TEACHER AND STUDENT CHOICES OF GENERALISING STRATEGIES: A TALE OF TWO VIEWS?

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Abstract

This paper reports a study whose aim was to examine teacher and student choices of generalising strategy to determine what teachers would prioritise and what students would judge as the most helpful strategy for expressing generality. Data were collected through the administration of a worksheet and a questionnaire to 16 secondary school mathematics teachers and 47 Secondary One students. The worksheet task concerned finding the functional rule underpinning a linear pattern. The questionnaire involved selecting and justifying the strategy that would best help students to establish the rule. The data were analysed to distinguish the kinds of views the teachers and students held about the various generalising strategies. The findings revealed differences in the judgements of teachers and students regarding the best-help generalising strategy.

Keywords: Pattern generalisation, teachers' pedagogical content knowledge, student generalising strategy

Background

Using examples to demonstrate how the mathematical tasks in question can be worked out appears to be an important teaching activity that features strongly in mathematics lessons. But selecting the tasks and solutions that will facilitate student understanding is not a straightforward process. It is often guided by many factors, one of which is the teachers' pedagogical content knowledge, which includes knowledge of content and students, knowledge of content and teaching as well as knowledge of content and curriculum (Ball, Thames, & Phelps, 2008). When a mathematical task can be solved in a few ways, choosing an appropriate solution that is easy for students to follow can be challenging. The selected solution could have been picked based on the teachers' beliefs about what students are capable of understanding and how they will learn best. But what do students think of their teachers' solution? Are they able to understand it?

The purpose of the present study was to compare and contrast teacher and student choices of strategy in the topic of pattern generalisation. This study was motivated by the findings of our previous work, which examined the generalising strategies used by a group of prospective secondary school mathematics teachers to solve a quadratic generalising problem. We had found that the teachers used a range of different strategies to establish a rule between the term and its position in the pattern (Chua & Hoyles, 2010). Given that there are numerous approaches to solving a generalising problem, we wanted to discover which strategies teachers would employ in class to show students how to work out this functional rule. Additionally, which ones would they believe best helped students to work out the rule? What justifications underlie their choice of strategy? Now imagine asking students to pick a strategy that would best help them to work out the rule. Which one would they select? How do student choices of generalising strategies compare with teacher choices? These are some questions that this paper seeks to answer. It is hoped that the findings of the present study will enhance the teaching and learning of pattern generalisation.

Theoretical Framework

Pattern generalising problems are a common feature of school mathematics in many countries. Generally, researchers concur that such problems are a powerful vehicle not only for introducing the notion of variables (Mason, 1996) but also for developing two core aspects of algebraic thinking: the emphasis on relationships among quantities like the inputs and outputs (Radford, 2008) and the idea of expressing an explicit rule using letters to represent numerical values of the outputs (Kaput, 2008). A typical generalising problem involves facilities like identifying a numerical pattern, extending the pattern to make a near and far generalisation, and articulating the functional relationship underpinning the pattern using symbols.

There is a wealth of research that examines students' generalising strategies and reasoning when they deal with pattern generalising problems. Students had been found to use a variety of strategies for constructing the functional rule underpinning the pattern depicted in the problems. For instance, Rivera and Becker (2008) established three types of strategy that students employed: (1) *numerical*, which uses only cues established from any pattern that is listed as a sequence of numbers or tabulated in a table to derive the rule, (2) *figural*, which only applies in generalising tasks that depict the pattern using diagrams, and relies totally on visual cues established directly from the structure of the figures to derive the rule, and (3) a combination of both the numerical and figural approaches.

The *figural* solutions were further distinguished into two different categories by Rivera and Becker (2008): (1) *constructive generalisation*, which occurs when the diagram given in a generalising task is viewed as a composite diagram made up of non-overlapping components and the rule is directly expressed as a sum of the various sub-components, and (2) d*econstructive generalisation*, which happens when the diagram is visualised as being made up of components that overlap, and the rule is expressed by separately counting each component of the diagram and then subtracting any parts that overlap.

