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An Investigative Approach to Mathematics Teaching and Learning

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Abstract: *In this paper, the investigative approach to mathematics teaching and learning as a mathematical inquiry that stems from the constructivist belief is examined. The implications on the nature of mathematical knowledge and activities that the approach requires are also considered and are followed by suggestions on how teachers might use schemes or textbooks to help children gain experiences in investigating. Finally, examples of the social norms expected of teachers and pupils in 'investigative' classes which differ from that of 'traditional' classes are provided.*

Introduction

When working with a class of 11-year-olds on an investigation, The Curious 1089 (Orton & Frobisher, 1996, p. 39), with the aim of trying to use an open-ended activity with children who had no experience in learning mathematics investigatively (see Appendix 1), the following is an example of an exchange that occurred between students as they discussed how they might find the trick/rule to the investigation.

S1 : Hold on. I think that every beginning number below 500, put it into 99, not 198 (secondary number). Take 452 and take away 254 equals 198. It works.

S2 : Maybe it has something to do with even numbers.

S1 : Try in sequence between numbers between 100 and 500. Try in sequence with either even and odd numbers. (Pause). It's consistent.

S2 : Hang on. Keep on going. (Pause) I think the 100s must be more than the ends sets (ones).

S1 : Yeah.

S2 : So if 456, you can't take away 654. But 654 to start off with, you can take away 456 (Pause) equals 198 (Pause). Hold on. It will always get you 99 or 198.

S1 : There must be link between them. That happens when the 100s go up and the ones go down.

The above exchange and written evidence (see Appendix 2) show that S1 and S2 were not only able to observe that not all numbers below 100 and 200 end up as 99 or 198 respectively, but were also able to generalise that the trick worked when the secondary numbers were factors of 99 if the finishing number was 1089.

While working on the above investigation, the pupils were involved in high levels of mathematical thinking (Desforges & Cockburn, 1987) as they overtly exhibited the mathematical processes such as guessing, making conjectures and testing their conjectures, generalising, and recording. They were also focused and were deep in discussion while encouraging each other to find Samantha's trick. During the task even the pupil who was the least motivated at the beginning of the task participated in the investigation and concluded with a fine generalisation of Samantha's trick. This experience supports Bird's (1991) view that there is much value, both cognitive and affective, as teachers teach and children work at mathematics in such an active and open way. Does other research support this view?

The current belief in mathematics education is that pupils need to be active learners rather than passive recipients for mathematical concepts to be learnt meaningfully. During the past three decades, a number of highly regarded reports (for example, Cockcroft Report, 1982) in the mathematics education community have recommended an increased emphasis on more creative aspects of mathematics learning such as learning via problem solving and investigational work, as shown by the above example where the children worked through an investigation. Teachers have also been made aware of the importance of pupils participating actively in class discussions, and having pupils engage in problem posing and solving open-ended activities or investigations. Jarworski (1994) listed rationales for having investigational work as a form of active learning in the classroom. First, the activities are more fun than 'normal' mathematical activities. Second, they appear to promote more truly mathematical behaviour in pupils than doing traditional topics and exercises. They also appear to promote the development of mathematical processes which can be applied to other mathematical work. Finally, they might be a more efficient means of bringing pupils up against traditional mathematical topics. Frobisher (1994) also noted the value of the investigative approach:

> "*A problem centred or investigative approach to learning maths should be perceived by pupils as an ideal way to learn the subject. An investigative style is a pedagogical approach to the mathematics curriculum, not just something which occurs when the routine of the normal curriculum becomes dreary and tiresome*". (p. 169)

In Britain and America, a radical move was made to modify their mathematics curricula to accommodate and promote active learning in the classrooms. For example, 'Using and Applying' is a strand in the England and Wales Mathematics National Curriculum where teachers address such learning approaches. In addition, the General Certificate of Secondary Examination (GCSE) required course work to be graded. This impetus towards engaging pupils in active learning had an impact on teachers' teaching approaches and on pupils' learning experiences in the country. Singapore is no exception. In the Revised Mathematics Syllabus (Ministry of Education, 2001), the conceptualisation of the mathematics curriculum is based on a framework where active learning via mathematical problem solving is the main focus of teaching and learning. Greater emphasis is now placed on meaningful understanding of concepts through activities; mathematical thinking and the processes of doing mathematical tasks like mathematical communication through oral work, group discussion and presentation; and investigative work and problem solving (Chang, 1996). The revised syllabus required a change in the teacher's role: from that of an instructor to that of a facilitator and listener.

