STUDENTS' ERRORS IN MATHEMATICS AT THE JUNIOR COLLEGE LEVEL

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Abstract

This study involved 69 students of mixed ability from a Junior College in Singapore. The aim of the study was to document the types of errors made by the students and the reasons why such errors were made. The test instrument had 15 free response items covering various topics from the syllabus. The errors made by the students were documented. Further information on the students' reasoning processes was obtained through semi-structured interviews of a selected group of students from the sample. Based on the test and the interviews, the errors were then analyzed. Five areas of mathematical reasoning errors emerged from the analysis. Many of the errors arose because students failed to bridge the cognitive gaps in their learning, which invariably contributed to a lack of structural awareness in mathematical ideas. Thus, mathematical knowledge gained from earlier experiences was inappropriately applied to new contexts having different structures, giving rise to errors in the use of analogies.

Key Words: reasoning errors, Junior College mathematics, mathematical reasoning

The development of students' ability to reason is an important aspect of any school curriculum, and mathematics is widely regarded as the primary means to accomplish this. The Singapore curriculum is no different. It has reasoning as one of the major goals of mathematics education. The Singapore mathematics curriculum document introduces mathematics as "an excellent vehicle for the development and improvement of a person's intellectual competence in logical reasoning, spatial visualisation, analysis and abstract thought" (Ministry of Education [MOE], 2006, p. 1). Mathematics is, after all, largely about reasoning, and as such it is sensible to expect one's reasoning skills to be inculcated by doing mathematics. However, as much as mathematics education improves reasoning skills, the converse is equally true: good reasoning is needed for doing mathematics. Unlike science which can depend on observations to verify some of the hypotheses, mathematics is wholly reliant on deductive reasoning. Reasoning is integral to mathematics.

In 2000, the National Council of Teachers of Mathematics [NCTM] (NCTM, 2000) advocated that "systematic reasoning" be a defining feature of mathematics. *Reasoning and proof* was made one of five process standards for all grade levels. Mathematical reasoning had not always enjoyed such prominence in the education curriculum of the United States. The Professional Standards for Teaching Mathematics (NCTM, 1991) advised that students be given frequent opportunities to engage in mathematical discussions in which reasoning is valued. Polya (1957) also suggested that students work independently as much as possible when attempting problem solving activities. In Singapore too, teachers are encouraged to use more discussions and interactive approaches, to emphasize more the understanding of concepts rather than just mastery of procedures and routines (Lim, 2002). However, this is not necessarily a regular feature of our mathematics classes.

There are disciplines where the acquisition of knowledge is a smooth progression from the elementary to the advanced. Mathematics is not one such subject. New mathematical knowledge may not readily latch onto previous knowledge. Learning mathematics requires significant reconstruction in thinking (Tall et al., 2000).

In elementary mathematics, students explore and perform operations on lower-level mathematical constructs with pre-existing meaning and properties, but as they advance to more formal mathematics, students are presented with definitions which give rise to theorems and mathematical objects with new meanings through a sequence of logical deductions. They need to make the cognitive shift from following rules and procedures to constructing mathematics through logical inferences. Some of the specific details that formed the main focus of their attention in the initial exploratory stage must give way and be replaced by expressions of generality (English & Sharry, 1996). Some rules which students learned and applied so faithfully are challenged in the face of new mathematical contexts. Students are forced to consider if these rules, which had served them so well, are still applicable in the new contexts. Even if mathematical ideas were learned in a meaningful way in one context, conceptual difficulties arise when the old meanings no longer hold in new contexts (Tall et al., 2000).

Consequently some students, who achieved high scores when they were in the lower levels, do very badly at higher levels when they are confronted with abstract mathematics, generalisations and the demands of deductive proofs (Lim, 2002). Many students find themselves struggling to cope with the increased depth and rigor of the Junior College mathematics curriculum. Accordingly, the focus of this study was to investigate the kinds of reasoning errors that students at the Junior College make and the reasons behind such errors. More specifically, the research questions were:

1. What are the mathematical reasoning errors made by students?
2. What may be some causes of the errors?

Answers to these questions will provide teachers with insights into the reasoning errors behind the mistakes in a student's problem solving strategy. Such knowledge will help teachers anticipate students' errors, bearing in mind the misconceptions at the root, so that they can either pre-empt these errors in instruction, or include questions that might surface such errors in assessment for learning.

NCTM has mentioned some domain specific reasoning, such as spatial reasoning, geometric reasoning, algebraic reasoning, proportional reasoning, probabilistic reasoning, and statistical reasoning in their literature (NCTM, 1989; NCTM 2000). We will not be discussing such domain specific reasoning in this study. Instead, we have chosen to focus on generic mathematical reasoning processes that lead to errors in students' solution strategies in problem solving. So *errors of mathematical reasoning*, in this study, refer not just to mistakes made by students, but to errors which are due to lapses in a student's mathematical reasoning process.

