Is $\frac{0}{0}$ undefined or not well-defined?

SPARIO Y.T. SOON

Quite often students ask about questions involving division by 0. This is a subtopic found in the lower secondary syllabus. The students were told that they should not divide a number by 0. It is undefined for it will give absurd results (for example 'E' is the answer for $1 \div 0$ if one uses a calculator. Please refer to [1] for calculator activities on this subject.)

Using the concept of a part of a set, n/b means the number of pizza a person will receive if n number of pizzas are to be shared amoung these b people. We may wonder how much a person will receive if there is no one there to share the n number of pizzas. Clearly this question is absurd as there is no one around and a person is there 'to receive $n/0$ number of pizzas'.

A neater way to show that zero cannot be a divisor is to use algebra. It is quite clear why $n/0$ is undefined when n is not zero. We may argue as follows:

Suppose $n/0$ is a number, let it be m. By following the law of division where a/b = c whenever a = c x b, we have:

$$\frac{n}{0} = \mathrm{m} \quad \text{means} \quad n = \mathrm{m} \times 0.$$

But for every m, the product is 0 which is not equal to n (a contradiction). Hence $n/0$ is undefined when n is not zero.

Taking a closer look at the above argument, we may wonder what will happen if $n = 0$? In this case with 0 = m x 0, one could assign any value m for $0/0$, e.g. $0/0 = 11$ because 0 = 11 x 0 or $0/0$ = 1000 as 0 = 1000 x 0. This means that $0/0$ does not have a unique value. Thus $0/0$ is not well-defined. Note that it is not undefined because we can fix a value for 0/0 and the value we have fixed will satisfy the law of division. This is the usual argument teachers used in explaining why 0 cannot be a divisor.

Alternatively, based on the fact that a/b is the only number that satisfies the linear equation $y \times b = a$, $n/0$ should be the only value to satisfy $y \times 0 = n$.

As $y \times 0 = 0$ for all values of y, every number will satisfy the equation if $n = 0$ and no number can be found to satisfy the equation if $n \neq 0$. Thus, through solving the linear equation, no number is associated with $n/0$ if $n \neq 0$ and every number is associated with $0/0$. Therefore $n/0$ is undefined for nonzero n and is not well-defined for $n = 0$.

The question is: is 0/0 really not well-defined (meaning we can define it the way we want) or is it actually undefined? This problem is raised because what we have just examined before is just the fulfillment of only one law, that is the law of division, and only one fact based on the linear equation. To be a well-defined definition in arithmetic, one needs to verify that the definition does not contradict each and every other laws in arithmetic. Let us now examine further the value of 0/0.

Let

$$\frac{0}{0} = v,$$

then

$$2 \times \frac{0}{0} = 2v.$$

By the product rule

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d},$$

we have

$$2 \times \frac{0}{0} = \frac{2}{1} \times \frac{0}{0} = \frac{2 \times 0}{0} = \frac{0}{0} = v.$$

Thus, $2v = v$ and so $v = 0$. Therefore, 0/0 can only be defined as zero and not any other values as we have said before. In this case

$$\frac{0}{0} + 1 = 1$$

By following the law of addition

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd},$$

we have

$$\frac{0}{0} + 1 = \frac{0}{0} + \frac{1}{1} = \frac{0 \times 1 + 1 \times 0}{0 \times 1} = \frac{0}{0} = 0.$$

Thus

$$1 = \frac{0}{0} + 1 = 0,$$

wich is absurd. Hence 0/0 cannot be defined as a numerical value which is consistent with the laws of arithmetic. Well, can *010* be defined as a logical quantity, say, for example, infinity? As we know that infinity 'behaves' differently from numbers, we should not introduce it to secondary students by saying something like '*n*/0 equals infinity' as students may then have an idea that infinity is a sort of very, very large number. Such a misconception may lead to confusion when they start to learn "limits". Anyway, whether 0/0 can be defined as infinity depends on how 'successful' the operational rules for infinity is constructed. This can be a challenging problem for students to work out.

The argument presented here on why 0/0 is undefined can be used as a teaching idea for developing and raising students' abstract thinking to a higher level which is similar to the third or even fourth level of Van Hiele's geometry model (please refer to *[2]* for the Van Hiele's model). It is hoped that by working through the deductive process together with the students either in the form of a worksheet or classroom discussion, students may be able to appreciate the process of abstraction which is an essential part of the nature of mathematics. This can also lead them to understand more about mathematics. Eventually, students may realise that it is not only the answer (which is an end product) that matters but that the processes in mathematics are equally if not more important.

References

Lim S.K. (1989). Division by Zero. Teaching and Learning, v.9 no.2. p. 65-74.

Hoffer, A. (1983). Van Hiele-Based Research. In R. Lesh & M. Landau (Eds.). Acquisition of mathematics concepts and processes. Academic Press, Inc.