzas or cake where fractions were meaningful, and,
(c) The story problem should be about sharing of entities which cannot be cut up such as marbles or balloons or people.

There were also partially correct answers where the stories were appropriate but contained minor problems such as not asking the question appropriately or not asking any question at all. Some story problems involving the division of a certain weight or volume were also acceptable for (a) and (b) although they do not illustrate the difference in context as well as those mentioned above. Some of the inappropriate answers were those which were not a story problem at all such as "Divide 23 by 4, giving your answer in decimals."

As there were various combinations of appropriate, partially correct and inappropriate story problems, the scoring depended upon the combination as given in Table 2.

Table 2: *Scores for combinations of answers for Item 6*

Answers	Points
All 3 stories are appropriate	4
2 stories are appropriate and 1 is partially correct	3
• Exactly 2 stories are appropriate and the third is inappropriate • 1 story is appropriate and the other two are partially correct	2

• One story is appropriate, one inappropriate and the third partially correct • Exactly one story is appropriate and the other two are inappropriate • Two are partially correct and the third is inappropriate	1
All other combinations	0

Item 10

(a) Describe an activity to teach pupils about the static view of “angle”.
(b) Describe an activity to teach pupils about the dynamic view of “angle”.
(c) If you are introducing the concept of angle to your pupils, which view would you introduce first? Give reasons for your answer.

The primary syllabus in Singapore schools first introduces the concept of angle at Primary 3 level. The approach at this level is for pupils to learn the concept as amount of turning. Subsequently, at the same level, pupils learn to identify right angles, to compare angles to right angles and to identify angles in 2-dimensional shapes. The measurement of angles in degrees with the use of a protractor is only learnt at Primary 4 level. Thus, it is up to the teacher when introducing the concept of angle whether to teach the static view of what is an angle between 2 straight lines or the dynamic view in terms of turning.

This item primarily tests the teachers’ understanding as to which concept was easier for children. The syllabus had intentionally chosen the dynamic concept first to illustrate something concrete such as amount of turning. Measurement of angles for comparison of size was thus left to a higher level. Part (a) and (b) of the item also tested the teacher’s wider understanding of the concept which would lead to their choosing an appropriate activity to illustrate either view.

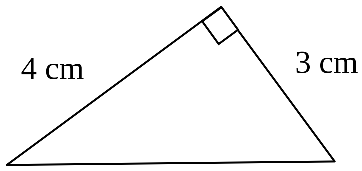
Full-scoring responses to (a) would include activities which would compare sharpness, the use of 2-dimensional objects in reality (for example, pizzas cut into different sizes would have different angles at centre) or the measurement of angles. The last activity could be included as part (a) did not require them to use the described activity to ***introduce*** the concept. Using figures and diagrams would be acceptable but accorded a lower score because of the more abstract nature of diagrams. Expected responses to (b) would be the turning objects (e.g. hands of a clock, turning the arm from the elbow, getting pupils to turn on the spot, etc.) or orientation and directions to illustrate the amount of turning. Expected responses to (c) are “from dynamic to static” and reasons could include (i) the dynamic activity being more concrete, more easily understood, more conceptual etc or (ii) that it could illustrate angles greater than 180°. Answers which chose to introduce the static view first because they wanted to familiarise the pupils with measuring angles or that the static view was easier to visualise were also acceptable but accorded a score one point below the expected response.

Item 16

The following diagram was given to Primary 6 pupils for finding the area of the triangle. Paul said that the area cannot be found because the height is not given.

(a) Why do you think he says that?

(b) Taking your *answer in (a) as a misconception/ learning difficulty* which many pupils may have, *give one example of a teaching activity* in the initial teaching of the area of triangle formula which can possibly reduce such a misconception.



This item tests the respondent's ability to diagnose the misconception in the context of van Hiele levels[1] in learning geometry and to choose appropriate teaching approaches to avoid such misconceptions. Each of the parts had a maximum of 2 points.

There are three categories of answers which merited a 2-point score for (a). The first is that Paul was looking for a vertical height, only recognising the 5 cm length as base. Some of the respondents drew in the perpendicular line from the top vertex to the base and indicated that Paul was looking for this line and its length. Another much less common but full-scoring answer was that Paul failed to recognise at the 3 cm length and the 4 cm length could be taken as a base-height pair. The third answer is to do with the Orientation of the triangle and that Paul could not visualise the height and base in this "tilted" view or that he could not "turn the triangle" to the standard view. Incomplete or less appropriate answers with some merit such as "he was taught that the height should be inside the triangle", "he does not understand the concept of height" were accorded 1 point while vague and inappropriate answers and those which merely restate the situation as "he cannot find the height" had zero scores.

