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ASSESSING MATHEMATICAL COMPETENCIES USING DISCIPLINARY TASKS

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The Singapore Mathematics Assessment and Pedagogy Project (SMAPP) is a research project conducted by the National Institute of Education and funded by the Ministry of Education. It aims to make assessment practices an integral part of teaching and learning, and broaden student learning outcomes by using authentic disciplinary tasks. As part of the project, some guidelines are provided for designing disciplinary tasks which have the distinctive features of their emphasis on contextual aspects. One of the criteria of a good disciplinary task is its ability to assess multiple mathematical competencies of students. In this paper, we will present some examples to illustrate how these competencies can be assessed. Another aim is to find out to what extent these tasks serve the purpose of assessing these competencies, by analyzing the students' performance in a sample SMAPP task.
Key words: Assessment, multiple mathematical competencies, disciplinary tasks

INTRODUCTION

The Singapore Mathematics Assessment and Pedagogy Project (SMAPP) is a research project conducted by the National Institute of Education and funded by the Ministry of Education. It aims to enhance the quality of mathematics teaching and learning by developing formative and summative assessment practices, making these assessment practices an integral part of teaching and learning, and broadening student learning outcomes by using authentic disciplinary tasks.

Fan, et al (2010) has given a summary of some of the tasks designed in the early stage of the project. In Zhao, et al (2011), some guidelines are provided for designing disciplinary tasks which have the distinctive features of their emphasis on contextual aspects. One of the criteria of a good disciplinary task is its ability to assess multiple mathematical competencies of students. The competencies we usually focus on include (i) understanding the problems and extracting information from the problems; (ii) computation; (iii) reasoning skills such as deductive and inductive reasoning; (iv) communication using appropriate representations and means such as graphs, tables, algebraic expressions, functions; (v) the use of mathematics to formulate and solve real life problems.

In this paper, we will present some examples to illustrate how the above competencies can be assessed. Another aim is to find out to what extent these tasks serve the purpose of assessing these competencies, by analyzing the students' performance in a sample SMAPP task.

ASSESSING COMPETENCIES: A CASE STUDY

In the last three years, we have designed eight disciplinary tasks and some of them have been tried out in four Singapore secondary schools. A special feature of the disciplinary tasks that distinguishes them from the traditional assessment problems is their contextualized content.

A sample SMAPP task

In this paper, we consider a task on paper recycling. The task scenario is given in Figure 1. Notice the relevance of the scenario to students' daily life. Based on this scenario, a series of questions are formulated to assess various mathematical competencies. These questions are given in the Appendix.

Arithmetic:

Paper Recycling

Paper is the most common type of waste in Singapore. About 1.26 million tonnes of paper waste were generated in 2008. Recycling paper conserves forest resources, and produces fewer pollutants than conventional pulping and bleaching processes.

Shamila is a member of her school's Green Club. Her school is using paper made from virgin pulp for printing teaching materials. She wants to find out how many trees can be saved each year if the school uses only recycled paper.

She also wants to present some facts and figures to her teachers and schoolmates, on the benefits to the environment if the school recycles paper and uses recycled paper.

In this task, you will make calculations to help Shamila complete the above tasks.

Figure 1. Scenario for paper recycling task

Competencies assessed in the sample task

For ease of reference, we label the mathematical competencies listed in the Introduction as follows: (MC1) problem understanding and information extraction; (MC2) computation; (MC3) reasoning skills; (MC4) communication using appropriate representations; (MC5) the use of mathematics to formulate and solve real-life problems.

Table 1 summarizes the mathematical competencies assessed in the paper recycling task. We wish to point out that not every competency is adequately assessed in every task. For example, MC4 is only assessed in this paper recycling task by one question (Q3e). It can be further assessed in the task on statistics in this project. The solution of this task requires mainly MC1 and MC2, and to some extent, MC3 and MC5. For MC3, inductive reasoning is assessed through pattern recognition in Q1b, Q1c(ii) and Q3b, while proportional reasoning is assessed in Q1e and Q1f. To assess the competency of deductive reasoning, we can employ a task on geometry. To incorporate the principle of assessment as learning, Q1e, Q1f, Q2b and Q2c are designed so that through solving these problems, students also learn how mathematics can be used to formulate real-life problems. Another example of such formulation is given in Zhao et al (2011), in which the area of a reservoir is approximated using polygons.

