Paper Title Overcoming Challenges in Mathematical Problem Solving in Student-Pair Work Settings: Cases From Singapore

Author(s) Luis Tirtasanjaya Lioe, National Institute of Education, Singapore; Yanping Fang, National Institute of Education, Nanyang Technological University, Singapore

Session Title Studies of Discourse and Student Learning in Mathematics

Session Type Paper

Presentation Date 4/30/2010

Presentation Location Denver, Colorado, USA

Descriptors Student Behavior/Attitude, Cooperative Learning, Mathematics Education

Methodology Qualitative

Unit SIG-Research in Mathematics Education

Overcoming Challenges in Mathematical Problem Solving in Student Pair Work Settings – Cases from Singapore

Luis Tirtasanjaya LIOE
Centre for Research in Pedagogy and Practice, National Institute of Education,
Nanyang Technological University, Singapore
Email: luis.lioe@nie.edu.sg

Yanping FANG
Curriculum, Teaching and Learning, National Institute of Education,
Nanyang Technological University, Singapore
Email: yanping.fang@nie.edu.sg

Abstract
This study aims to understand students' mathematical learning in a collaborative setting from real cases of Singaporean students' problem-solving pair work. Videotaped observations, with a think-aloud instruction, were conducted on 7 Fifth Grade and 8 Seventh Grade pairs of Singaporean students. The data were analysed using compare-and-contrast method (Glaser & Strauss, 1967) across all student pairs to generate cases of pair interactions that fostered or hindered students in overcoming the challenges that they encountered. Findings reveal at least four aspects of the pair interactions that either fostered or hindered these processes. Important implications were drawn to support teachers in developing insights about the enabling and disabling factors for effective collaborative work in their classrooms. This paper presents the detailed analysis of all these cases.

Overview and Objective
The importance of students' collaboration in learning mathematics is emphasized in 21st-century-reformed curriculum around the world, such as the NCTM standards (National Council of Teachers of Mathematics, 1991, 2000) in the U.S, the Board of Studies NSW (2002) in Australia, and the Singapore's problem-solving-based curriculum (Curriculum Planning and Development Division, 2000, 2005). A growing body of literature has noted the wider range of learning opportunities in collaborative work than in individual work (e.g. Artzt & Armour-Thomas, 1992; Goos, Galbraith, & Renshaw, 2002; Johnson, Johnson, & Holubec, 1998). However, many teachers face challenges in employing collaborative work in their classrooms, especially in East-Asian countries, including Singapore (Fang, Ho, Lioe, Wong, & Tiong, 2009) where teacher-centreed instruction has a strong tradition (Fan, Wong, Cai, & Li, 2004). Teachers have had various concerns in implementing collaborative work (Foong, Yap, & Koay, 1996) when they found that not all collaborative work would lead to good quality interactions (Carr & Biddlecomb, 1998) and some may even diminish students' problem-solving performances (Stacey, 1992). Therefore, helping teachers develop insights of the enabling and disenabling factors for effective collaborative work is urgently needed. Particularly for Singaporean teachers, analysis of cases of students' collaborative problem-solving in the local classrooms is important to help teachers understand how to adapt collaborative work

featured in the Western literature to fit into the local cultural practices of teaching and learning. Our study is designed to meet this need based on our project, Developing Repertoire of Heuristics in Mathematical Problem Solving in Singapore, funded by Centre of Research in Pedagogy and Practice (CRPP), National Institute of Education, Singapore(2004 – 2009).

Theoretical Framework

Mathematical Problem Solving

In Singapore, mathematical problem solving takes the central place in the national curriculum framework (See Figure 1 below). With some minor revision in 2001, and later in 2007, the framework of the current curriculum is presented as a pentagon model with five interdependent components – concepts, skills, processes, attitudes, and metacognition serving for mathematical problem solving.

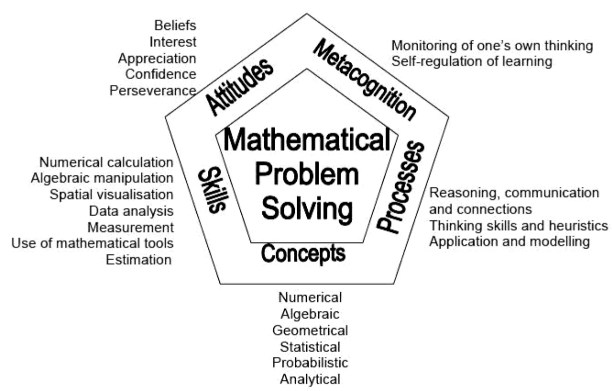


Figure 1: The pentagonal framework of the Singapore Mathematics Curriculum 2007

This framework sees problem solving as its overall goal and organizer rather than as one of the components of successful mathematics learning. Therefore, one can see that mathematical problems serve as both a means and an end in school mathematics instruction. Hence, it expects students to be competent problem solvers who are able to acquire and apply "mathematics concepts and skills in a wide range of situations, including non-routine, open-ended and real-world problems" (Curriculum Planning and Development Division, 2005, p. 6). Stacey (2005) noted that placing MPS at the centre of curriculum makes the Singapore's curriculum distinctly different from the mathematics curriculums in other countries such as those of the U.S., U.K., and Australia.