Part (b) of the question had two categories of answers which are awarded the full 2-point score. The first referred to activities which exposed pupils to triangles in different orientations. The second category, somewhat related to the first, include activities which promote recognition of base-height pairs in a triangle. Both these categories activities are applications of an understanding of the van Hiele theory which are covered in all pre-service mathematics methodology courses. Partially acceptable answers include teaching particular recognition methods only applicable to right-angled triangles, explaining and reinforcing that the height is perpendicular to the base or teaching students to rotate the triangle or the paper.

Conclusion

So far, the instrument has been used to measure teachers' MPCK among various groups of teachers, both pre-service and more experienced teachers. Some findings on the performance of pre-service teachers from two different programmes can be found in Lim-Teo et al (2006) and in Yeo, Cheang & Chan (2006). These papers discuss the overall

[1] Details of the van Hiele theory, which is well-known to the mathematics teacher educator community, can be found in various mathematics methodology textbooks.

performance and the difference in performance at the beginning and end of the student teachers' initial teacher preparation programmes. Further research is needed on the data already collected: to carry out detailed analysis of the responses across various topics or constructs so as to use the findings to affect the design and delivery of our mathematics methodology courses.

References

Darling-Hammond, L. (2000). Teacher quality and student achievement. A review of State policy evidence. *Education Policy Analysis Archives*, *8*(1). http://epaa.asu.edu/epaa/v8n1. Accessed 19 April 2005.

Grossman, P.L. (1990). *The making of a teacher: teacher knowledge and teacher education*. Teachers College Press.

Lim-Teo, S.K., Chua, K.G., Cheang, W.K. & Yeo, K.K. J, (2006). The Development of Diploma in Education Student Teachers' Mathematics Pedagogical Content Knowledge, International Journal of Science and Mathematics Education, (to appear).

Quigley, M,, Lim-Teo, S.K. & Fan, L. (2003), *Teacher trainees' mathematics-specific pedagogical content knowledge – a snapshot*. Unpublished conference paper. Asia-Pacific Conference on Education. Singapore.

Shulman, L.S. (1987). Knowledge and teaching: foundations of the new reform*.* *Havard Educational Review, 57*, 1 – 22.

Yeo, J.K.K., Cheang, W.K. & Chan, C.M.E., (2006). Development of MPCK in Pre-service Teachers. APERA Conference Paper.

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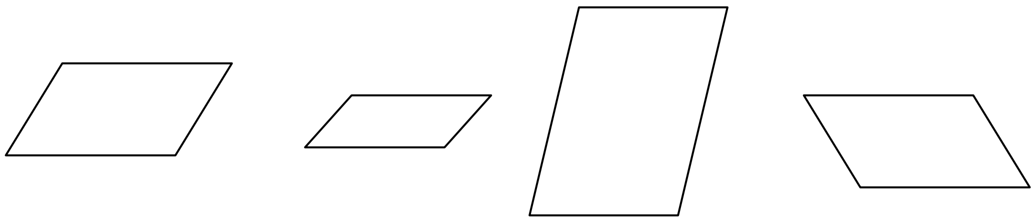
Appendix 1: Items of the MPCK Instrument

