CHILDREN'S STRATEGY CHOICE AND SUCCESS IN SOLVING ALGEBRA WORD PROBLEMS: INTERPLAY BETWEEN COGNITIVE VARIABLES AND KNOWLEDGE

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Abstract

Across cultures and curricula, it is commonly observed that many beginning learners of formal algebra revert to familiar methods rooted in arithmetic reasoning to solve algebra problems. Though a prevalent cause for frustration in teachers trying to get students to practice and master letter-symbolic algebra, some have argued that such displays of "flexibility" in strategy choice should not be discouraged. Some even found children to be more successful in solving algebra problems when they use arithmetic methods (e.g., Nathan and Koedinger, 2000a). Are arithmetic methods more effective than algebraic methods in solving algebra word problems? Are students truly being "flexible" when they use arithmetic methods in situations calling for algebraic methods? Is it a case of flexibility or inflexibility when students persist in using arithmetic methods when they are no longer appropriate or effective? Are students' strategy choices and their resulting success in solving algebra word problems contingent upon their cognitive capabilities and their understanding of algebra? We examined these questions by giving 157 Secondary 2 students a set of algebra word problems under specific instructions to use letter-symbolic algebra, as well as tests of their algebraic knowledge, intellectual ability, working memory, and inhibitory ability. Results revealed that (i) despite the specific instructions, a substantial number of students persisted in using arithmetic methods, suggesting a reluctance or inability to use letter-symbolic algebra; (ii) students were not more successful in solving algebra word problems when they used arithmetic methods; (iii) algebraic knowledge, intellectual ability, working memory, and inhibitory abilities contribute in both unique and overlapping ways to both strategy choice and success in solving algebra word problems.

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In most mathematics curricula, children would have had many years of experience with arithmetic by the time they are introduced to formal algebra. It is a common observation that beginning learners of algebra frequently revert to familiar arithmetic methods of problem solving when solving algebra problems (e.g., Nathan & Koedinger, 2000a; Ng, 2003; Stacey & MacGregor, 1999). Some have argued that children are merely employing strategic flexibility in using arithmetic methods to solve algebra problems, finding them to be more successful when they used arithmetic methods (e.g., Nathan & Koedinger, 2000a). However, are students truly being flexible when they revert to arithmetic methods? Are they choosing arithmetic methods because they are truly more effective or appropriate? What are some of the factors influencing children's strategy choices and their consequent success in solving algebra word problems?

Algebra Word Problems

There are two general classes of word or story problems – arithmetic problems involving result-unknown quantities and algebra problems involving start-unknown quantities. Consider the question *"A cow weighs 230 kg. A dog weighs 150 kg less than the cow. A goat weighs 130 kg less than the cow. How much do the three animals weigh altogether?"* Questions of this type are considered arithmetic as the unknown values (weight of the dog, weight of the goat, and total weight of the three animals) can easily be derived by directly applying arithmetic operations described in the problem on the known value (weight of the cow). The start-state is given (weight of the cow) while the result-state is unknown (total weight). Conversely, consider the question *"A cow weighs 150 kg more than a dog. A goat weighs 130 kg less than the cow. Altogether the three animals weigh 410 kg. How much does the cow weigh?"* The result-state (total weight) is now known while the start-state (weight of the cow) is unknown. The known value is a combination of the three unknowns and the unknown(s) cannot be easily solved for by directly applying the arithmetic operations described in the problem on the known value. Such start-unknown problems are considered algebraic as letter-symbolic algebra "or more sophisticated modeling" are usually required to solve them (Nathan & Koedinger, 2000a; see also Bednarz & Janvier, 1996).

Children in Singapore start solving word problems in primary or elementary school (for 6 – 12 year-olds), starting with end-unknown arithmetic word problems. In their last two years of primary school, they learn to solve start-unknown algebra word problems with arithmetic methods of problem solving. When they progress to secondary school (for 13 - 17 year-olds) they are taught to solve structurally identical problems using algebraic methods.

Arithmetic versus Algebraic Problem Solving Strategies

Students in the local curricula are taught arithmetic methods such as "listing", "guess-and-check", "unwinding" ("working backwards"), "grouping" and the "model" (Ng, Lee, Ang & Khng, 2006; see Table 1) in primary school and letter-symbolic algebra in secondary school. With letter-symbolic algebra, one works directly with unknown quantities. An equation comprising both known and unknown values is first formed using "forward operations" (Usiskin, 1988). To solve for the unknown, one then operates on the equation in a way that maintains its symmetric equivalence – terms are transposed while keeping the left-hand side of the equation balanced with the right-hand side (see Table 1vi). In contrast, when arithmetic strategies are used, one works only with known quantities. The unknowns are found by the direct application of arithmetic operations on known values, often involving the inversion of operators described in a problem (see Table 1i-v).