METHODOLOGY

A qualitative approach was deemed to be more appropriate for this study, as qualitative studies seek to gain understanding from the observed data and build towards a theory from the intuitive understandings gained (see Merriam, 1998). This qualitative study was carried out with 69 students (Year 12 students) spread over four classes in a Junior College (JC) in Singapore. These students had completed the A-level curriculum and were preparing for their Preliminary Examinations, which is a school summative assessment at the end of the JC course and in which the students are tested on what they are taught over the two years. There were 28 female and 41 male students in the sample. These students were about 18 years of age and had all taken Additional Mathematics, a harder O-level course (end of Year 10) in their respective secondary schools. The sample can be considered to be of mixed abilities.

The participating students had to sit for a 2-hour test consisting of 15 free response items that covered a range of topics including: proof by mathematical induction, inequalities, transformation of graphs, functions, sequences, calculus, trigonometry, complex numbers, permutations and combinations. These topics were chosen to represent a spread of the A-Level Pure Mathematics syllabus. Questions were constructed to test mathematical reasoning skills, not their ability to manipulate expressions. The test items were carefully selected and checked by an expert in the field. The instrument was piloted with a small group of 6 students (other than those in the sample) and as a result some of the questions were revised. The test was marked and the errors documented. The common errors were then further analyzed in the light of the literature to understand the mathematical reasoning behind these errors. Semi-structured interviews were conducted with 11 selected students based on the solutions they had provided to either confirm, refine or to reject the initial understandings. This required an insight into the students' thought processes and an understanding of the meaning that the students had constructed for themselves. Six of the interviewed students were from the top third, two from the middle and three from the bottom third of the students in the sample.

RESULTS

Five areas of mathematical reasoning errors emerged from the analysis of the responses received in this study. They are (1) errors from an over-reliance on instrumental understanding, (2) errors from the inappropriate use of inductive reasoning, (3) errors from inadequate analogical reasoning, (4) errors from a lack of structural awareness, and (5) errors from ineffective concept images. Because of space constraints, actual solutions from students and interview extracts will not be shown in this section. However, examples will be highlighted during the presentation.

1. Errors from an over-reliance on instrumental understanding

Many of the errors that surfaced in this study could be attributed directly or indirectly to an over reliance on instrumental understanding, where students acquire fixed plans and procedures without understanding why the procedure works. Here, in this study, rules for the transformations of graphs, and conditions for the existence of composite and inverse functions were memorised. The problem with such instrumental learning is that students cannot fall back on any other processes should their memory fail. Moreover, these fixed plans derived through instrumental learning are not very versatile and often fail the student when he is faced with a non-routine problem.

Consistent with Harel and Sowder's (1998) observations, there are indications from this study that instrumental understanding gives rise to something analogous to the authoritarian proof scheme, the ritual proof scheme and the symbolic proof scheme. Unable to provide reasons for their mathematical procedures, participants in this study had on a number of occasions pointed to the notes or to the words of their teacher as their reasons – the exhibition of the authoritarian proof scheme. The ritual proof scheme was evidently manifested in the participants' association of the proof by mathematical induction with the format of the proof. In fact, there were some participants who thought that the inductively presented solution constituted a valid proof, but it was not a proof by mathematical induction because it did not adhere to the format. They had judged the correctness of an argument by its form rather than by the correctness of the reason. Furthermore, some participants were so engrossed with remembering the form that they had placed emphasis on unessential points, and missed crucial information, like the necessity of n to be an integer in any proof by mathematical induction. The lack of relational understanding had also hampered students' ability to understand the meaning behind symbols. In this study, we saw glaring mistakes made by the participants in their work regarding the "| |" symbol. Many see "| |" as a reflection of negative (x or y) values, and did not think of "| |" as absolute value. So some drew the $|f(x)|$ graph thinking it represented $f(|x|)$, while some others reflected different portions of the graph of $f(x)$.

2. Errors from the inappropriate use of inductive reasoning

This study also revealed students' lack of ability in constructing rigourous mathematical proofs. They are limited by their possession of the inductive proof scheme, and do not know the difference between formulating a conjecture and establishing a mathematical truth. Some participants did not understand that a proof must be general. They generalised using discrete points and accepted inductive reasoning as an acceptable mathematical proof. In this study, it was also noted that the students' weakness in proofs may be due to their lack of understanding of "implication" (i.e., modus ponens). Some participants felt that one can only "assume" if he has a reason to do so. As such the base case of the proof by mathematical induction was taken to be the "basis" for assuming that the $p(k)$ statement is true.

