Using a Multi-Dimensional Approach to Understanding to Assess Primary Students' Mathematical Knowledge

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Mathematics assessment should involve more than just assessing students' knowledge of content. How students understand the content is important and should also be a part of the assessment process. For robust knowledge, students need a multi-dimensional understanding of content, including a focus on skills, properties, uses, and representations related to the content under study. This paper discusses the importance of assessing students' understanding across various dimensions and illustrates how their understanding can vary by dimension. Specific examples of assessment items come from the International Project on Mathematical Attainment, a longitudinal study of primary students' achievement in which the United States of America and Singapore were participants. Achievement results from these two countries illustrate how a focus on overall results, without consideration of an analysis by dimension of understanding, can mask important similarities and differences in achievement.

Key words: assessment, comparative study, dimension of understanding, mathematics, primary students

Many mathematics educators have argued for considering multiple perspectives in the learning of mathematics content. For instance, Freudenthal (1983) considers different ways in which a topic might be used and how those different perspectives lead to different understandings. The synthesis of research about children's understanding of mathematics in the United States compiled by Kilpatrick, Swafford, and Findell defines mathematical proficiency as consisting of five intertwined strands: procedural fluency, adaptive reasoning, conceptual understanding, productive disposition, and strategic competence (National Research Council, 2001). These strands are interconnected and interdependent. Kilpatrick et al. argue that students need to develop competence in all five strands concurrently to develop a robust understanding of mathematics. Krutetskii (1976) showed that, at least among gifted students of mathematics, one can identify students who regularly use algebraic or analytic approaches to solve problems and others who use geometric or spatial approaches. Hence, curriculum materials and assessments that use a multi-dimensional perspective present a balanced view of mathematics that accommodates classrooms with a range of students having different mathematical strengths and learning styles.

One approach to a multi-dimensional perspective on understanding is that of SPUR, denoting skills (those procedures that students should master with fluency), properties (the principles underlying the mathematics), uses (the applications of the concepts to the real world or to other concepts in mathematics), and representations (graphs, pictures, symbols, and other visual depictions of the concepts) (Thompson & Senk, 2008; Usiskin, 2007). Students who have a robust understanding of mathematics should possess understanding in each of these dimensions. Although originally used as a basis for developing curriculum materials in the United States, SPUR can be a

powerful tool for assessment. If assessments consistently determine students' achievement in these four dimensions, they have the potential to provide educators with insights into how students' development of mathematical understanding progresses.

The purpose of the study reported in this paper is to answer the following research questions:

- How does the mathematics achievement of primary students vary across different dimensions of understanding?
- How do the growth patterns in mathematics achievement differ according to dimension of understanding?

Sample and Procedures

The results and discussion reported in this paper come from a secondary analysis of achievement from students who participated in the International Project on Mathematical Attainment (IPMA) in the United States and Singapore. IPMA was a multi-country longitudinal study that tracked students' growth in mathematics from their first year of schooling through the end of primary schooling, generally 5 to 6 years. Many previous large-scale international studies (e.g., Trends in International Mathematics and Science Study (TIMSS) or Programme for International Student Assessment (PISA)) have focused on student achievement at a particular point in time, or on the development of mathematical understanding over time using different groups of students (Dossey, 2003; de Lange, 2007). Although the sample size for IPMA was much smaller than the samples for TIMSS or PISA, IPMA contributes to the research base by providing a longitudinal perspective of the mathematical progression of the same group of students across the entire span of primary school.

The IPMA tests are unusual in their construction in that the tests grow over time with items administered from the first time they appear until the final test. That is, Test 1 is embedded within Test 2 (i.e., Test 2 contains the same 20 items as Test 1 as well as an additional 20 new items), Test 2 is embedded within Test 3, and so on through Test 6. The embedded nature of the test makes it possible to track growth in student achievement on a core subset of test items across the primary grades. For a copy of the IPMA test in its entirety, see http://nsdl.org/resource/2200/test.20061004102347933T.