Table 1: Mathematical competencies assessed in paper recycling task

	Q1a	Q1b	Q1c(i)	Q1c(ii)	Q1d(i)	Q1d(ii)	Q1e	Q1f	
MC1	√	√	√	√	√	√	√	√	
MC2		√	√		√	√	√	√	
MC3		√		√			√	√	
MC4									
MC5							√	√	
	Q2a	Q2b(i)	Q2b(ii)	Q2c	Q3a	Q3b	Q3c	Q3d	Q3e
MC1	√	√	√	√	√	√	√	√	√
MC2		√	√	√		√			√
MC3						√			
MC4									√
MC5		√	√	√					

Findings and discussion

A total of 407 Secondary One (grade 7) students from 11 classes across 4 schools attempted the paper recycling task. Table 2 summarizes the performance of students in individual questions.

To help students better understand the contextualized content and extract the relevant information, summary questions are provided, as shown in the Appendix. We expect these summary questions to be “easy”, but only 67.8% were able to score the full 3 marks and 28.7% scored zero. Possible reasons are careless omission or students did not realize that they are required to answer them.

For Q1a, 82.8% knew how to find the area of A0 paper (a rectangle) as 0.841 × 1.189. For students who chose the wrong option (0.841 + 1.189) × 2, they might have confused between the formulae for the perimeter and area of rectangle.

Table 2: Percentage of students by marks for individual questions

Question	Marks						
[Full marks]	0	0.5	1	1.5	2	3	4
Q1a [1]	17.2		82.8				
Q1b [4]	13.3		0.7		3.7	9.1	73.2
Q1c(i) [1]	47.9		52.1				
Q1c(ii) [1]	32.7		67.3				
Q1d(i) [2]	46.4		2.0	2.7	48.9		
Q1d(ii) [2]	48.9		13.5	1.2	36.4		
Q1e [3]	61.7	0.5	9.6		4.7	23.6	
Q1f [2]	81.6	1.5	10.1		6.9		
Q2a [2]	27.5		15.5		57.0		
Q2b(i) [1]	30.5		69.5				
Q2b(ii) [1]	28.3		71.7				
Q2c [2]	45.7	22.1	2.0	2.5	27.8		
Q3a [2]	34.4		4.2		61.4		
Q3b [3]	28.5		7.6			63.9	
Q3c [1]	39.6		60.4				
Q3d [1]	73.0		27.0				
Q3e [1]	47.7		52.3				

To answer Q1b correctly, students have to understand the problem and be able to extract the relevant information. Some basic computation and reasoning skills (recognition of number pattern) are also needed. 73.2% were able to fill in all the four blanks in Q1b correctly. The following are responses from one student (with the correct answers given in parentheses):

Ans1: 0.05 (0.125) Ans2: 16 (16) Ans3: 0.125 (0.03125) Ans4: 64 (64)

Ans1 and Ans3 are incorrect, as this particular student did not see the number pattern exhibited in the estimated area. This may suggest that he did not understand the relationship between the different paper sizes.

In comparison to Q1b, only 52.1% and 67.3% respectively could answer correctly Q1c(i) and Q1c(ii). This is somewhat surprising, as Q1c(i) is simply a conversion from m^2 to cm^2: $0.03125 \times 10000 = 312.5$ cm^2. This might be due to a lack of practice with this type of

computation in school. The solution for Q1c(ii), 32 ÷ 4 = 8 sheets, requires a reasoning that is slightly more than the recognition of number pattern.

In Q1d(i), students encountered an unusual form of rate (g/m^2) taken from real-life context but not commonly seen in textbook. Only 48.9 % could answer Q1d(i) correctly. The correct answer for the mass of one sheet is 80 × 0.0625 = 5 g. For the incorrect response 80 ÷ 0.0625 = 1280 g, where ÷ was used instead of ×, the student possibly thought that the "/" sign in the unit g/m^2 means division is needed. This provides an example that a misconception (of rate) hinders the understanding of a problem. Here is another incorrect response:

$$80 \div 10000 = 0.008, \quad 0.008 \times 0.0625 = 0.0005 \text{ g}.$$

It seems that the student was trying to convert m^2 in the rate 80 g/m^2 to cm^2, indicating a lack of understanding in the unit associated with a numerical value.

