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A THEORETICAL FRAMEWORK FOR UNDERSTANDING THE DIFFERENT ATTENTION RESOURCE DEMANDS OF LETTER-SYMBOLIC VERSUS MODEL METHOD

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The Singapore model method was introduced as a heuristic to support the problem solving capacity of upper primary pupils, but has been found to impact on their subsequent acquisition of letter-symbolic algebra. This is consistent with recent neuroimaging studies which demonstrated that letter-symbolic algebra draws more heavily on the attention resources of the students but which were unable to account for the difference between the two methods. This paper provides a theoretical framework which explains all of these results, providing teachers with a deeper understanding of the difference between the two approaches, in order to guide students through a more confident transition from model to algebraic solution methods.
Keywords: solving algebra word problems, multiple solutions, transition

BACKGROUND AND THEORETICAL FRAMEWORK

Research shows that solving algebra word problems continue to challenge students. A variety of reasons were offered to account for the challenges faced by students. Studies (e.g. Clement, 1982; Clement, Lockhead & Monk, 1981, Wollman, 1983) showed that students found it difficult to translate text presented in English into appropriate mathematical expressions and their corresponding equations. Wollman (1983) showed that these difficulties could be ameliorated through appropriate intervention strategies. Koedinger and Nathan (2004) challenged the assumptions that solving word problems was more problematic that solving corresponding algebraic equations. They found that differences in external representations affected high school students' performance in solving algebra word problems. The students were more likely to solve algebra word problems when the representations were meaningful and easier to comprehend than when representations were purely of a symbolic nature. Stacey and MacGregor (2000) examined the written solutions of 900 students aged 13 – 16 to four word problems. Their findings suggested that students failed to understand the logic underpinning the algebraic problem solving method. Students' prior experiences in solving arithmetic word problems where the objective was to obtain a single numerical solution influenced their choice of methods. This was manifested in the way they used letters, the meanings they gave to unknowns, their interpretation of equations and the methods they chose to solve equations. Students "were deflected from the algebraic path by reverting to thinking grounded in arithmetic solving methods" (Stacey and MacGregor, 2000, p. 149).

The Singapore mathematics curriculum puts a premium on students' capacity to solve problems (Curriculum Planning & Development Division (CPDD) 2001, 2006). To cultivate this capacity, students are taught to use various problem-solving heuristics. The policy to introduce problem-solving heuristics at the primary level (CPDD, 1990) has shown positive results. The heuristic 'draw a diagram' known locally as the model method seemed to be particularly effective (Ng & Lee 2009). With this heuristic, rectangles are used to represent numbers and the resulting schematic representation or model drawing (known as the model), captures the information in a problem. The construction of the model draws on the most important relationship that can be developed about numbers, namely part-part-whole nature of numbers, (Van de Walle, 2001). The model helps students "visualise" or "see" (Ng, 2003) a problem. The model method can be used to solve arithmetic as well as linear algebraic word problems involving one or two variables. In arithmetic word problems each rectangle represents a given numerical value. Translation of the information captured in the model results in arithmetic equations which may result in the solution of the problem. With algebraic word problems, the rectangles can be used to represent numbers or unknowns. The model drawing becomes a pictorial equation (Ng, 2003). Should the textual information be translated into an algebraic equation, then there is a correspondence between the resulting equation and the model drawing.

Their progress to secondary school meant that Secondary 1 and 2 students were expected to use letter-symbolic algebra to solve algebra word problems that would require the construction of no more than two linear simultaneous equations. Many students, however, continued to rely on their prior knowledge of problem solving heuristics, in particular the model method to solve algebra word problems. Studies with secondary students (Khng & Lee, 2009, Ng, 2003) showed that their prior knowledge of problem solving heuristics continued to influence their choice of methods.

It was hypothesised that the model method was easier of the two methods. To test the veracity of this hypothesis, neuroimaging methodology (Lee et al. 2010, 2007) was used with adult participants. Although both methods activated similar areas of the brain, the neuroimaging studies (Lee et al. 2010, 2007) found that the letter-symbolic method directed more resources towards attention orientation or retrieval of information specific to generating algebraic equations.

This paper reviews the strategies used by secondary 2 (14+) students who successfully solved the algebra word problems and to support the neuroimaging findings, discusses why the letter-symbolic method could be more resource intensive of the two methods.

METHODS

One hundred and twenty-four Secondary 2 (14+) students from five schools who participated in the study were given an hour to respond to the ten algebra word problems. Because of limited space, only solutions to three problems which students succeeded in solving are discussed here. See Table 1 for problems to be discussed.

RESULTS

The study showed that although students were able to use the letter-symbolic method to solve algebra word problems, there were students who continued to use problem solving heuristics, in particular the model method to solve algebra word problems.