For the purposes of the secondary analysis, the researchers from the United States and Singapore categorized each of the items according to its SPUR dimension of understanding. Table 1 contains sample test items for each of the four dimensions of understanding. The sample items have been chosen from two content areas to illustrate that this using a multi-dimensional approach to understanding is applicable across the major content areas in the primary curriculum.

Table 2 reports the number of items in each of the SPUR dimensions for Tests 1 through 6. Achievement was analyzed for each test by dimension of understanding as long as there were five items in a given dimension of understanding on that test.

Table 1. Sample Test Items According to the SPUR Classification

Content Strand	Skills	Properties	Uses	Representations
Number	70 × 0.3 =	Give a number to two decimal places, which lies between 122.257 and 122.263	6 tickets cost $2.10. What is the cost of 13 tickets?	What is the number shown on the number line? [picture of number line with labels at 0.0 and 0.1 and marks for every 0.01]
Data/Chance	An unbiased dice is thrown. What is the probability of throwing a six?	The probability of it raining tomorrow is $\frac{1}{4}$. What is the probability of it **not** raining tomorrow?	A football team scored the following number of goals in 10 matches: 3, 0, 2, 2, 4, 2, 5, 1, 0, 1. What is the mean number of goals scored per match?	[Bar graph of hours of rain for each day of the week] (a) On which day did it rain for the longest time? (b) How many hours of rain were there on Friday?

Note: Items are taken from the IPMA test developed by Professor David Burghes at the University of Exeter. Items were modified slightly in each country to use appropriate cultural spellings and monetary usage.

Table 2. Item Breakdown by Test and SPUR

Test	Number of Items				
	Skills	Properties	Uses	Representations	Total
1	7	4	0	9	20
2	17	8	3	12	40
3	23	11	6	20	60
4	28	15	7	30	80
5	40	21	11	38	110
6	56	25	17	42	140

In the United States, students from six schools in two different states were solicited to participate in the study. Although 372 students began the study at the beginning of kindergarten (roughly age 5 years), there were only 181 students who completed all the tests by the end of fifth grade (roughly age 10 years) (Bleiler & Thompson, 2010; Thompson, 2004). In Singapore, students from three different schools participated from primary one (roughly age 6 years) through primary five (roughly age 10 years). Overall, 1016 students began the study but only 856 took all the tests (Kaur, Koay, & Yap 2004). Data were collected from Fall 1999 through Spring 2005 in the United States and from Spring 1999 through Fall 2003 in Singapore.