However, Desforges and Cockburn (1987) found that there were teachers who rejected this teaching strategy not because they rejected its virtues but because the goals of an investigative approach were prohibitively difficult to implement. One pressing problem noted by them was that teachers were constantly weighed down by the increasing pressures of the curriculum and assessment. As an erstwhile mathematics teacher, it appears to me that Singapore teachers also face this problem. Hence, one might ask how teachers can resolve the tension between trying to complete the syllabus and at the same time trying to engage pupils in more active learning which requires more time? Is active learning the only way to enable children learn effectively? Is active learning the only way to teach effectively? If so, how can the pressures be rationalised? In Bird's (1991) terms: 'How can we cover a syllabus and yet at the same time involve children in carrying out their own mathematical thinking?' (p.11)

This paper aims to highlight features of the investigative approach[1] and discuss implications for mathematics teaching and learning. The tension noted above will be explicitly addressed as it has implications for implementation in context's such as Singapore's. We begin with a description of how 'current' problems used in English and Welsh schools evolved before formalising a framework for the investigative approach to mathematics teaching and learning. The investigative approach framework calls for a change in the conceptualisation of mathematics instruction as inquiry. In the description, there is an attempt to identify the mathematics which children engage themselves in during the mathematical thinking process. Additionally, some teachers' and pupils' actions in investigative classes which are different from that of the 'traditional' classes are identified. Finally, it questioned whether or not the investigative approach is the only approach to optimise pupil learning.

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[1] The phrase 'an investigative approach to mathematics teaching and learning' will be frequently required and for brevity, it will be shortened to 'an investigative approach' or sometimes 'investigative class'.

Theoretical Perspective of the Investigative Approach to Mathematics Teaching and Learning

Children of this century meet different mathematical problems, including investigations, compared to those encountered by their parents when they were at school. An understanding of the evolution of problems helps in formalising a framework for the investigative approach to mathematics teaching and learning later. Let us examine this understanding in the context of English and Welsh mathematics curriculum given its tradition since 1960 in engaging pupils in active learning with problems.

Evolution of 'Problems'

'Word problems', when first identified as part of the early mathematics curriculum in the English and Welsh mathematics curriculum, took the form of problems set in words where the mathematics was contextualised in pseudo-real situations. They were considered important because pupils could practise the knowledge and skills which had been previously taught in a 'real' context. Exposition was the usual teaching mode in this situation where teachers demonstrated how to solve an exemplar problem followed by pupils practising the techniques learned on similar problems (Orton & Frosbisher, 1996). Evidence (Ernest, 1991) showed that children developed weak constructs such as learning number concepts by rote where, when mathematics was learned in this manner, many pupils developed 'misconceptions' as the mathematical structure they had constructed in their minds differed from their teacher's.

Developing problem solving strategies to tackle these "contextualised" problems appeared next on the scene. Problem solving was intended to imply 'a process by which the learner combines previously learned elements of knowledge, rules, techniques, skills and concepts to provide a solution to a novel situation' (Orton, 1992). Polya (1973) believed that people became better problem solvers and established a routine to help them. Criticisms have been made regarding this routine and Orton (1992) felt that the danger here was that in trying to establish a routine, there was a possibility of moving towards a more algorithmic approach to learning problem solving. He also asserted that problems could not be routine. If the learner was able to solve the problem, the solution was dependent on the learner not only having the required knowledge and skills, but also being able to draw from them to establish a structure.

In the 1960s, a new and different perspective of mathematics problems emerged. In 1967, the Association of Teachers in Colleges and Departments of Education (ATCDE) in England asserted that there existed mathematics problems which have a different nature from the general problems studied in the past. These problems have intrinsic interest and there are no immediate obvious techniques for their solution. According to Orton and Frobisher (1996), 'The objective was to foster in students a belief that they made mathematics their own through exploration' (p. 24).