In line with this emphasis, the Ministry of Education (MOE) recommends a list of eleven heuristics for primary students and thirteen heuristics for secondary students to help them solve mathematical problems. The eleven heuristics recommended for Primary students are: act it out, use a diagram or model, make a systematic list, look for patterns, work backwards, use before-after concept, use guess and check, make suppositions, restate the problem in another way, simplify the problem, and solve part of the problem. For secondary students, the following two heuristics are added: look for similar problems and use equations. Among Singapore primary schools, the heuristic, 'draw a diagram', is widely used in Primary classroom teaching through a technique called "model method" from Primary two to six.

Developed by Kho, model method (Kho, 1987; Kho, Yeo, & Lim, 2009) is used in elementary schools as reasoning based on pictorial representations in contrast to the reasoning based on abstract representations emphasised from secondary schools onward, starting with solving algebraic equations in Secondary 1 (Seventh Grade). It should be noted that the term "model" as in "model method" must not be confused with a "mathematical model" as, for example, defined by Niss, "a combination of one or more mathematical entities and the relationships among them that are chosen to represent aspects of a real-world situation" (Zbiek, 1998, p. 184). Rather, it refers to "a structure comprised of rectangles and numerical values that represent all the information and relationships presented in a given problem" (Ng, 2004, p. 42). Figure 2 shows an example of a model to represent the information and relationships in a word problem in the Primary Five textbook.

> Mrs Lim made 300 tarts. She sold three quarters of them and gave one third of the remainder to her neighbour. How many tarts had she left? (CPDD, 1999)

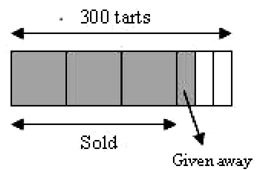


Figure 2: A model to represent the network of quantities and quantitative relationships

The rectangles are often called "units". Four big units represent 300 tarts, the total number of tarts made, three big shaded units represent the number of tarts sold, which is three quarters, and one unit represents the number of remaining tarts. This one quarter of tarts is further divided into three smaller equal units with the shaded one indicating the tarts given away and the two small unshaded units represent the remainder. From here, it could be seen that the model can be used to represent the situations and quantitative relationships given in *algebraic word problems* (Bednarz & Janvier, 1996) that involve unknowns.

Then, what constitutes mathematical problems? In general, a problem is viewed as "a situation in which a goal is to be attained and a direct route to the goal is blocked" (Kilpatrick, 1985, p. 2) and a problem can serve as a vehicle to various pedagogical purposes in mathematical learning. Such roadblocks pose challenge to students' mathematical understanding, which, if well used, is able to reveal students' misconceptions and facilitate the construction of *new* knowledge (Glasersfeld, 1995). Thus, overcoming these challenges is at the heart of students' learning *via* problem solving and the challenges can be studied in terms of how students reason about a problem and build its solution routes.

Given the widely recognized difficulty to investigate students thinking in problem-solving activities, various ways to study different dimensions of students' thinking in problem solving have been proposed and tried out to reveal students' exhibited behaviours, actions, and reasoning processes. For example, studying the mistakes and misconceptions surfaced by the students opens up a window to infer how students understand certain mathematical ideas (Confrey, 1991; Greer, Verschaffel, & De Corte, 2002; Ng & Lee, 2004; Thompson, 1995). Meanwhile, thinking aloud (Ericsson & Simon, 1993; Foong, 1993; Teong, 2000) and peer discussions in group settings (Artzt & Armour-Thomas, 1992; Goos et al., 2002; Schoenfeld, 1985) are other approaches to study students' thinking through analyzing their discourse and verbalized thoughts. . In this study, Thompson's *quantitative reasoning* is used to characterise students' challenges in reasoning quantitatively in solving word problems and how analysis of the peer discourse in student pair work provided the source of the challenges and how they were overcome.

Quantitative reasoning

Thompson (1988, 1993, & 1995) characterizes the kind of reasoning essential to mathematics learning as *quantitative reasoning*, an ability needed to develop higher-order thinking (Kieran, 2004; Schmittau & Morris, 2004; Smith & Thompson, 2007). Such reasoning is grounded in making sense of the problem situation as a network of *quantities* and their *quantitative relationships,* and using these relationships to establish the solution. Students' lack of ability in reasoning quantitatively is manifested in students' stronger tendency to work on arithmetic operations than *quantitative relationships*, and their limited beliefs and heuristics (e.g. Foong & Koay, 1997; Lampert, 1990; Nesher, 1980; Schoenfeld, 1985, 1992). This inability constitutes major challenges in solving word problems and often such challenges are difficult to spot by students when they work individually.