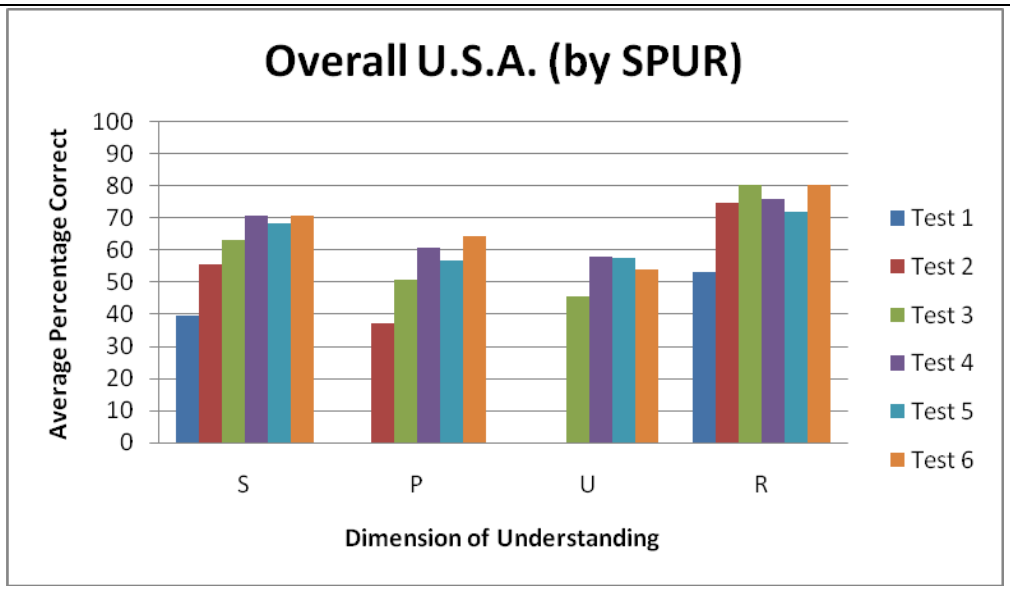
Results and Discussion

Much is often made about differences in achievement between the United States and Asian countries, such as Singapore, on international assessments. For instance, on the final IPMA Test 6, the mean percent correct for U.S. and Singapore students was 71% (s.d. = 11%) and 76% (s.d. = 12%), respectively. Similar differences in

achievement can be found for the other tests as well. Although these overall test results suggest that students in Singapore score somewhat higher than those in the United States, the overall means provide little to no insight on what accounts for these differences. Hence, there is little information for educators to use to determine what changes in curriculum or instruction should occur to enhance achievement.

Differences in Achievement by SPUR Dimension

Figure 1 shows the achievement by dimension of understanding for each of the tests taken by students. By Test 6, differences in achievement are due primarily to differences in performance on items dealing with skills or with uses of mathematics. On the final test, students in both countries performed comparably on items dealing with properties and with representations.



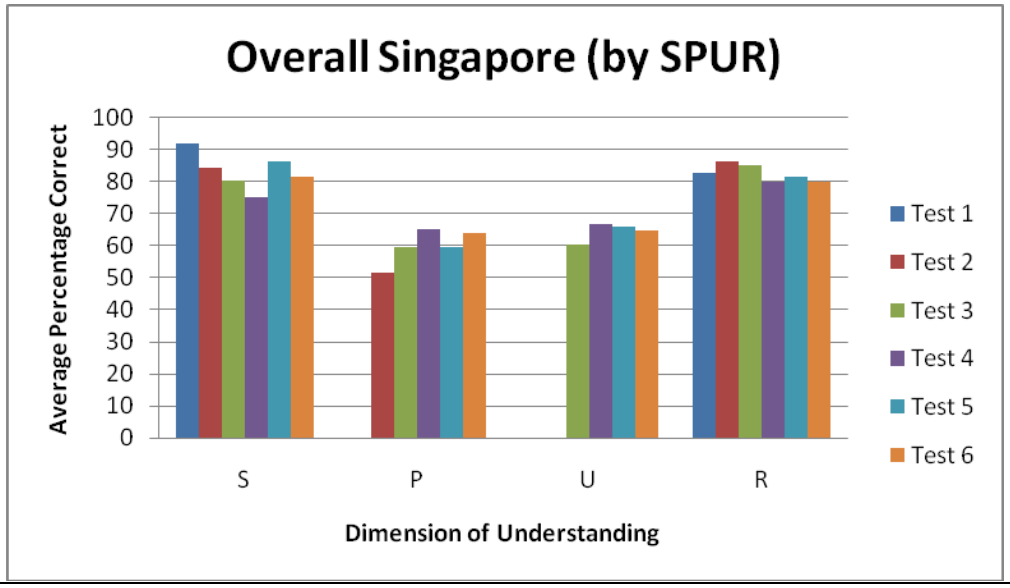


Figure 1. Achievement by students in the United States and Singapore on IPMA Tests 1 through 6 by SPUR dimension of understanding (Note: Results by a given SPUR category are only reported when there are at least five items in that category. For Properties, analyses were conducted beginning with Test 2; for Uses, analyses began with Test 3.)

In the United States, achievement on skills and properties grows over time, achievement dealing with representations is typically the highest, and achievement on uses is typically the lowest. The figure suggests a need to focus instruction on properties (i.e., the principles underlying the mathematics) and on how the various content topics are used to solve real-world problems.

In contrast, in Singapore, achievement on skills begins high, drops somewhat, and then increases. Achievement on representations is consistently high; achievement on properties and on uses, although lower than for the other dimensions, is relatively consistent across the tests.