Hence, a good starting point to think about the difference between the various types of problems rests in terms of what pupils do with the problem given to them. For example, consider again the 'The Curious 1089' problem. If a child's response to the problem is to explore the mathematical structure in order to try to understand the relationships between the mathematical elements, the child is said to be investigating an investigation. The assumption is that the child has no prior experience in dealing with this problem and needs to go down an investigative route to solve it. In contrast, if children recognise the problem as one of 'those' which requires them to go by a certain familiar route in order to solve it, they are not investigating in the proper sense. It has become a procedure to them and thus the 'danger' to which Orton referred.

The changed perspective of problems created an impact on how the public viewed learning and assessment in English schools. The Certificate of Schools Examination (CSE) introduced the idea of assessing pupils' work completely outside the examination room. This new approach was later given added weight and strong emphasis when investigative and problem solving approaches to mathematics learning and teaching was encouraged in the Cockcroft Report (1982). Proponents of this recommendation saw opportunities for practical work, problem solving and investigational work pupils to be appropriately assessed which were not possible in time-limited written tests or examinations.

The investigative approach to mathematics teaching and learning deployed the use of problems which promoted mathematical behaviour. Thus, it developed into a favourable approach which has the direct association with the 'normal' curriculum (Orton & Frobiser, 1996) and claimed to provide pupils with a fertile experience of the processes involved in mathematics and mathematical teaching:

> '*The essence of an investigative approach is the application of communication, reasoning, operational and recording processes to a study of the core topics which make up the content of a mathematics curriculum*'. Frobisher (1994, p. 169)

Understanding the Investigative Approach to Mathematics Teaching and Learning

An inherent assumption for the investigative approach to mathematics teaching and learning is to view *knowledge as inquiry* as developed by Sanders Pierce (1839-1914) (Siegel & Borasi, 1994). More specifically, an inquiry epistemology challenges popular myths about the truth of mathematical results and the way in which they are achieved and suggests that mathematical knowledge is fallible. Mathematics is thus created through a non-linear process in which the generation of hypotheses plays a key role; the production of mathematical knowledge is a social process that occurs within a community of practice; and the truth value of mathematical knowledge is constructed through exploration (Siegel & Borasi). Problems eventually become a vehicle for other learning which Wheeler (1988) called *mathematization*, the process by which mathematics is brought into being. Consequently, there are implications with regard to the nature of mathematical knowledge and activities that the approach requires. Taken together, these considerations suggest different activities, goals, values and social norms than those characteristics of traditional classrooms where the teachers play a dominant and authoritative role in disseminating knowledge and pupils play a passive role in receiving knowledge.

Mathematical Knowledge

Investigative classrooms emphasise the full complexity of knowledge production and expect pupils to see the doubt arising from ambiguity, anomalies and contradiction as a motivating force leading to the formation of questions, hunches and further exploration (Siegel & Borasi, 1994). This stems from the constructivist belief that 'children construct their own knowledge of mathematics over a period of time in their own unique way building upon their pre-existing knowledge' (Ernest, 1989, p.151). For example, consider the pair of tasks shown below (Yackel, Cobb, Wood, & Merkel, 1990) where three, second-grade pupils, Brenda, Consuela and Dai-soon, working together were asked to give the solution to Task 2.

The following is what the researchers (Yackel et al, 1990) observed:

> '*Brenda's problem was relating the two tasks (because she said that the first and second tasks were the same as six and six make twelve for the second task). Consuela and Dai-soon both solved the problem adding 46 and 46 but their solutions indicate differences in their interpretations and in their personal mathematical conceptions. Consuela's solution shows that she is capable of operating on two-digit numbers by thinking them as made up of tens and ones. Dai-soon's solution of attempting to count on forty-six ones is more primitive.*' (p. 35)

The three pupils' solutions for Task 2 showed that although the three pupils were