Apart from these two kinds of *figural* strategy, Chua and Hoyles (2010) introduced a new strategy, called *reconstructive generalisation*, into the existing classification scheme developed by Rivera and Becker (2008). This strategy occurs when one or more components of the original diagram are rearranged into something more familiar. The newly reconfigured figure then unveiled the pattern structure and facilitated the construction of the functional rule.

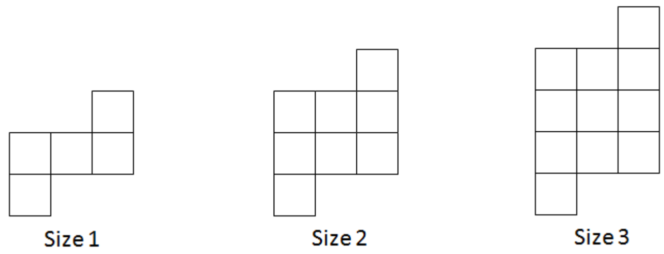
To sum up, the literature review leads us to recognise the diverse ways of constructing the functional rule that underpins the pattern presented in a generalising problem. But do teachers themselves realise which way would best help their students to perceive the underlying pattern structure and then express the functional rule using symbols? Hence, the present study aims to add to the body of work on pattern generalisation by extending the research to examine what both teachers and students judge as the best-help generalising strategy for detecting pattern structure and expressing generality.

Methods

The present study involved both teachers and students. Teacher data were gathered through two instruments – a worksheet and a questionnaire – administered to 16 secondary school teachers, 12 female and 4 male coming from 13 different schools. 13 of the teachers taught mathematics for not more than five years and three for six to ten years. The worksheet comprising the *Birthday Party Decoration* problem, presented in Figure 1 below, was distributed to every teacher prior to discussing the different strategies for dealing with such a figural generalising problem. The teachers were asked to work out the functional rule individually using the strategy that they would employ in their classroom demonstration, and also to justify how they derived the rule. A discussion of the various ways of constructing the rule then followed.

Mary used identical square cards to make several birthday party decorations of different sizes.

The diagrams below show three party decorations she made.



As the size number became larger, more square cards were used.
Mary wanted to find the number of square cards she had to use to make any size.
She used a rule to find this number.

Figure 1. *Birthday Party Decoration*

Subsequently, a questionnaire containing the *Birthday Party Decoration* problem, together with four possible student solutions to it, was distributed to each teacher. Figure 2 below shows the four distinct student solutions. Set in a context of a discussion amongst four students, each student solution represented a different way of constructing the rule based on the classification scheme described above: *constructive* (Method 1), *numerical* (Method 2), *deconstructive* (Method 3), and *reconstructive* (Method 4). The teachers had to choose the method that they believe would best help students to construct the functional rule. Similarly, they had to provide justifications for their choices of the best-help method.