DISCUSSION

These Secondary 2 students seemed to have three solution methods available to them: (i) letter-symbolic method, (ii) semi-algebraic method and (iii) the problem solving method using the heuristics they were taught in primary school. Figure 1 shows how different students successfully applied these methods to solve the same problem. Solutions to the Parade Problem provide an example of what is described as a semi-algebraic method where the student used a combination of the model method and the letter-symbolic method.

Table 1

Word Problems Presented To Secondary 2 Students and the Related Success Rates as a Function of Letter-Symbolic Methods (A) versus Model Method (MM)

	The Algebra Word Problems	A (C)	A (IC)	MM (C)	MM (IC)
1	Parade Problem: There are 900 people at a parade. There are 40 more men than women. There are twice as many children as there are men. How many children are there?	39 52[p]	36 48	11 26	32 74
4	Spending Problem: Ahmad has four times as much money as Betty. After Ahmad spent $160 and Betty spent $40, they each had equal amount of money. How much money did Ahmad have at first?	59 81	14 19	13 30	30 70
9	Age Problem: In 4 years' time, Mr Wong will be 3 times as old as his son. 4 years ago he was 5 times as old as his son. How old is Mr Wong now?	22 35	41 65	1 5	21 95

*C represents correct responses and IC, incorrect responses. [p] represents proportions reported as percent who used the method. The total does not sum to 100% because other methods used are not recorded here. Problems are named to facilitate discussion.

The solution to an algebra word problem can be found by selecting any of the unknowns. With the Parade Problem the solution on the extreme left shows a letter-symbolic method which uses the unknown number of children as the generator represented by the letter *x*. This allowed for the number of children to be found directly. The solution in the far right panel shows how another student continued to use the model method to find the number of children. In this case the unknown number of women was the generator. The centre panel shows how a combination of the two methods could be used to solve for the number of children. The solution was accessed by using the unknown number of women as the generator. By inserting the letter *x* into each rectangle suggests that the student knew that the rectangle represented the unknown *x*. However a mixture of methods comprising doing-undoing and letter-symbolic methods was used to solve for the unknown *x*. Perhaps the student thought that the solution was algebraic because the letter *x* was used to announce the answer.

The solution to the Spending Problem tells a different story. The solution in the left panel shows how the letter-symbolic method was used to solve for the unknown amount of money held by Ahmad. The solution in the right panel shows how two methods were used to help solve the problem. The model drawing correctly depicted the amount held by Ahmad and Betty. Instead of constructing a system of equivalent equations to find the value of x, the amount of money held by Ahmad was found by a series of doing-undoing processes. To the left of the model drawing although the letter x was used to construct the amount of money held by each friend, it was never featured again in the solution.

Either the letter-symbolic method or the guess and check heuristic could be used to solve the Age Problem. Although it is not wrong for students to use alternative methods other than letter-symbolic algebra to solve these problems, students' continued use of these alternative problem solving strategies may not serve them well in the long term. It is vital for students to acquire a sound knowledge of letter-symbolic algebra and the related procedures and skills should they wish to engage with higher mathematics and other disciplines such as the sciences and economics.

WHY LETTER-SYMBOLIC METHOD IS MORE RESOURCE INTENSIVE OF THE TWO METHODS

With the model method, the value represented by rectangle representing the unknown can be found by applying the doing and undoing process, a method well established in primary school. With the letter-symbolic method information in the text is translated into an algebraic equation. The value of the unknown is evaluated by construction of a series of equivalent equations. Thus the transition from the model method to letter-symbolic method requires students to know that the role of the rectangle is taken over by the letter.

(I) The transition to letters as unknowns requires an expansion of the knowledge related to the use of letters. Letters have different meanings in different situations (Booth, 1984; Kieran, 1989, Küchemann, 1981; Stacey & MacGregor, 2000; Schoenfeld, & Arcavi, 1993; Usiskin, 1988). Solving for the unknown x in any equation may require first simplifying the algebraic expressions on either side of the equal sign and then transforming the simplified expressions into simpler equations. Such knowledge construction is no mean feat as it requires attending knowledge associated with operating on letters.