1. A pupil tells you that when you multiply two numbers together, the product is always larger than either of the two numbers. How do you respond to the pupil?
2. One way to illustrate the meaning of division by 3 using *equal sharing* is to tell a story: "*Benny had 12 grapes and distributed them equally among his three sons. How many grapes did each son receive?*" Write a story to illustrate the meaning of division using *repeated subtraction*.
3. Here are two problems. Do not solve them. (a) Ali is selling melons at 3 for \$5. How much would 9 melons cost? (b) Leni is selling melons at 3 for \$6. How much would 9 melons cost? Do you think that children in P4 would find these two problems *equally difficult*, or is one *easier* than the other? Explain your answer carefully.
4. Timmy gets some of his addition questions correct, but gets some of the simplest ones wrong. Here are five of the questions he did. If Timmy makes the same mistake with the *sixth question* below, *fill in the answer* you think Timmy would have got. $\begin{array}{r} 46 \\ +\ \ 3 \\ \hline 13 \end{array}$ $\begin{array}{r} 18 \\ +30 \\ \hline 48 \end{array}$ $\begin{array}{r} 8 \\ +16 \\ \hline 15 \end{array}$ $\begin{array}{r} 42 \\ +56 \\ \hline 98 \end{array}$ $\begin{array}{r} 85 \\ +\ \ 6 \\ \hline 19 \end{array}$ $\begin{array}{r} 72 \\ +\ \ 5 \\ \hline \end{array}$ Give one suggestion that might help Timmy get his questions correct.
5. A child approaches you with her examination results. She scored 11 out of 15 for paper 1, and she scored 20 out of 25 for paper 2. She writes both results as fractions, as usual, and carries out the following computation: $\frac{11}{15}+\frac{20}{25}=\frac{31}{40}$ She says that she is puzzled because she showed this calculation to her elder brother who said that it was incorrect, yet nevertheless she *was* given 31 out of 40 on her report card. How do you resolve this situation *for the girl*?
6. When 23 is divided by 4, three possible answers are given (a) 5.75 (b) $5\frac{3}{4}$ (c) 5 with remainder 3. For each of them, write one story problem for which that answer is most appropriate.
7. If you were introducing how to convert a decimal to a fraction, and had to use the following three decimals: 0.2, 0.03, and 0.23, for your introduction which of them would you use first, which second, and which third? Explain your choice.
8. A pupil showed the following working to convert $\frac{1}{8}$ to $12\frac{1}{2}$ % $\frac{1}{8} = 0.125\times100$ $= 12.5\%$ Do you see anything wrong in what the pupil wrote? If so, explain what it was.

9. (a)	For each of the following, state whether the statement is true or false. Justify your answer. (i) All squares are rectangles. (ii) There is no quadrilateral which is a rectangle as well as a rhombus.
(b)	X is a quadrilateral whose diagonals do not bisect each other. Which of the following five shapes is a possibility for X? (i) Rhombus (iv) Kite (ii) Rectangle (v) Square (iii) Trapezium You may choose *more than one* of the above.
10.	(a) Describe an activity to teach pupils about the static view of "angle". (b) Describe an activity to teach pupils about the dynamic view of "angle". (c) If you are introducing the concept of angle to your pupils, which view would you introduce first? Give reasons for your answer.
11.	In an introductory lesson on parallelograms, the following diagrams were given to P4 pupils as examples of parallelograms. If you were the teacher, draw two more difficult parallelograms which you would use so that your pupils will be better able to identify parallelograms. 
12.	Picture the following scenario: *Miss Aishah is teaching her primary 1 class the topic "Shapes" and had asked her pupils to bring objects to identify various shapes seen in these objects. Maria brought a tin of condensed milk and pointed out circles at the top and the bottom of the tin. Miss Aishah said "Good, Maria." Then Tony brought out a conical Christmas hat and said that there was a circle and a triangle in his hat, tracing the shapes with his fingers. Miss Aishah again said "Yes, good." Then Maria raised her hand and asked, "Teacher, can I say there is a rectangle in my tin?"* (a) Why do you think Maria asked the question she did? (b) If you were Miss Aishah, would you have accepted Maria's answer of a rectangle? Why?
13.	Explain carefully the difference between *mass* and *weight*. Give *an example of an activity* you could do with children to help them come to understand this difference.
14.	When teaching children about measurement for the first time, Mdm Ho prefers to begin by having the children measure the width of their book using paper clips, then again using pencils. Why do you think she prefers to do this rather than simply teaching the children how to use a ruler?
15.	Suppose you wish to know if your pupils really understand the formula *the area of a rectangle = the length* × *the breadth*. Here are two relevant problems. *Problem **A***: If a rectangle is 4 cm long and 3 cm wide, what is its area? *Problem **B***: Sketch two rectangles each having an area of 12 cm^2. If you can *only use one* of the problems, which one would you choose? Why?

16. The following diagram was given to Primary 6 pupils for finding the area of the triangle. Paul said that the area cannot be found because the height is not given.

(a) Why do you think he says that?

(b) Taking your *answer in (a) as a misconception/learning difficulty* which many pupils may have, *give one example of a teaching activity* in the initial teaching of the area of triangle formula which can possibly reduce such a misconception.