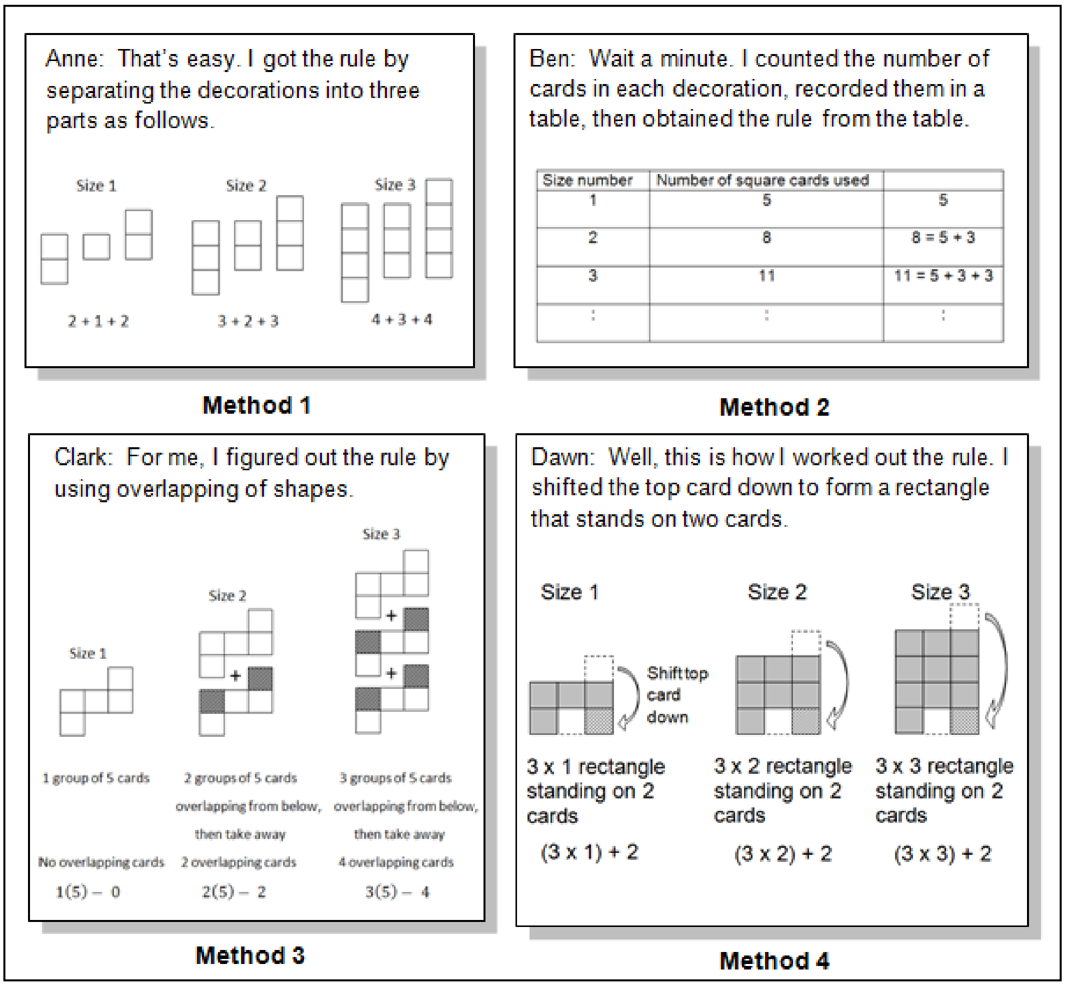


Figure 2. *Student solutions to Birthday Party Decoration*

Student data were collected from 47 Secondary One students (aged 13 years) from a secondary school, with 28 of them from the Express course and 19 from the Normal (Academic) course. The students were placed in these courses based on their performance at a national examination taken at the end of their primary education when they were 12 years old. These students, 23 boys and 24 girls, were selected by the school according to their Mathematics grade in the national examination. Amongst the Express students who were considered academically more able than the Normal students, 15 scored an A or A* (high distinction) for Mathematics while the remaining 13 scored a B or C. All the 19 Normal (Academic) students scored a B or C.

The students were asked to do the *Birthday Party Decoration* problem in the worksheet first before attempting the student questionnaire. This was to prepare and familiarise them with the problem so that they could better understand the four student solutions provided in the questionnaire. The student questionnaire was similar to the teachers' except for a slight change in the question, now asking students to choose the method that they believed would best help them to work out the rule in the problem.

All 16 teacher worksheets and questionnaires were collected and analysed to determine the generalising strategies that the teachers would use in class to derive the functional rule for the *Birthday Party Decoration* problem, as well as their choices of method that they thought would best help students to work out the rule. The teachers' strategies were then coded accordingly using the classification scheme described above. As for their choices of *best-help* method, the frequencies of the four methods given in the questionnaire were counted. Similarly, all 47 student questionnaires were also collected and analysed. As well, the frequencies of the four methods for the students' choices of the *best-help* method were counted. Both the teacher and student data were re-analysed and coded again a few days later to ensure consistency in the analysis process.

Results

This section presents the findings to the following three questions that guided this study.