(II) Algebraic expressions are legitimate forms of answers. When the model method is used to solve algebraic word problems, the information captured by the model is translated into arithmetic expressions which meant that they could be evaluated and a single answer was the result from operating on the arithmetic expression. For example, the number of children participating in the parade was found by first writing down this arithmetic expression: $(195 \times 2) + 40 + 40$ and the resulting sum was 470. With the letter-symbolic method, however, students must be able to accept algebraic expressions as legitimate answers (Collis, 1975, Davis, 1975). This algebraic expression $x + x/2 + (x/2 - 40)$. actually represents the number of children at the parade. Because the operators are still visible, many novice learners of algebra experience a product-process dilemma (Davis, 1975) where they do not treat this algebraic expression as a legitimate answer. Thus students have to accept that the algebraic expression $x + x/2 + (x/2 - 40)$ does represent the number of children at the parade, and this expression is a single entity. Early work in algebra begins with students being able to accept that there is a lack of closure in algebraic forms Collis (1975).

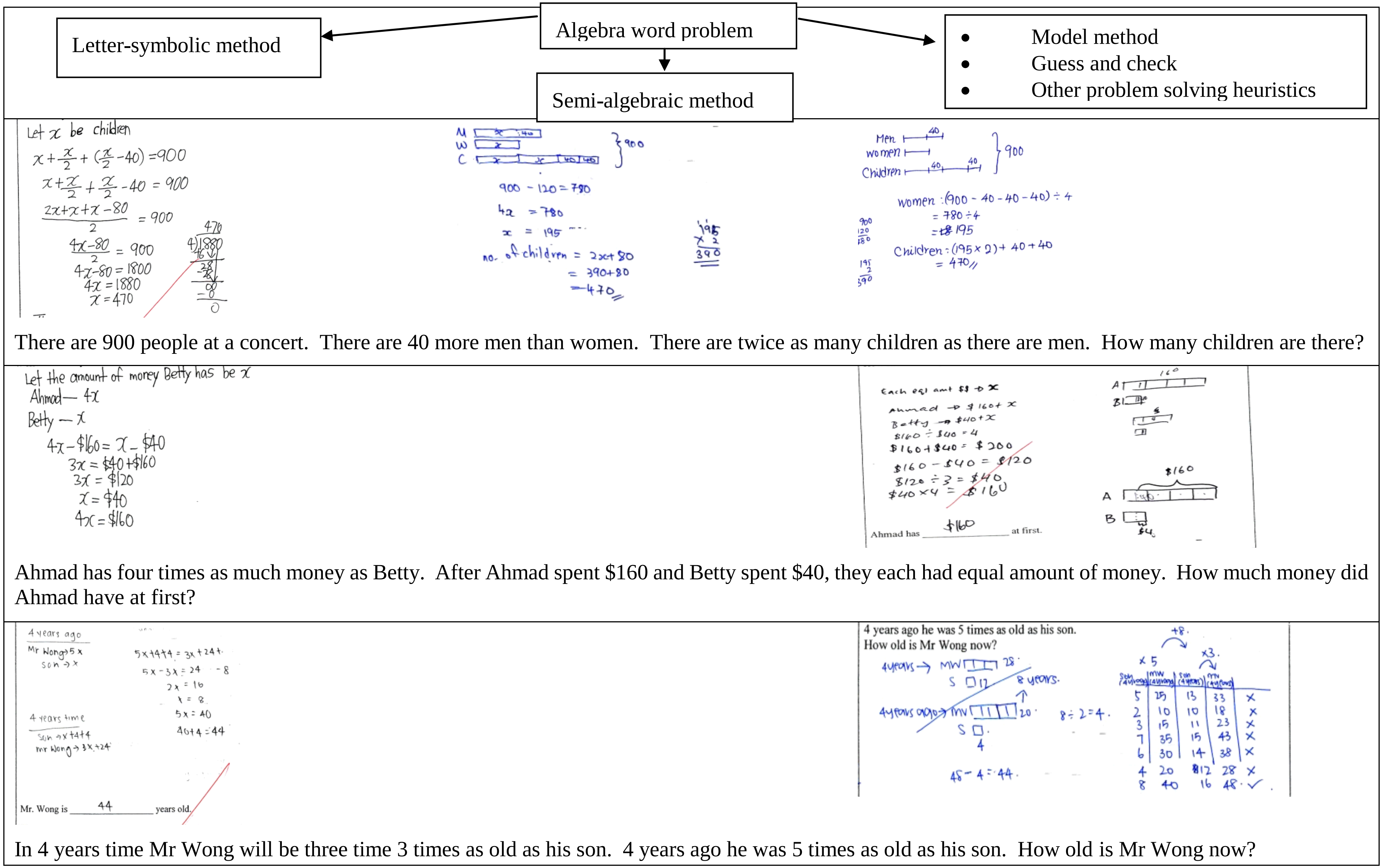

Figure 1: Possible problem solving methods adopted by proficient problem solvers to solve algebra word problems