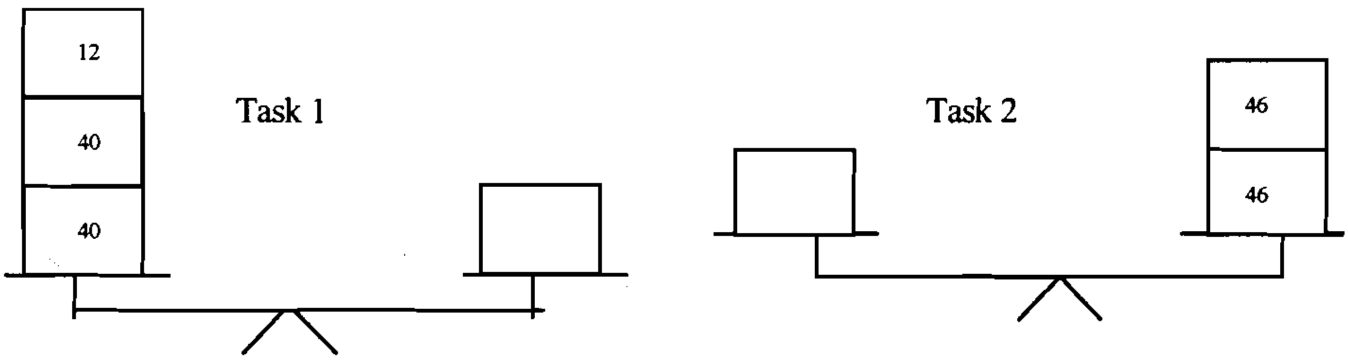

Figure 1 Yackel, Cobb, Wood and Merkel (1990) task

working closely together, they constructed their own mathematical understandings which were very different from each other's. This appears to indicate that mathematical learning involves active manipulation of meanings, not just numbers and formulas, and rather than passively receiving and recording information, the learners can actively interpret and impose meanings from their existing knowledge structures. Thus teachers and learners need to accept that mathematical knowledge is 'corrigible and can never be regarded as being above revision and correction' (Ernest, 1991, p. 3). But what is this mathematical knowledge children possess while investigating?

In mathematics education, we are concerned that pupils should acquire the skills and techniques that we identify as doing mathematics or thinking mathematically. More often than not, we insist that pupils must acquire the mathematical 'content' in the syllabus as well as focus on the 'processes' (Lerman, 1989). Bird (1991) saw a difference in these mathematical terms: 'content' and 'processes'. According to her (Bird, p.108), content is associated with the view of mathematics as a body of knowledge which consists of certain skills and concepts to be learnt. On the other hand, processes are associated with a view of mathematics as an activity. Bird also asserted that 'what today has acquired the status of '*content*' was brought about yesterday through activity'. Bird related many anecdotes of how young children engaging in mathematical activities exhibit a variety of processes that occur in the syllabus. She claimed that given the freedom, children frequently set themselves challenges which enable them to explore unknown situations. Children are thus led to structuring their activities in ways which involve various aspects of the mathematics through their own volition, developing the situation and controlling it until they create mathematics, which they themselves are not aware of. The case studies in Bird's book showed that if we concentrate on teaching the 'content' from the syllabus in an investigative way, it will not work as children will exhibit more than what is in the syllabus. In fact, there will also be overlap of content areas when learning is done in an investigative way. This implies that implementing the investigative approach will lead to a syllabus which does not have a hierarchy yet includes the traditional curriculum with additional aspects which children might learn as a process of being involved in the activity.

Using an Appropriate Scheme

The investigative approach appears to facilitate teaching and learning in a content-oriented mathematics curriculum. However, as teachers generally have to rely heavily on published schemes or textbooks for activities to be used in the classrooms, published schemes or textbooks still encourage teachers to teach and children to learn using more conservative approaches.