For Q1d(ii), only 36.4% could find the mass of one ream: 5 × 500 = 2500 g = 2.5 kg. The following is one incorrect response:

$$80 \div 10000 = 0.008, \quad 0.008 \times 500 = 4 \text{ kg}.$$

This student seemed to think that 1 sheet of paper has a mass of 80 g. He could not see that the solution of Q1d(ii) needed the answer in Q1d(i). Another incorrect response, 0.0625 × 500 = 31.25, also indicates the difficulty in linking Q1d(ii) and Q1d(i).

From Table 2, we see that Q1e and Q1f are the two most difficult questions with 61.7% and 81.6% respectively scoring zero. (Note that with 73.0% scoring zero, Q3d seems to be more difficult than Q1e. As explained later, the poor performance in Q3d is probably not because Q3d is difficult, but is rather the result of misconception.) This is not unexpected as 4 competencies (MC1, MC2, MC3, MC5) are needed in answering Q1e and Q1f. To solve Q1e, students needed first to find the mass of paper used: 3 × 1200 × 2.5 = 9000 kg = 9 tonnes. Then the number of trees saved is 9 × 17 = 153. The following is an incorrect response:

$$3 \times 500 \times 1200 = 1800000, \quad 1800000/17 = 105882.35.$$

This suggests that the student could not see the relationship between Q1e and Q1d(ii).

Q1f can be solved in three ways. The first is to repeat the steps in Q1e using 70 g/m^2 instead of 80 g/m^2. The second is to repeat the steps in Q1d and Q1e using the difference 80 – 70 = 10 g/m^2. The third way is to use proportional reasoning: 153 × (10/80) ≈ 19 trees. However, none among those 6.9% who answered Q1f correctly used this third method, which is the desired way. The only student (out of 407) who used proportional reasoning scored 1 mark because the answer is not correct:

$$153 \times \frac{7}{8} = 133\frac{7}{8} \approx 134, \quad 152 - 134 = 18.$$

This suggests that although students have been solving many questions involving ratio and proportion in primary school, many of them still could not transfer the skills learned to solve real-life problems. One possible reason is that students solve problems by first identifying the "standard" type in which the question falls, and then applying the corresponding strategy. Once a real-life problem does not fit into one of these types, they could not solve the problem.

For Q2a, the information (benefits of recycling papers) that students needed to extract from a

search engine is not mathematical in nature. Nevertheless, we think that MC1 can still be assessed by Q2a, as the competency of extracting relevant information from the internet is an important skill in solving real-life problems. For the 27.5% who scored zero for Q2a, some simply "copied and pasted" the information without checking its relevance or whether it makes sense. An example of an answer that does not make sense is "The Recycled Paper can be turned into new paper so new trees don't have to be cut down, thus decreasing wildlife."

For Q2b(i) and Q2b(ii), 30.5% and 28.3% scored zero respectively. The answers to these two questions can be obtained by just multiplying two numbers given in the questions: $150 \times 16 = 2400$ kg (carbon dioxide absorbed), and $150 \times 10 = 1500$ kg (oxygen produced). The majority of those who scored zero did not give any answer. One possible reason is that these students have difficulty in formulating the mathematical expression needed to answer these real-life questions. For Q2c, it is then not surprising that the percentage who scored zero is higher at 45.7%, as the solution of Q2c makes use of the answer of Q2b(i): $2400 \div 0.16 = 15000$ km. Some students answered Q2b(i) correctly but gave $2400 \times 0.16 = 384$ km as the answer for Q2c, suggesting that they did not understand the meaning of "0.16 kg for every km".