(III) Algebraic representations no longer adhere to the same set of conventions underpinning the use of numbers. In arithmetic, operating on whole numbers can be expressed as a single entity as illustrated by this arithmetic equation: $(195 \times 2) + 40 + 40 = 470$. To complicate matters further the sum of two fractions can be expressed as the concatenation of two numbers. For example the sum of these two numbers $2 + \frac{1}{3}$ can be written side by side: $2\frac{1}{3}$. There is no one-to-one correspondence between how numbers can be expressed and how algebraic expressions can be simplified. For example with letter-symbolic algebra $x + \frac{x}{2}$ cannot be simplified to $x\frac{x}{2}$. Thus the shift from model method to letter-symbolic algebra as a problem solving tool requires construction of new knowledge, namely that algebraic representations no longer adhere to the same set of conventions underpinning the use of numbers (Kieran, 1989).

(IV) Knowledge of equality-equivalence of algebraic expressions (Kieran, 1989; 1997) is crucial. Since algebraic equations are necessary for solving algebraic word problems, the shift from the model to letter-symbolic method as a problem solving tool requires construction of knowledge that an algebraic equation is (i) a structure which links two different algebraic expressions and (ii) two or more different algebraic expressions can be constructed to represent the same situation. This then requires sound knowledge of equivalence: reflexive (the same is equal to the same), symmetric (equality of the left and right sides of each equation) and transitive.

In arithmetic, the equal sign is seen as a procedural symbol that announces the answer after a series of operations have been conducted (Kieran, 1981; MacGregor & Stacey, 1999; Stacey & MacGregor 1997). For example, when the model method was used to solve the amount of money held by Ahmad, the equal sign at the end of the arithmetic expression $160 - $40 was used to announce the answer, that is the difference of $160 and $40 is $120. Contrast that meaning of the equal sign with that required when letter-symbolic algebra is used to solve the same problem. Research (Kieran, 1997; Sfard & Linchevski, 1994) shows that there are two main sets of conceptual demands associated with solving equations: simplifying expressions and working with equality-equivalence (Kieran, 1997). The cognitive demands needed to simplify expressions have already been discussed in the preceding section: Hence the acceptance of algebraic expressions as legitimate answers will not be repeated here. Solving algebraic equations require two significantly different conceptualisations of equality-equivalence.

Reflexive-equivalence (equality of the left and right sides of each equation): In a conditional equation, for a specific value of x, the resulting values of the expressions on either side of the equal sign are the same. For example, in the spending problem, the equation $4x - \$160 = x - \$40$, the expression 4x - $160 is equal to the expression $x - \$40$ when x has the value of $40.

Equivalence of successive equations in the system of equations constructed to solve the problem: To solve a given equation, the conventional procedure is to construct the vertical chain of equivalent equations that will result in the resolution of the unknown value. The equivalence is achieved in one of two ways. First equivalence is maintained by replacing an expression with an equivalent expression. In the parade problem, the expression $x + \frac{x}{2} +$

$\left(\frac{x}{2} - 40\right)$ in the equation $x + \frac{x}{2} + \left(\frac{x}{2} - 40\right) = 900$ was transformed into the expression $(4x - 80)/2$ and was replaced by latter to yield the subsequent equation $\frac{4x - 80}{2} = 900$ in the equation solving chain. Second equivalence is achieved by replacing an equation with an equivalent equation (i.e. one having the same solution as the previous equation in the chain) without having to replace the preceding expression with another equivalent expression. The equation $\frac{4x - 80}{2} = 900$ is equivalent to the equation $\left(\frac{4x - 80}{2}\right) \times 2 = 900 \times 2$. In this case equivalence of the previous equation is maintained by multiplying both sides of the equation by the same amount. The expression $\left(\frac{4x - 80}{2}\right)$ is not replaced by any other expression.

Comparing and contrasting the solution of the parade problem using letter-symbolic algebra against the model method suggests that major cognitive adjustments – 'accommodations' rather than assimilation are needed to solve algebraic equations. Such major accommodations may help in some way explain why neuroimaging studies found, of the two methods, letter-symbolic method tended to activate the procedural part of the brain and why it required more attention resources. This was because to maintain equality-equivalence of the series of equations in the equation solving chain appropriate rules and procedures specific to operating on letters were operationalized. These rules and procedures were no longer identical to those used to operate on numbers. With the model method, because solving for the unknown values involved working only with numbers there was no conflict in the conventions used and hence may be less procedurally driven. In the model method, solving for the unknown involves the processes of doing and undoing. With the letter-symbolic method the solving for the unknown requires using forward operations and mandates the maintenance of equivalence of each single algebraic equation.

CONCLUSION

The theoretical framework offers possible reasons why of the two methods, letter-symbolic method requires the participants to draw more on attention resources to solve algebra word problems.

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