Insert Table 1 about here

By the time our students are in their second year of secondary school, they will have a repertoire of both arithmetic and algebraic strategies at their disposal. Secondary school students are required to use only letter-symbolic algebra to solve algebra word problems. However, in a local study, Ng (2003) found a substantial proportion of secondary school students still relied on arithmetic methods to solve algebra word problems (see Nathan and Koedinger, 2000a for US-based findings). Persistent use of such methods poses a problem as it can hinder the acquisition of algebraic strategies. Algebra word problems are given to beginning algebra students to help them learn letter-symbolic algebra which is not only a more sophisticated method, but also essential for progression to higher mathematics (e.g., National Mathematics Advisory Panel, 2008). The more children persist in using arithmetic strategies, the less chance they will have to practice and master letter-symbolic algebra.

The Present Study

The present study examined the relationship between children's strategy choice and their resulting efficacy in solving algebra word problems. We also examined how individual differences in the cognitive capabilities children bring to the classroom interacts with the subject-specific knowledge they acquire to influence both their strategy choice and success in solving algebra word problems.

Method

Participants

A total of 162 Secondary 2 students from six schools in Singapore participated in the study. Secondary school students in Singapore are streamed according to their results in their Primary School Leaving Examinations. Participants from all streams, except the Gifted Elective Program, were included in the sample. Students in the Gifted Program represented the top 1% of the cohort and were excluded as they followed a different syllabus. The next 9% go to the Special stream, the following 50% into Express, the next 25% Normal (Academic), and the remaining 15% Normal (Technical). Students from each of these 4 streams were included in proportions approximate to national distribution. The final data set contained 157 participants (73 boys, mean age = 14.7 years, range = 13.5 – 15.7 years, *SD* = 0.47). Five students were excluded because of missing data due to file corruption or equipment failure.

Materials

A set of algebra word problems, measures of algebraic knowledge, working memory capacity, intellectual ability, and six tests of inhibitory ability were administered. Inhibitory measures were chosen that provided indices of participants' ability to suppress a dominant, over-learned, or highly automatized response (e.g., Friedman & Miyake, 2004; Nigg, 2000). Such abilities are hypothesized to be involved in preventing well-learnt or practiced prior knowledge (such as arithmetic strategies in this instance) from interfering with the acquisition or execution of newer knowledge (letter-symbolic algebra in this instance). Measures included the proactive interference measure from the California Verbal Learning Test for Children (CVLT-C; Delis, Kraner, Kaplan, & Ober, 1994), number of perseverative responses from the Wisconsin Card Sorting Task (WCST: CV3; Heaton, Chelune, Talley, Kay, & Curtiss, 1993), the adjacency index from the Random Number Generation (RNG; Evans, 1978), total number of conflict-move errors from the Tower of Hanoi (TOH), number of Stroop errors from the Number-quantity Stroop (Stroop, 1935), and the proportion of perseverative categorization responses from the Stop-signal (Logan & Cowan, 1984).

Algebra word problems. Nine start-unknown word problems modified from questions in local mathematics textbooks (Teh & Looi, 1997, 1998) were administered. All were solvable with both simple letter-symbolic algebra and arithmetic methods. Participants were specifically instructed to use letter-symbolic algebra and to show all procedures. The number of problems answered correctly served as the accuracy score. Arithmetic intrusions served as an indicator of strategy-use and referred to the percentage of attempted questions in which non-algebraic methods were used. Each question's solution was coded exclusively as "algebraic", "arithmetic", "mixed", or "no strategy" based on the overall solution. Non-algebraic methods included both "arithmetic" and "mixed" solutions. Classification was based not on the initial problem representation, but the steps taken to arrive at the answer.

Algebraic knowledge. An adaptation of the Küchemann Algebra Test from the Chelsea Diagnostic Mathematics Tests (CDMT; Brown, Hart, & Küchemann, 1985) was administered as a standardized measure of participants' algebraic knowledge. The test assessed students' understanding of substitution, simplifying expressions, and constructing, interpreting and solving equations. Participants' level of algebraic understanding was scored in accordance to standardized procedures (Brown et al., 1985).

Working memory. A complex working memory span score was derived from the Counting Recall and Backward Digit Recall subtests from the Working Memory Test Battery for Children (WMTB-C; Pickering & Gathercole, 2001).