For Q3a, students were only required to extract the correct information 26495 and 1438 from a poster. However, 34.4% scored zero. Again, the majority of those who scored zero did not give any answer. For Q3b, 63.9% were able to correctly complete the factor tree with the factors 3, 3, 3785, 5. For Q3c, 60.4% were able to recognize that given 757 is prime, it has only 2 factors (1 and itself). However, only 27.0% were able to write down the prime factorization of 238455 as $3^2 \times 5 \times 7 \times 757$ in Q3d. Some common mistakes are: (1) regarding 1 as a prime number as in "$3^2 \times 5 \times 7 \times 1 \times 757$"; (2) listing the factors instead of writing down a product of primes as in "5, 3^2 , 7, 757". This suggests that students have a misconception that 1 is prime and an incorrect understanding of prime factorization.

For Q3e, only 52.3% chose the correct pictogram to represent 238455 litres. For students who selected the first option, it seems they did not realize that 240000 litres provides a better approximation than 200000 litres. A misconception of pictogram (that only "complete" pictures are allowed) might also be a reason.

CONCLUSION

Disciplinary task, as a form of assessment, can be used to assess "integrated" mathematical competencies which cannot be achieved through conventional school tests and examination. From this study, we give some suggestions for improving the assessment of competencies:

(1) Complete the competency table while the task is being designed so that a sufficient range of competencies can be assessed.

(2) As we have seen, there are not enough questions in this task that need reasoning and communication skills. We need a few more tasks involving a variety of contexts in order to have a complete assessment of the competencies.

(3) We have at least 73.2% answering Q1b correctly, but for Q1c(i) which depends on Q1b, only 52.1% answered it correctly. As students were only required to key in the numerical answers for these questions, it is not possible to identify the reason for the mistake without interviewing them. This is perhaps one drawback of online assessment. In order to improve

the assessment of competencies, we may need to ask students to provide intermediate steps so that the reason for any mistake can be identified.

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Appendix: Task on paper recycling

Read the text below carefully. Then fill in the blanks in the summary box with numerical values.

There are 1200 students in Shamila's school. She estimates that each student uses about 3 reams of paper per year (for notes, test and exam papers, etc.).

Looking at the packaging of the printing paper, she sees '80 g/m^2' (80 grams per square metre) printed on it. Each ream of paper contains 500 sheets of printing paper.

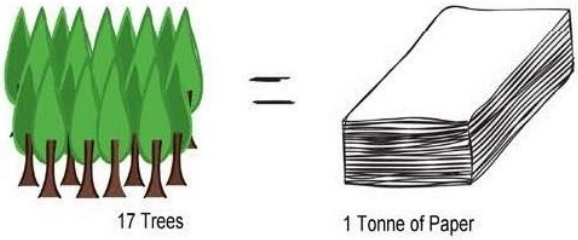
She learns from a National Environment Agency (NEA) poster that it takes about 17 mature trees to make a tonne (1000 kg) of printing paper.

If she can calculate the weight of the printing paper, excluding the packaging, she will be able to know the number of trees that can be saved by converting to recycled paper.

She does not have a weighing machine. You are approached to help her solve the problems.

SUMMARY [3]

- *There are _____ students in Shamila's school.*
- *Each student uses about _____ reams of paper per year.*
- *Each ream of paper contains 500 sheets of paper.*
- *Weight of paper used = 80 grams per square metre.*
- *17 mature trees are required to produce 1 tonne (_____kg) of printing paper.*



Number of Trees to be saved

1.

(a) The length and breadth of an A0 paper is 0.841 m and 1.189 m. Which of the following expressions is correct to get the area of an A0 paper?

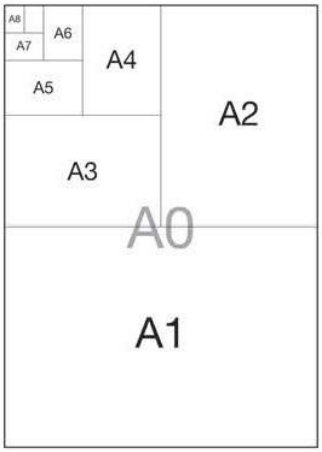
(1) (0.841 + 1.189) × 2

(2) 0.841 × 1.189

(3) 1.189 ÷ 0.841

(4) 1.189 – 0.841 [1]

(b) An A1 paper is obtained by folding an A0 paper into two equal halves.