3. Errors from inadequate analogical reasoning

Instrumental learning approaches also give rise to analogical reasoning errors. In analogical transfer errors, procedures are detached from their meanings in source systems and are inappropriately applied to other contexts. Without adequate relational understanding on both the source and the target systems, students had simply focused on the surface similarities and had neglected the structural differences. The responses in this study exhibited a variety of analogical reasoning errors: the distributive property was misapplied in the context of logarithms and in the arguments of complex numbers, differentiation procedures meant for polynomials were applied on exponential functions, methods for solving equations were directly transported to solve inequalities and the BODMAS convention was inappropriately used to decide on the order of compound transformations. Another analogical reasoning error which surfaced in this study was the use of $\pi\int[f(x)-g(x)]^2\ dx$ to find the volume generated by completely rotating the area bounded by two curves $y = f(x)$ and $y = g(x)$ about the x-axis. This is due to an erroneous transfer in applying the formula of finding area to that for finding volume, as illustrated in Figure 5.1.

Source System	➔	**Target System**
Area = $\int y\ dx$	➔	Volume = $\pi\int y^2\ dx$
Area = $\int[f(x)-g(x)]\ dx$	☒	Volume=$\pi\int[f(x)-g(x)]^2\ dx$

Figure. 1. Analogical transfer error in finding volume

4. Errors from a lack of structural awareness

Many participants in this study did not account for the difference in structure between equations and inequalities and directly applied the solution structure of equations on inequalities. They do not see that in algebraic transformations of inequalities (and for that matter also in equations) the truth value must be conserved. This led students to misapply operations like squares or square roots to both sides of inequalities. It was also common that students write meaningless inequalities, which contained the "$\pm$" symbol or complex numbers. In addition, this study showed that students generally do not approach problems from multiple perspectives. Consistent with Tsamir, Almog and Tirosh's (1998) findings, this study saw that in solving inequalities, the algebraic method was the most popular method among students, yet it proved to be the most unreliable method, especially for the weaker students. Similar observations were made in the participants' solution to complex numbers. Most chose the algebraic method over the use of an Argand diagram, which was more reliable. In addition to the structural ignorance of equation-inequality, participants in this study also showed a lack of awareness in the structural difference between vectors and scalars. Some had applied the Ratio Theorem which is meant for vectors on scalars.

5. Errors from ineffective concept images

Finally, there is evidence from this study that some participants did not have effective concept image on functions. It surfaced during the interviews that the participants had regarded graphs of circles to represent functions. One participant even used the term “many-to-one functions”. Moreover, the notion of the domain on which a function is defined is starkly absent in students’ concept image of functions. Some readily accepted $y = \ln x,\ x \in \Re$ as a function. “Equations”, “graphs” and “$f(x)$ = something” are some other notions included in the participants’ concept image of functions. The univalence property was conspicuously absent from their concept image. Such notions are consistent with what Even (1993) had observed in her study on prospective secondary school teachers’ concept on functions. Without effective concept images, students rely on their memory for rules to check for the existence of composite functions and procedures to find inverses. In computing the inverse, more than a quarter of them had ignored the domain on which a quadratic function is defined, while some others wrote an expression which violated the univalence property of functions. Their understanding of the functions was largely instrumental.

Causes of the Errors

This study also sought to explain the possible causes of the errors identified.

1. Errors from an over-reliance on instrumental understanding

The data gathered from this study showed that many students learn procedures and rules without much thought to the reasons behind them. According to Sfard’s (1991) model of concept formation, processes must be reified into structural objects for students to understand relationally (Linchevski & Sfard, 1991). Rules without reasons are like processes without objects; any new processes that are built on these will be detached from a developed system of concepts and left dangling without a foundation. Students’ understanding will remain instrumental without reification (Sfard, 1991). The problem is that reification is inherently difficult, so much so that the structural approach may be beyond reach for some (Linchevski & Sfard, 1991). In this regard, there is evidence from this study that there are Junior College students with only an operational understanding of the “=” symbol, which was also observed by Kieran (1981). This may have also contributed to their inability to appreciate that the algebraic transformations applied must conserve the truth value of inequalities.

It may be convenient to direct the cause of instrumental understanding to the lack of ability on the part of the students, but as Skemp (1978) observed, it is also possible that some teachers make a reasoned choice to teach instrumentally. Students may also choose to learn instrumentally. For students to learn relationally, there need to be a match in the teaching and learning styles of the students and the teachers. Students must want to learn relationally and teachers must choose to teach relationally.