General intellectual ability. The Brief Intellectual Ability (BIA) test from Woodcock-Johnson III Tests of Cognitive Abilities (Woodcock, McGrew, & Mather, 2001) was used as a brief measure of general intelligence. A BIA score was calculated with the test's Compuscore and Profiles Program.

Procedure

All participants were tested over two sessions in their respective schools: a group session where they completed the algebra word problems followed by the Küchemann Algebra Test in a classroom, and an individual session in which they were administered the CVLT-C, BIA, WMTB-C, Number-quantity Stroop, Stop-signal, RNG, WCST, and TOH in a computer lab. Participants were given up to one hour to complete the algebra word problems and 45 minutes for the Küchemann Algebra Test. Unless stated otherwise, there was no time limit for the other tasks. Practice trials were given for the Küchemann Algebra Test, BIA, WMTB-C, Number-quantity Stroop, Stop-signal, RNG, and TOH. The order of task administration was randomized with the following constraints. As recall interference was assessed in the CVLT-C, it was always administered first to avoid interference from other tests, as well as degraded memory due to fatigue.

Results

Correlation analysis revealed a negative relationship between arithmetic intrusions and algebra word problem solving accuracy. Of all questions, 36% were attempted with algebra. Non-algebraic methods (arithmetic intrusions) were used 52% of the time: 38% arithmetic methods; 14% mixed arithmetic-algebra. An analysis of the proportion of questions correctly solved using each strategy showed the highest success rate for algebra strategies (76%). In contrast, only 50% of the solutions with arithmetic intrusions (arithmetic and mixed) yielded correct answers. Within arithmetic intrusions, the success rate for mixed strategies was higher than that for arithmetic strategies (70% & 42%, respectively). A detailed breakdown of the strategies used and their success rates is provided in Table 2.

Insert Table 2 about here

To examine the interplay between cognitive and knowledge factors in influencing strategy choice and success, path analyses were conducted using LISREL 8.30 (Jöreskog & Sörbom, 2000). Principal components analysis revealed that the 6 inhibitory measures loaded onto two inhibitory components. Component 1 consisted of the Stop-signal, CVLT-C, and WCST measures. Component 2 comprised of the RNG, Stroop, and TOH measures. Inhibitory ability was thus entered into the path analysis as two variables using factor scores generated from the principal components analysis.

The final model provided a good fit to the data, χ^2 (8, N = 157) = 5.21, p = .74, CFI = 1.00; SRMR = .036; RMSEA < .001 (see Figure 1). Use of arithmetic strategies (arithmetic intrusions) had a direct negative path of β = -0.18 to accuracy. Intellectual ability and algebraic knowledge had the biggest influence on both accuracy and strategy choice. Working memory had only indirect paths to arithmetic intrusions and accuracy through intellectual ability and algebraic knowledge. Both inhibitory components had only indirect effects on accuracy, with component 2 operating through strategy choice (arithmetic intrusions), and component 1 operating through intellectual ability. Inhibitory component 2 had a direct effect on strategy choice while component 1 had an indirect effect of through intellectual ability.

Insert Figure 1 about here

Discussion

In the present study, despite specific instructions to use algebra, a substantial number of students persisted in using arithmetic methods, displaying either an inability or reluctance to use symbolic algebra. Because the use of arithmetic strategies was actually associated with lower success rates, we do not think that students were displaying “strategic flexibility” but were unable to use algebraic strategies. Furthermore, “flexibility” is only advantageous if children can call upon different strategies for different types of problems and situations. The students, on the contrary, seemed to have displayed inflexibility by being unable to use letter-symbolic algebra despite having been asked to do so. Contrary to Nathan and Koedinger’s (2000a) findings, children in this study were more successful when they applied algebra (76%) than when they used arithmetic strategies (42%). As the efficacy of arithmetic methods decreases with increasing problem complexity (e.g., Tabachneck, Koedinger, & Nathan, 1995), this difference is probably due to the inclusion of more difficult problems in our study, such as those involving simultaneous equations.

Both cognitive variables and acquired knowledge influenced strategy choice and problem solving success in unique and overlapping ways. Expectedly, algebraic knowledge and intellectual ability were the strongest predictors of both strategy choice and problem solving accuracy. However, the finding of a direct influence from inhibitory component 2 to strategy choice may have pedagogical implications. Measures that loaded on component 2 (Number-quantity Stroop, RNG, TOH) involved the inhibition of well-practiced or well-entrenched processes that have been reified into the concept image of the stimuli (see Tall, Thomas, Davis, Gray, & Simpson, 1999). In these tasks the association between the stimulus and to-be-inhibited response/process is not dissimilar to the relationship between arithmetic methods and algebra word problems, which, through practice, are likely to have become intimately related. Perhaps what explains the direct effect of component 2 on arithmetic intrusions is that the underlying measures index children's abilities to inhibit reified processes.