Similarly, an A2 paper is obtained by folding an A1 paper into two equal halves.

Table 1 shows the estimated area of the "A series" paper size and the sheets of paper that can be obtained from an A0 paper.

Fill in the blanks with the correct answers. [4]

Paper Size	Estimated Area (m^2)	Sheets of paper that can be obtained from an A0 paper
A0	1	1
A1	0.5	2
A2	0.25	4
A3		8
A4	0.0625	
A5		32
A6	0.015625	
A7	0.0078125	128
A8	0.00390625	256

Table 1

(c) i. Based on your answer to the area of A5 paper, find its estimated area in square centimetres. ($1 m^2$ = 10 000 cm^2). [1]

ii. Find the number of sheets of A5 paper that can be obtained from a sheet of A2 paper. [1]

(d) Looking at the packaging of the printing paper, Shamila sees '80 g/m^2' (80 grams per square metre) printed on it.

i. What is the mass of one sheet of A4 paper? [2]

ii. One ream of paper contains 500 sheets of printing paper. What is the total mass of one ream of A4 paper (excluding the mass of the packaging)? Give your answer in kg. [2]

(e) There are 1200 students in Shamila's school. It is estimated that each student uses about 3 reams of A4 paper per year (for notes, test and exam papers, etc.).

If the school uses only recycled paper, how many trees can be saved each year? (Recall that 1 tonne of paper ≈ 17 trees) [3]

(f) If the school uses a lighter type of paper (70 g/m^2 instead of 80 g/m^2), how many more trees can be saved each year? Give your answer as a whole number. [2]

Benefits of using Recycled paper

Shamila wants to make a poster to present some of the facts and figures on the benefits of recycling paper and using recycled paper. Besides the number of trees that can be saved, she plans to include other information on the benefits to the environment if the school converts to recycled paper. You are approached to help her solve the rest of the problems.

2.

(a) By searching for "advantages of recycling paper" on a search engine, find and write down 2 benefits of recycling paper. [2]

(b) It is estimated that each mature tree can absorb 16 kg of carbon dioxide and produce 10 kg of oxygen each year. If the school saves 150 trees per year by only using recycled paper,

i. how much carbon dioxide could have been absorbed by those 150 trees per year? [1]

ii. how much oxygen could have been produced by those 150 trees per year? [1]

(c) On average, a car emits about 0.16 kg of carbon dioxide for every kilometre travelled (http://www.carpages.co.uk/co2/). If it emits the same amount of carbon dioxide as found in **b(i)** above, how far has it travelled in km? [2]

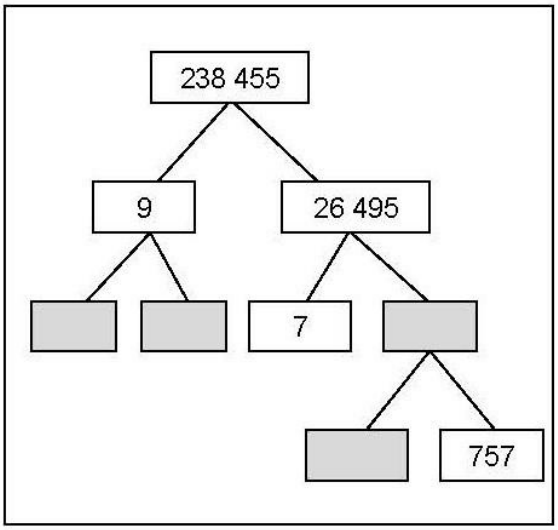
Save Water!

3.

(a) Fill in the blanks based on the information from the poster. [2]

A wall poster at Changi Airport states that every tonne of paper recycled can save _____ litres of water and _____ litres of oil.

(b) Shamila calculated that the school uses about 9 tonnes of paper per year and will therefore save 238 455 ℓ of water.
Complete the factor tree below. [3]



(c) Given that 757 is a prime number, what are the factors of 757? [1]

(d) Write down the prime factorization of 238 455. [1]

(e) Shamila wants to draw a pictogram to show the amount of water saved (238 455 ℓ).

Which pictogram below should she use? [1]

One bottle represents 50 000 ℓ of water.

(1)

(2)

(3)

(4)