Nevertheless, the analysis of the achievement results for both countries suggests that achievement across the different dimensions of understanding varies. If understanding in each dimension is important, then such analyses provide insights into where teachers need to focus instruction in order to increase students' mathematical proficiency.

Growth by Dimension of Understanding

Because each test was embedded in subsequent tests, it is possible to consider the growth in students' achievement on a given set of items over time. In particular, there were three testing administrations for which students answered items 1 through 42b, a total of 80 questions overall. At the first administration, the overall percents correct for U.S. and Singaporean students on these items were 70% (s.d. = 13%) and 74% (s.d. = 12%), respectively; at the second administration, the percentages were 78% (s.d. = 12%) and 86% (s.d. = 11%), respectively; and at the final administration, the overall percents were 88% (s.d. = 10%) and 90% (s.d. = 10%), respectively.

Although it is clear that the overall achievement for students in both countries increased over time, at issue is the extent to which the increase was consistent for all four dimensions of understanding. Figure 2 highlights students' growth in achievement on these items over these three administrations according to the dimension of understanding.

In both the United States and Singapore, students' achievement within each dimension of understanding progressively increases over time. For U.S. students, by the final administration at fifth grade, achievement across the four dimensions is roughly balanced, so that the difference in performance across skills, properties, uses, and representations at this grade varies by at most 10%. Students appear to be developing a broad understanding of mathematics in its varied facets.

On the final administration of these items for Singaporean students, achievement across skills, properties, uses, and representations varies by as much as 20% (e.g., between skills and properties). It appears that students are much more proficient at skills and representations than at properties and uses. So, although overall achievement was high, the achievement was not as balanced across the four dimensions as might be desired.