1. *What generalising strategy would teachers use in class to show students how to derive the rule?*

All the 16 teachers derived the linear rule underpinning the *Birthday Party Decoration* problem correctly. 10 of them produced a *numerical* solution while the remaining six yielded a *figural* solution that employed the *constructive generalisation* strategy. As for the *numerical* solutions, we noticed that the rule was obtained either algebraically or inductively once the number of cards in each array was counted and recorded as a sequence of terms or in a table, and the constant difference between consecutive terms was detected. In the algebraic approach, the linear rule was found by first letting it be *p*n + *q*, followed by formulating algebraic equations using the numerical cues to solve for *p* and *q*. Whereas in the inductive approach, the rule was established following the spotting of a consistent pattern structure between the size number and the corresponding number of cards used in each array. By drawing on the first term 5 and the common difference 3, the subsequent terms were found by adding the common difference $(n-1)$ times to the first term, thus resulting in the rule $5+(n-1)\times 3$.

Unlike the numerical solution, the *figural* solution presented in Figure 3 shows how a teacher relied totally on visual cues established from the diagrams to make a *constructive* generalisation. As clearly shown in the solution, the structure of each original diagram was viewed as being made up of a n by 3 rectangular array of cards placed in between the two topmost and bottommost cards. Hence the rule is $(n\times 3)+2$.

2. *Which method do teachers believe would best help their students to work out the rule?*

As Table 1 shows, the majority of the 16 teachers seemed to believe that a *figural* solution would best help their students in constructing the rule. Eight of them picked Method 4 (reconstructive), four picked Method 1 (constructive) and one picked Method 3 (deconstructive). Only three teachers chose Method 2 (numerical).

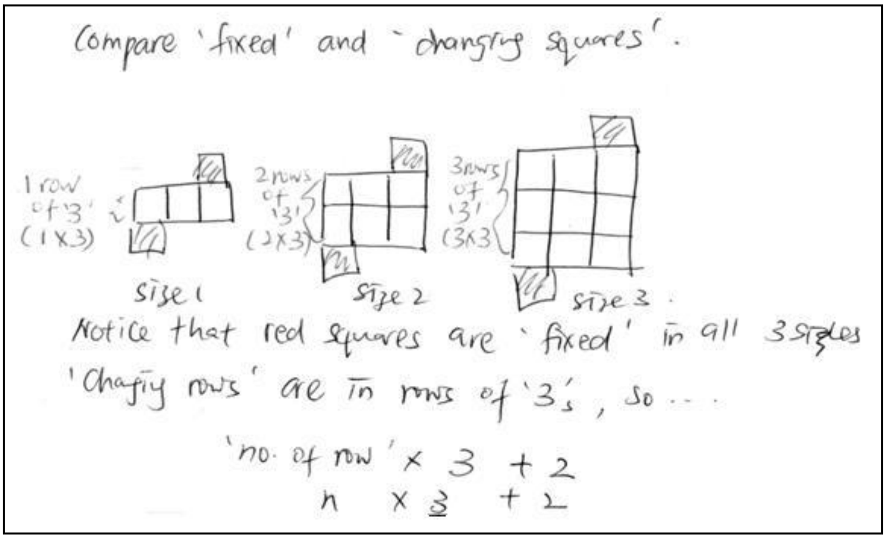


Figure 3. *A figural solution*

3. *Which method do students believe would best help them to work out the rule?*

Table 1 shows that 15 of the 28 Express students in this study believed that a figural method would best help them to derive the rule, with nearly an equal number of them choosing Methods 1 and 4 and two students preferring Method 3. The remaining 13 students picked the numerical solution in Method 2. In contrast, a significant number of the Normal (Academic) students (13 out of 19) found Method 2 most helpful. Amongst the rest, four chose Method 1, two selected Method 4 and none opted for Method 3.