There are current published schemes that are exercise-oriented and focus on the learning of 'content'. If 'investigations' are part of these schemes, they are usually closed and guide children to a specific goal which is not the true spirit of engaging children in investigational work. The example in Appendix 3 is supposedly an investigation. Instead of letting children use their own methods to find the trick, examples are shown and questions are asked to guide/lead children to the trick. In fact, all the numbers are 'invented' for the children to work on. In contrast, the activities in the Harcourt Brace Jovanovich (HBJ) scheme (1992) encourage children to explore the basic mathematical concepts. As in Appendix 3, the example in Appendix 4 (HBJ Y5, 1992, p.138) enables children to practice decimal notation and calculations involving money. Instead of providing children a page of publisher's invented numbers, children throw dice and invent their own amounts (biggest and smallest sum of money). By playing the game with dice and recording their findings, they are encouraged to look for strategies in getting the highest possible sum and the lowest possible sum. Children engaging in the tasks from HBJ may develop the use of mathematical processes as well as the acquisition of skills and knowledge. The processes include asking questions, collecting and displaying data, estimating, making hypothesis and testing conjectures. The activities are open and allow children to define their tasks, plan their method of working and decide on how to record their findings. HBJ authors recommended that teachers encourage children to ask their own questions and to follow their own lines of enquiry. They also believed that when children are given such opportunities to learn, they link their new learning to their present comprehension. HBJ authors also warned teachers not be the source of all information but rather become a resource to be used: instead of being providers, be facilitators. The HBJ scheme, as designed, appears to reflect the spirit of the investigative approach to mathematics teaching and learning.

As teachers plan tasks which are appropriate for their pupils, pupils will respond and develop understanding at their own pace. William and Holmes (1993) gave evidence on how a teacher developed tasks which could be attempted by the weakest pupils while challenging and motivating the high achievers. The teacher had the SMP 11-16 scheme in his class, but confessed that he rarely used them. Sometimes, he asked his pupils to transform textbook type work into puzzles for the rest of the class to solve. For example, pupils were asked to make up one Function puzzle to which only they would know the solution. One pupil wrote:

1	x	99	=	99	7	X	99	=	693
2	x	99	=	198	8	X	99	=	792
3	x	99	=	297	9	X	99	=	891
4	x	99	=	396	10	X	99	=	990
5	x	99	=	495	11	X	99	=	1089
6	x	99	=	594	12	X	99	=	1188

Figure 3. Shuard, Walsh, Goodwin, & Worcester, (1990). 'Investigating 99' task.

When the class had the chance to solve all the 'puzzles' created by their peers, the solutions were discussed. Some pupils even gave generalisations such as $n \rightarrow 2n + 2$ or $n \rightarrow 2(n+1)$ for the above puzzle. According to the teacher, this style of working appears to produce enthusiasm unlikely to be stimulated by textbooks and worksheets. This style of teaching, without a published scheme, is only possible if the teacher truly understands the philosophy of teaching and learning mathematics investigatively.

The Investigative Classroom Ethos

Teachers' and Pupils' Actions in Investigative Classrooms

Teachers in investigative classrooms no longer spend a great deal of time transmitting information via talking or reading but by taking up a more challenging job of supporting pupil inquiry. This means establishing a radically different set of social norms and values in the classroom as well as finding ways to invite pupils into the inquiry process in a conducive learning environment.

In essence, teachers in investigative classrooms should

- demonstrate how to approach various aspects of the investigative processes;
- become the socialising force in helping pupils become mathematically literate by encouraging them to question, to challenge and learn anything about the real mathematical behaviour;
- listen to pupils so that teachers can understand pupils' beliefs about learning, the experiences they bring to specific inquiries, and to gain insight into the meanings and connections pupils construct during inquiries; and
- initially give children 'short' investigations which provide short term rewards (Orton & Frobisher, 1996). Note: Gradually, when children are more open to investigations, 'longer' investigations where the goals are not so apparent can be introduced.

In addition, when pupils are investigating, they should

- become active members of a community of practice who share the responsibility of planning, conducting and reflecting on their inquiries with other members, namely their peers and the teacher;
- listen and negotiate with others; and
- must have mutual trust between peers and the teacher so that mathematical thinking is shared freely. This can only be accomplished when the classroom ethos is conducive.

Choice of Teaching Approaches to Optimise Learning

An investigative approach has been presented as an ideal approach to help children develop their mathematical thinking and learning, yet one must question the need for children to 'invent' formulas and theories that mathematicians take years to construct. Though there are disadvantages to using the 'traditional' mode, that is, expository teaching, it seems perfectly reasonable that when there is a need to teach mathematical conventions, teachers should just use the expository approach to show pupils what the conventions mean. Noddings (1990) stated that if it is clear that performance errors are getting in the way of concentrating on more significant problems, straightforward practice/telling may actually facilitate genuine learning. Asking children to explore or discover conventions appears to be an inefficient method because ultimately, there is only one particular way of representing conventions accepted by the mathematics community.