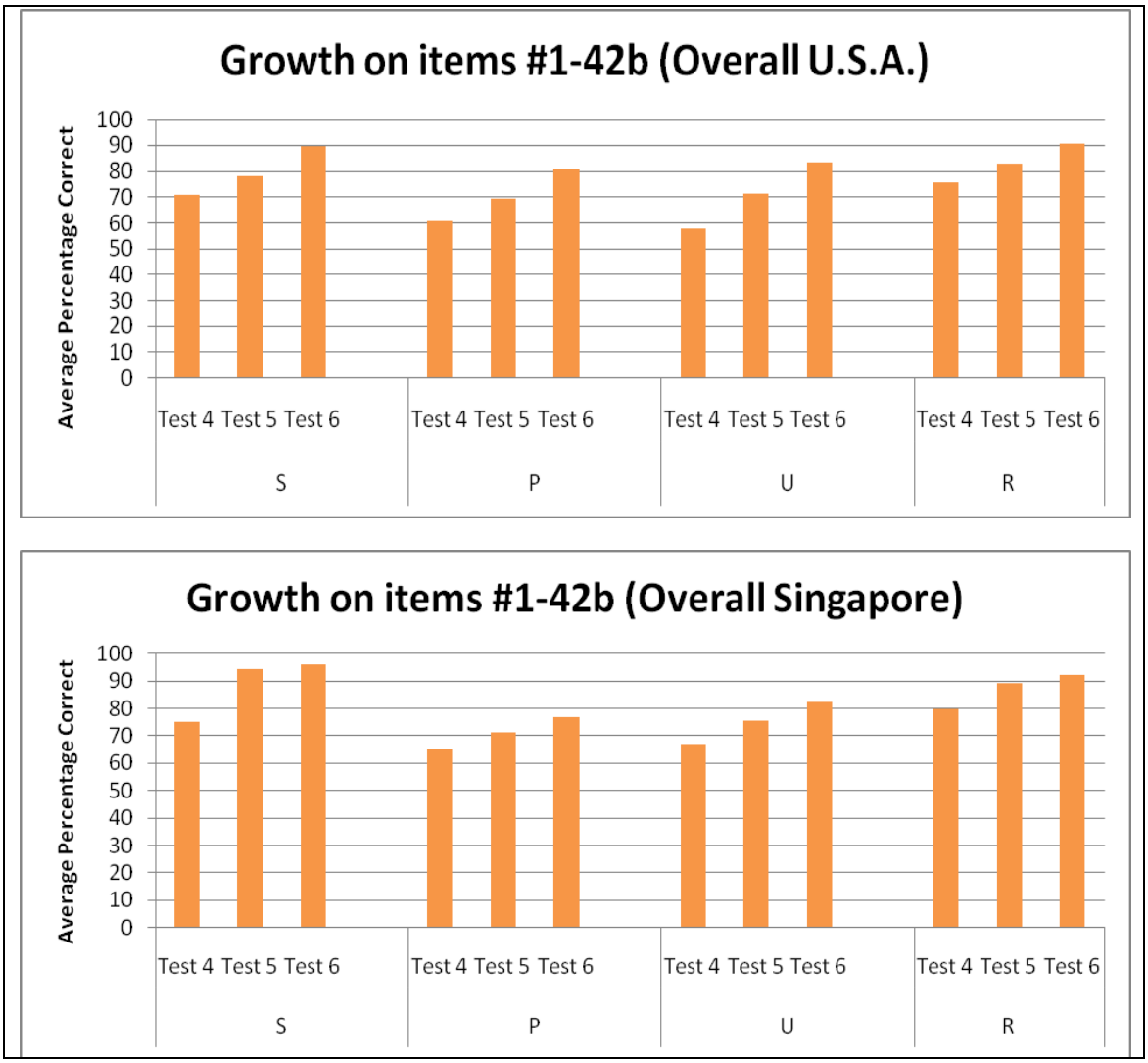


Figure 2. U.S. and Singaporean students' growth in achievement on common items 1-42b for three administrations of the test by dimension of understanding (Because of the embedded nature of the tests, items 1-42b were administered as part of Tests 4, 5, and 6.)

Items 43 through 56, a total of 30 questions, were administered twice. There were only 4 items in the dimension of uses, so results for these items were not analyzed. Figure 3 highlights students' growth in achievement on the 26 questions comprising the three dimensions of skills, properties, and representations.

On these items, for U.S. students, growth in achievement for each dimension of understanding increases over the two test administrations. For Singaporean students, achievement decreases slightly for the dimension of properties. Although U.S. students performed slightly better on the Properties dimension, Singaporean students performed a good bit better on the Skills and Representations dimensions.