Pedagogical Implications

The learning of algebra begins in primary school where students solve start-unknown word problems with arithmetic procedures. However, teaching methods that emphasize the procedural acquisition of heuristics can present obstacles to students when they have to learn a more sophisticated method. Well-practiced arithmetic methods can become reified as part of the "meaning" encapsulated in algebraic word problems. In the mathematical literature, it has been observed that rote learning of procedures "may work at one stage, yet create met-befores that interfere with later development" (Tall, 2004). The findings of our present study suggest that failure to inhibit entrenched arithmetic methods can cause added interference in the acquisition of algebraic methods.

In mathematical development, concrete operations at one level become embodied in structural mental objects at a higher level, which then form the basis of operations for building structural objects at yet higher levels. One pedagogical pitfall in the current teaching of algebra is that students are taught to solve algebraic word problems with arithmetic methods in primary school; in secondary school, instead of building algebraic methods on students' existing knowledge of arithmetic methods, students are often expected to "abandon" their accustomed methods and to adopt a new one (Ng, et al., 2006). We advocate building new knowledge on existing knowledge and suggest more effort be made to emphasize the connection between arithmetic and algebraic methods, so that students can see the two as an *assimilated whole* instead of two competing strategies. One possible way to do this is to emphasize the equivalence between pictorial and symbolic representations of unknowns in the teaching of formal algebra (Ng, et al., 2006; see also Nathan & Koedinger, 2000b).

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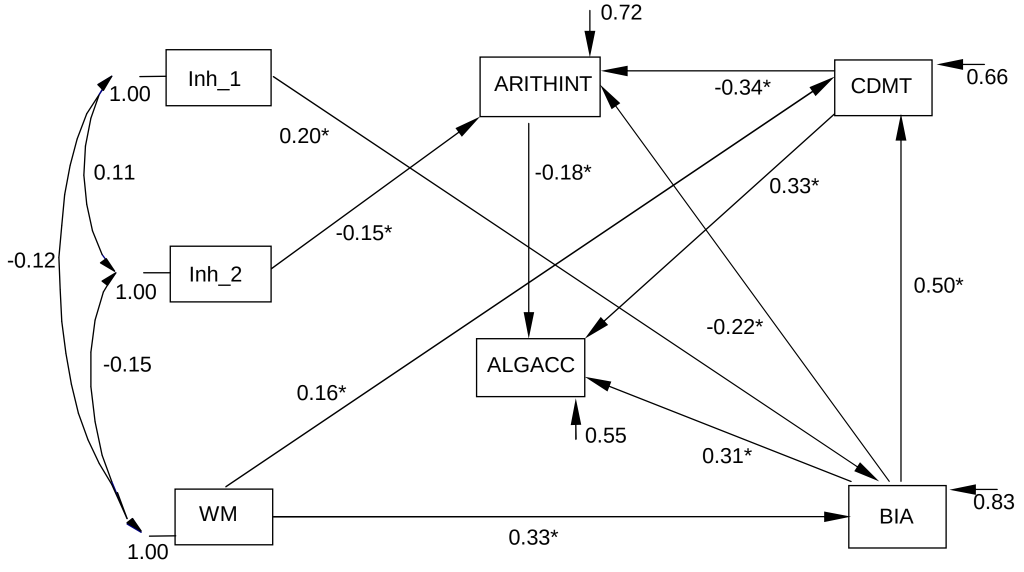
Figure 1



Chi-Square = 5.21, df = 8, P-value = 0.74, RMSEA = 0.000, SRMR = .036, CFI = 1.00

Note. Inh_1, Inhibition Factor Score 1 (WCST, CVLT-C, Stop-Signal); Inh_2, Inhibition Factor Score 2 (TOH, Stroop, RNG); WM, Working Memory Complex Span; BIA, Brief Intellectual Ability; CDMT, Chelsea Diagnostic Mathematics Test; ALGACC, Algebra Word Problem Solving Accuracy; ARITHINT, Frequency of Arithmetic Intrusions. Standardized parameter estimates are reported.

* Denotes reliable paths.