Jaworski (1994) saw the tension between teachers having some particular knowledge which they want pupils to gain and the belief that they cannot give them the knowledge as due to the influence of their belief of the philosophy of constructivism. Jaworski cited an example of a teacher, Ben, who recognised this tension when he was going to teach the lesson on the topic Vector (p. 159-168). He felt that no compromise could be made where conventions of vectors were concerned. Certain aspects of vector representations and definitions needed to be established, for example, the difference between AB and BA, or the meaning of 3AB and BA. Hence, he felt that his vector lesson was didactic where he needed to elicit or inculcate certain aspects of vectors pupils needed to know. However, Jaworski reported that she saw much that was investigative in Ben's vector lesson. Though 'it had very particular mathematical content, pupils were expected to focus on certain properties of vectors, had to understand what a vector was and be familiar with its many representations' (p.168). There were two important features which she felt were investigative: the time given to exploring meanings and developing intersubjectivity, and expectation that his pupils accept *the ways a mathematician writes things down* suggesting that he was not dictating the truth, merely stating a convention. In addition, Ben set a task investigative in nature for his pupils. For example, he asked his pupils to make their own vector questions and write their own answers. This challenged them to think about what they were doing rather than mechanically respond to given questions with a prepared technique.

Another teacher, Mike, used an alternative approach in dealing with conventions. He provided experiences for his pupils before introducing them to known conventions. Consider the two tasks (Jaworski, 1994, p. 118 &120) shown in Figure 3. By working on these tasks, Mike hoped that his pupils would start to be aware of relationships which would lead to the introduction of Pythagoras' theorem which could be stated as, 'the sum of the squares on/of the (lengths of the) two shorter sides of a right-angled triangle is equal to the square on/of the (length of the) hypotenuse' (Jaworski, p.123). The theorem could be seen to relate properties of numbers, or properties of triangles, or properties of both. The two tasks provided a good conceptual understanding of the Pythagorean relationship which included an awareness of all the related properties.

Square Sums	**Triangle Lengths**
$1^2 + 2^2 = 5$	Draw a triangle with a right-angle.
What other numbers can be made by adding square numbers together?	Measure accurately all 3 sides.
Investigate	Can you find any relationship between the three lengths?

Figure 3 Jaworski's (1994) study. Page 12

Atkinson (1992) stated that specific teaching (direct instruction) ought to be used to sort out how and when to teach standard notations and symbols, to work out what to do with children whose own methods have shortcomings, and to help children who are persistently stuck. It is usually more effective when the convention learned follows an exploratory or investigative piece of work in that domain (as seen from Ben's example) or by providing experiences which have embedded properties of the conventions (as seen from Mike's example) before introducing children to mathematical conventions. As a consequence, children can appreciate the conventions as a means of resolving their own problems (Jaworski, 1994). In teaching, getting the right balance between providing pupils with different teaching approaches is the secret of maximally effective teaching. We also see from Ben and Mike's teaching approaches that the use of telling and explanation is a part of an investigative approach but how much to tell and explain really depends on the teachers' sensitivity to the needs of their pupils. Teachers need to anticipate the skills which pupils will likely need to construct important concepts and principles, and to offer the right kind of teaching support at the right moment to lead pupils to the best learning outcome.

Conclusion

The investigative approach to mathematics learning and teaching appears to be a method that allows teachers to maximise their potential as mathematics teachers, and helps pupils of all abilities to learn mathematics meaningfully. Teachers need to provide opportunities for pupils to overtly express and freely construct their own ideas as well as encourage students to ask questions and follow their own lines of inquiry. Teachers need to be confident, flexible and be willing to explore whatever arises.

For pupils to work investigatively, teachers are required to make an effort to create a classroom ethos where there is mutual trust among teachers and pupils, and between pupils and pupils, where pupils work collaboratively, and take responsibility for their own learning and behaviour. Sensitivity to pupils' thinking and providing appropriate mathematical challenge are part and parcel of an investigative class. For example, teachers, sometimes, have to pose questions which alert pupils to ideas which they consider as important, and done in such a manner where they implicitly inform pupils that they still value their individual perceptions.