Table 1
Teacher and Student Choices of Best-help Method

Birthday Party Decoration Problem					
	Best-help Method to students				
Methods	**1**	**2**	**3**	**4**	**Total**
Teachers (n = 16)	4	3	1	8	16
Express Students (n = 28)	6	13	2	7	28
Normal (Academic) students (n=19)	4	13	0	2	19

Discussion

The moderately high percentage of *numerical* solutions seems to suggest that such an approach of obtaining the functional rule underpinning the pattern is a widely used teaching method amongst the teachers. Its popularity lies in its simplicity for them to demonstrate in class without having to draw any diagrams, and also for students to understand. In particular, finding the rule from a table of values is neat and systematic. A teacher even remarked that such a method is so generic that it "sure works for all cases", referring to both numerical and figural generalising problems. As for the approach of deriving the functional rule algebraically by means of solving equations, it does not seem to lead to any algebraic thinking despite the use of letters. This is because students are not engaged in noticing a commonality amongst the particular, which is a trait that constitutes the core of pattern generalisation (Radford, 2008). It is thus important for teachers to realise that such an approach does not help to develop the notion of variables. So students will still not grasp the varying characteristics of a variable regardless of how adept they are at applying this approach.

A noteworthy finding that supports the widely held notion of how teachers' knowledge of content shapes their choice of strategy (Ball et al., 2008) emerged from personal conversations with the teachers. Three teachers admitted to employing the numerical approach for it was the only method that they knew of handling such generalising problems. They were unaware of any figural methods until they learnt about them later in the questionnaire. But it is encouraging to find many teachers who had initially used the numerical approach now believing that a figural method would best help the students. Their most common justification is that the use of diagrams illuminates the link between the size number and the number of cards used, thus facilitating students' recognition of the underlying pattern structure. It is then not surprising why so few teachers regarded Method 2 (numerical) in the questionnaire as the best-help method.

In very much the same way, the choice of strategy seems to be guided by the teachers' judgements about what students are capable of doing and learning. A few teachers were sceptical of students' ability to visualise the diagrams in *figural* solutions, hence they decided to use a numerical approach for teaching instead. One of them further shared that his students are "usually more comfortable with numbers than diagrams". On the other hand, some teachers assumed that the less able Normal (Academic) students are more visual than the Express students, thus leading them to think that the figural methods would better suit these weaker students. Concurring with this view, a teacher mentioned that the table of values, a common feature in the numerical method, "scares Normal (Academic) students". As for the Express students, many teachers believed that the simpler and more straightforward numerical method would appeal to them. But the present study offers some compelling evidence confirming that students hold differing views from their teachers. A significant majority of the Normal (Academic) students were found to prefer the numerical method to the figural method whereas the Express students tended to favour the figural method for working out the rule.

Some valuable insights have also emerged from the students' justifications of their choice of strategy. There were students who preferred the numerical method due to its clarity and simplicity, which made pattern detection and understanding easy. Subsequently, this led to the ease of obtaining a rule. Those who eschewed this method generally found it time consuming, confusing and tedious to set up a table of values. Those who opted for figural methods reported that they were helpful in finding the rule quickly. Moreover, there was an explicit link between the size number and the number of cards used. The pattern was also easy to visualise and eventually to obtain the rule. Like those who shunned the numerical method, students who disliked the figural method lamented that it was tedious to draw and difficult to visualise the diagrams. In particular, the *deconstructive* generalising strategy (Method 3) was the least helpful because students found it complicated and confusing to understand. In addition, some students were concerned about overlooking the subtraction of the overlapping cards, miscounting them, and having to spend more time to identify a consistent structure underpinning the pattern.

Conclusion

It is evident from the present study that there is incongruence between teacher and student choice of generalising strategies for establishing the functional rule in a generalising problem. Clearly, teacher judgements of what would be best-help strategies for students do not always match the reality of student choice of strategies. This lack of alignment can impact the efficacy of teaching and learning outcomes. What this then implies is that teachers will need to look into the assumptions that they are making when deciding on the kind of strategies to use in class. For pattern generalisation in particular, teachers will need to be keen observers of how their students express generality to find out how they process the strategies. This will be useful to teachers seeking to develop knowledge of what strategy might best help students. Additionally, teachers will also need to be equipped with adequate generalising strategies that lead students to develop algebraic thinking when they work with generalising problems. Finally, while the findings may be preliminary since the present study is still on-going, they appear to hold promise of creating a greater awareness amongst teachers of what students are actually capable of doing and learning. By making an attempt to understand how students visualise patterns can help teachers to plan more effective teaching and learning experiences to improve students' ability to make generalisations.

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