2. Errors from the inappropriate use of inductive reasoning

It is likely that the possession of the inductive proof scheme may be due to influences from everyday reasoning. As Polya (1941) said, “rigorous, precise, properly so-called logical reasoning is found in its pure form only in mathematics” (p. 450). This may be one reason why students are not able to differentiate between formal presentations of a proof and heuristic strategies in deriving conjectures. It was also surfaced in this study that students’ acceptance of inductive reasoning as a valid mathematical justification could be influenced by previous encounters with pattern recognition questions from secondary years. At that level, once students manage to obtain the correct general mathematical expression from given patterns, the problem is considered solved, but this is not so in a mathematical proof. Moreover, students may not appreciate the generality of a mathematical statement. Some were unable to see that the verification of a statement for discrete cases does not establish the truth of the general statement meant for all. Unable to reconcile the conflict between what they think is sufficient enough evidence from inductive reasoning and the formal form of the proof by mathematical induction, students resort to the ritual proof scheme. Hence they associate the method of proof by mathematical induction with the format of the proof. The problem has also been compounded by students' limited exposure to proofs. Constructing a mathematical proof is a new experience to students. In particular, the proof by mathematical induction is a new experience which presupposes many other concepts like the ordering of natural numbers, familiarity with logical implications and proficiency in algebraic manipulations (Ernest, 1984).

3. Errors from inadequate analogical reasoning

Errors from inadequate analogical reasoning revolve around students' incapacity to isolate and direct structural similarities from the source to target problems. One reason is that students do not have a relational understanding of the source and target systems. In this study, it was found that students apply the formula $\pi\int_a^b [f(x)-g(x)]^2\ dx$ to find the volume of the solid generated by a region bounded by curves $y = f(x)$ and $y = g(x)$. The participants admitted that they used the formula as such because $\int_a^b [f(x)-g(x)]\ dx$ gives the area bounded by the two curves $y = f(x)$ and $y = g(x)$ (see Figure 1). This may be due to the possession of the perceptual proof scheme where students cannot anticipate the results of transformations accurately (Harel & Sowder, 1998). They erroneously thought that the effect of squaring and multiplying by π always gives the volume that is generated by rotating the bounded area one revolution about the x-axis.

Besides a lack of relational understanding, analogical reasoning errors may also result from a lack of conditional knowledge (English, 1998). Students do not observe the conditions under which a procedure can or cannot be applied. Many participants had ignored conditions in the Fundamental Theorem of Calculus and applied the procedures for integration to functions discontinuous over a bounded interval. Some participants missed the condition that |z| must be 1 for $z^* = \frac{1}{z}$.

4. Errors from a lack of structural awareness

Progress in mathematics requires significant reconstruction in thinking (Tall et al., 2000). Tall (2004) postulated that students' confusion between equations and inequalities may be due to cognitive obstacles from their subconscious links to incidental properties from earlier experiences. Influences from such met-befores periodically surfaced in this study – inductive strategies obtained from their previous experiences in deriving generalized mathematical expressions from patterns were used in proofs, the BODMAS convention was used in transformations, solution structures meant for equations were applied to inequalities, the Ratio Theorem was used on scalars, and the domain in functions were habitually ignored. In all of these cases, the properties meaningfully learned and applied in former contexts no longer hold in the new. As Harel (1998) observed, students possessed a habit of mind of arriving at conclusions without examining their meaning and truth.

Students also tend to prefer the algebraic approach to solving inequality. However as we saw from the participants' responses, those that relied solely on algebraic method alone made the most errors. On the other hand, the graphical approach was the most reliable. This is consistent with Tsamir, Almog and Tirosh's (1998) observations. Harel (1998) suggested that it is advantageous to change ways of understanding a concept in one's attempt to solve a problem. He also advised that if a concept can be understood in different ways, it should be understood in different ways. Ways of understanding can impact ways of thinking (Harel, 1998). The lack of multiple perspectives may have contributed to the lack of structural awareness.

5. Errors from ineffective concept images

Consistent with Harel's (1998) observation, participants in this study had not built effective concept images on functions; rather, they had relied on their concept definitions. Some of them had memorized the rules for the existence of functions. As Even (1993) noted, students' concept images are determined by the functions they work with and not by knowing the definition. The students' concept images are shaped and moulded through their experiences with examples and non-examples. Students' ineffective concept images could be due to the narrow range of examples used in classes. The examples that were shown to students may have been limited to just "nice" graphs that can be expressed as "$f(x)$ = something". As Vinner (1983) noted, very often students acquire wrong concept images as a consequence of a specific set of examples given to them. Students with effective concept images will be able to communicate their corresponding concept definitions in their own words and can think about them in general terms and connect them to other concepts (Harel, 1998); those without effective concept images fall back on instrumental understanding and memory as this study showed.

CONCLUSION

This study had a relatively small sample and thus was limited in its size and reach. The participants were all from one school and the range of problems used was also limited. However, some interesting results surfaced through the study. As the responses to the test instrument show, it is difficult, in practice to make judgement about a student's mental processes from what he or she writes. Much of the instrumental understanding that surfaced in this study would have gone unnoticed if not for the interviews. As Skemp (1978) noted, the best way to help them would be to talk to the students. There is no short-cut to attaining relational understanding.

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