What has been articulated in this paper on the investigative approach to mathematics teaching and learning is only a fraction of all the research in this area. Nevertheless, the implementation of the investigative approach to mathematics teaching and learning appears to be the right move towards the direction of maximising pupils' potential and encouraging them to be independent thinkers.

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Appendix 1: The Curious 1089 Investigation

The Curious 1089

Samantha has a trick she does with numbers.
Here it is. How do you think it works?

$$\begin{array}{r} 854 \\ -\,458 \\ \hline 396 \\ +\,693 \\ \hline 1089 \\ \hline \end{array}$$

Samantha says that every time she does this trick the answer is always 1089. Do you agree with her?

Pupils were asked to investigate Samantha's trick.

Appendix 2: Students S1 and S2's Investigation

Numbers below 10 do not work because you cannot reverse them
~~Numbers below 100 always end up as 99.~~
~~Numbers below 200 always end up as 198~~ eg

$$\begin{array}{r} 192 \\ -\ 27 \\ \hline 045 \\ 45 \\ +\ 54 \\ \hline 99 \end{array}$$

All the factors of 99 that are secondary numbers work out as 1089 as the finishing number.

SECONDARY NUMBERS

099	198	297	396	495
990	891	792	693	594
1089	1089	1089	1089	1089
STARTING NUMBERS				
eg 120	220 221	320 321 322	420 421 422 423	520 521 522 523 524.
620 621 622 623 624 625	720 721 722 723 724 725 726	820 821 822 823 824 825 826 827	920 921 922 923 924 925 926 927 928.	
* Tens didgits can change				

~~Take~~ the beggining number away from the last didgit. If the last digit is less than the first, add 10 to the end number. Get the answer and look for the secondary number with the same last digit. That means that the number you have found is the secondary number And because it is a multiple of 99, it will go to 1089

~~320 reversed~~
~~23~~
~~it is 23 away from nearest factor~~

Appendix 3: The curious amount of £10.89 (Nuffield Maths 6, 1990)

Write an amount of money less than £10 to pay into a bank (no half pence). For example, £8.34. Reverse the digits to make another amount less than £10, like this: £4.38. Subtract the smaller amount from the larger.

$$\begin{array}{r} £\,8.34 \\ -\ £\,4.38 \\ \hline £\,3.96 \\ +\ £\,6.93 \\ \hline \end{array}$$

Reverse these digits and add.
What is the final result?

1 Do the same with these amounts:

a £7.42	**d** £2.75	**g** £4.05	**j** £5.75 does not work. Why not?
b £8.97	**e** £1.68	**h** £0.31	
c £6.33	**f** £2.93	**i** £0.01	

2 Instead of money, try other units:
a 4.86 m **b** 9.75 kg **c** 4.56 litres **d** 7.25 cm²

3 Most amounts countered in base ten except those like questions **1j** react in the same way; so do most 3-digit whole numbers.
Try these:
a 391 **b** 583 **c** 277 **d** 301

Without a decimal point, a number such as 352 is an exception:

$$\begin{array}{r} 352 \\ -\ 253 \\ \hline 99 \\ +\ \ \ 99 \\ \hline 198 \\ \hline \end{array}$$

But if the same digits are used with a decimal point:

$$\begin{array}{r} 3.52 \\ -\ 2.53 \\ \hline 0.99 \\ +\ 9.90 \\ \hline 10.89 \\ \hline \end{array}$$ it works.

4 **a** Try 473, then try 4 . 73 **b** Try 594, then try 5 . 94

5 Find the rule for those 3-digit numbers which do not work.
If the same digits are used with a decimal point to form a number less than 10, the trick does work. Why?

Appendix 4: Money Codes (HBJ Mathematics Y5, 1992)

This is a game for two people. You will need two dice numbered 1 to 6, and one die numbered 0 to 5.

Take turns to throw the three dice.
Use the numbers to make the **biggest** sum of money you can.

£ 6.53

and then the **smallest** sum of money you can.

£ 3.56

Find the difference between the biggest and smallest amounts and write this amount on a score card. Add to your score every time it is your turn. Decide what total you will aim for. The first person to get past that total is the winner.