Table 1

Examples of Arithmetic and Algebraic Methods Used to Solve Start-Unknown Word Problems

Question Type A: A shopkeeper bought some ducks and 4 times as many chickens. A duck costs $4 while a chicken costs $3. Altogether the shopkeeper spent $192. How many chickens did the shopkeeper buy?

i) Systematic listing

Ducks		Chickens		Total
1	$4	4	$12	$16
2	$8	8	$24	$32
3	$12	12	$36	$48
4	$16	16	$48	$64
5	$20	20	$60	$80
6	$24	24	$72	$96

The shopkeeper bought 48 chickens.

Solving by systemic listing entails listing down all plausible values for the unknown(s) until the correct tally is reached.

ii) Guess-and-check

1 duck and 4 chickens: $1 \times 4 + 4 \times 3 = \$16$

10 ducks and 40 chickens: $10 \times 4 + 40 \times 3 = \$160$

12 ducks and 48 chickens: $12 \times 4 + 48 \times 3 = \$192$

The shopkeeper bought 48 chickens.

A subset of systematic listing. Only certain guessed-at values for the unknown(s) are listed.

iii) Grouping

Cost of 1 duck and 4 chickens: $\$4 + (4 \times \$3) = \$16$

$\$192 \div \$16 = 12$

$12 \times 4 = 48$

The shopkeeper bought 48 chickens.

Unknowns are grouped into sets and solved for by unwinding from the given total.

Question Type B: A cow weighs 150 kg more than a dog. A goat weighs 130kg less than a cow. Altogether the 3 animals weigh 410 kg. How much does the cow weigh?

iv) Unwinding

$$410 - 150 - (150 - 130) = 240$$

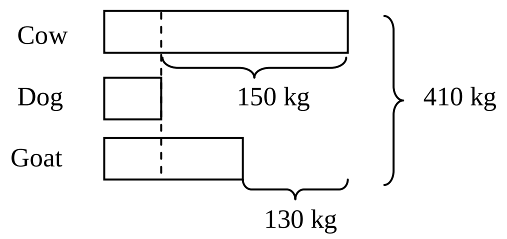
$$240 \div 3 = 80$$

$$150 + 80 = 230$$

The cow weighs 230 kg.

The child works backwards from a known result-value to an unknown start-value by inverting the arithmetic descriptors relating the known and unknown values.

v) Model



$$\left.\begin{array}{c} 410 - 150 - (150 - 130) = 240 \\ 3 \text{ units} = 240 \\ 1 \text{ unit} = 240 \div 3 = 80 \end{array}\right\} \quad \text{Unwinding}$$

$$150 + 80 = 230$$

The cow weighs 230 kg.

A diagram representing the problem scenario is first drawn. Rectangles are used to represent known and unknown quantities. A series of arithmetic expressions are constructed and the unknown is solved by unwinding the quantitative relationships represented in the diagram.

vi) Letter-symbolic algebra

Let x be the weight of the cow.

$$x + (x - 150) + (x - 130) = 410$$

$$3x - 280 = 410$$

$$3x = 410 + 280$$

$$3x = 690$$

$$x = 230$$

The cow weighs 230 kg.

Letters are used to represent unknown quantities. An equation representing elements described in the problem – known and unknown values and the quantitative relationships between them – is formed. The unknown is solved for by constructing a series of *equivalent equations* – each subsequent equation is an equivalent expression of the first equation; each equation is a structural embodiment of the problem scenario. Compare this with the series of procedural expressions in arithmetic methods (i-v).

Note. Different arithmetic methods (i – v) commonly used to solve 2 types of start-unknown word problems. Letter-symbolic algebra (vi) can be used to solve both types of problems.

Table 2

Percentage Distribution of Problem Solving Strategies and Success Rates Collapsed Across Questions

Strategy Type		Usage rate	Success rate
No Response		11	--
No Strategy		2	4
Algebra		36	76
Non-Algebraic		52	50
Types of Non-Algebraic Strategies			
Arithmetic		38	42
	Model	10	47
	Unwinding	15	30
	Guess-and-check	3	28
	Listing	3	64
	Grouping	7	56
Mixed		14	70
	Mixed-Model	2	64
	Mixed-Unwinding	8	72
	Mixed-Guess-and-check	0[a]	0
	Mixed-Listing	0	--
	Mixed-Grouping	3	76

Note. Figures in percentages, rounded off to the nearest whole number. Success rate refers to the number of questions correctly solved using a strategy out of the total number of questions attempted with that strategy, reported as a percentage.

[a] 0 when rounded off to nearest whole number.