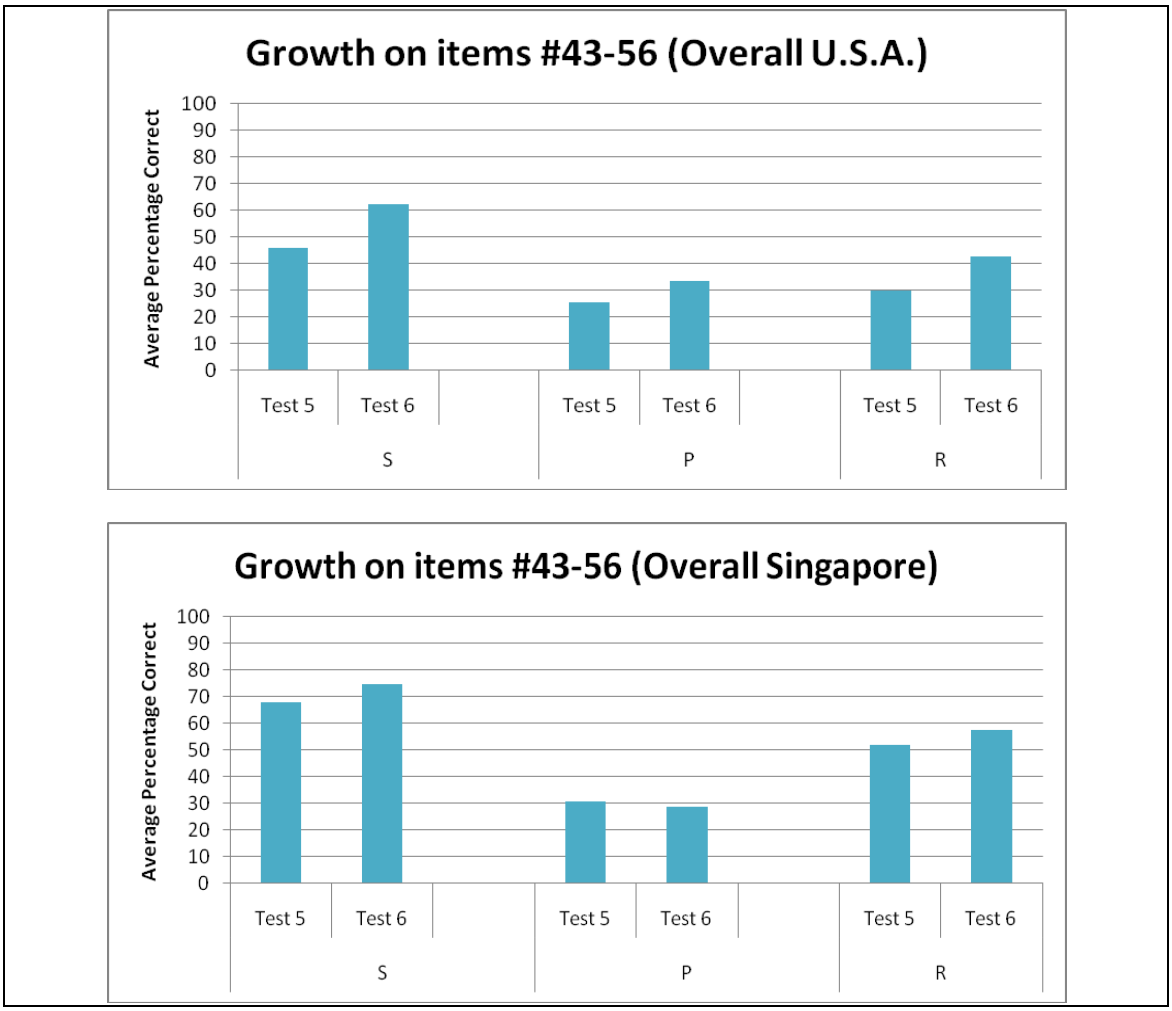


Figure 3. U.S. and Singaporean students' growth in achievement on common items 43-56 for two administrations of the test by dimension of understanding (Because of the embedded nature of the tests, items 43-56 were administered as part of Tests 5 and 6.)

Conclusion

The results shared in this paper suggest that analyzing achievement by dimension of understanding provides a more nuanced view of mathematical attainment than overall scores alone. Total scores fail to provide information to educators about where students might struggle, what aspects of their understanding are underdeveloped, or where students have hit a plateau in understanding. If it is important for mathematical proficiency that students develop ability to handle skills, properties, uses, and representations about a particular concept, then assessments should include items related to all four dimensions of understanding. Results should also be analyzed according to these dimensions. In this way, educators can customize instruction to ensure that students develop a robust and balanced understanding of mathematics.

Although it is not uncommon for test results to be disaggregated by content strand, it seems to be less common to disaggregate by dimension of understanding. The actual dimensions of interest to a particular country might be somewhat different, but, as indicated in this paper, exploring achievement by whatever dimensions of understanding are considered valuable can be informative for both research and

subsequent instruction. The results and discussion reported here suggest that this multi-dimensional perspective on assessment is applicable across countries.

The analyses of such results by dimension of understanding can also provide instructional insights for classroom teachers within or across schools (Bleiler & Thompson, 2010). Teachers can determine whether (1) instruction focused on all four dimensions but students failed to achieve as desired, or (2) instruction focused on only some dimensions and students performed as expected, but instruction needs to be modified to incorporate other important facets of mathematics. By better understanding the nature of students' mathematical proficiency, teachers can help them achieve at the highest possible level.

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