The following example shows how quantitative reasoning is applied in the course of solving an algebraic word problem on Fraction using model method that was presented earlier in Figure 2.

> Mrs Lim made 300 tarts. She sold three quarters of them and gave one third of the remainder to her neighbour. How many tarts had she left? (CPDD, 1999)

There are four quantities and three quantitative relationships involved in this problem situation. The quantities and their magnitudes are: 1) total # of tarts = 300, 2) # of tarts sold = unknown. 3) # of tarts given away = unknown, and 4) # of tarts left = unknown. The objective of solving this problem is to find the fourth quantity, which is the number of tarts left. The quantitative relationships are: 1) total # of tarts = # of tarts sold + # of tarts given away + # of tarts left, 2) # of tarts sold = ¼ × # of total tarts, and 3) # of tarts given away = 1/3 × (# of total tarts – # of tarts sold). The quantities and quantitative relationships can be represented in a model as shown in Figure 3:

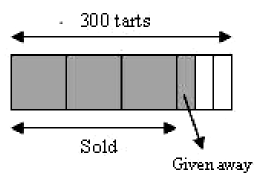
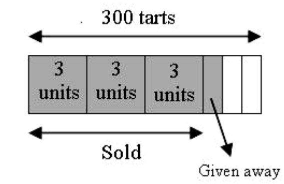


Figure 3: A model to represent the network of quantities and quantitative relationships

By representing in a model, the three quantitative relationships can be simplified and connected through proportional reasoning among the units of each quantity. For example, 1 unit of "# of tarts given away" is 1/3 of 1 unit of "# of tarts sold". Therefore, the series of quantitative operations to find # of tarts left is: 1) Partitioning the "# of tarts sold" into a common unit with "# of tarts given away", 2) find the magnitude of "# of tarts given away" by conceiving its proportionality with the total number of tarts (had been represented in common unit), and 3) find the magnitude of "# of tarts left" by conceiving its proportionality with the "# of tarts given away".

The quantitative operations can be executed in terms of the following series of pictorial equations as shown in Figure 4: 1) Thirding the "# of tarts sold", 2) 12 units → 300 tarts, hence 1 unit → 25 tarts which represents "# of tarts given away", and 3) 2 units → 2 × 25 = 50 which represent the number of tarts left.



12 units → 300 tarts
1 unit → 300 ÷ 12 = 25 tarts

The number of tarts left = 2 units → 2 × 25 = 50 tarts

Figure 4: Quantitative reasoning using model method

In other words, the quantitative reasoning from Figure 4 can be summarised as partitioning each unit of # of tarts to get a common unit with # of tarts given away, followed by proportional reasoning to determine # of tarts associated with 1 unit and multiplicative reasoning to determine the number of tarts left.

Note that in performing the above reasoning, the students must bear in mind what each rectangle represents and the relationship among the rectangles. In other words, in deriving the series of arithmetic equations that build the solution in Figure 4, their reasoning must be grounded in some *image* of quantities and quantitative relationships such that the reasoning consists of a series of quantitative operations that eventually leads to evaluating the magnitude of quantity that is intended to find. In this case, the model drawn serve as a visual *map* to record the *territory* of the whole network of quantities and quantitative relationships, and such a *map* assists them in building the required mental images in which the reasoning is based on.

Therefore, re-presenting the quantities and quantitative relationships is no less importance than analysing the quantities and quantitative relationships. In fact, re-presentation is a foundation to effectively reason quantitatively. Studies have shown that more students' mistakes were due to erroneous representation of the problem instead of to computational errors (De Corte, Verschaffel, & De Win, 1985; Ng & Lee, 2009). Hegarty, Mayer, & Monk (1995) also show that *good* problem solvers were different from the *poor* ones in terms of the quality of representation, both mentally and drawn, with regard to its ground in quantities and quantitative relationships. This assures us that the mental representations conceived by *good* problem solvers have strong basis in the network of quantities and quantitative relationships. This fact is significantly important in

Singapore mathematics educations since the model method, grounded in representations, is used in solving most of the word problems at upper elementary level in Singapore.

Collaborative work

In a collaborative setting, students' mathematical problem solving can be viewed as "a social and cultural phenomenon that is constituted by a community of actively cognizing individuals" (Wood, Cobb, & Yackel, 1995, p. 402). This community has a shared goal of successfully solving the problems. From sociocultural perspective, the collaborative learning process takes place in Vygotsky's notion of *zone of proximal development* (ZPD) defined as "a window of potential learning that lies between what he or she can manage to do unaided and what he or she can achieve with help" (Wells, 1999, p. 296). Vygotsky (1978) described that such a potential learning can be achieved through adult assistance or peer assistance. In collaborative problem solving settings, peer assistance can be termed as *collaborative zone of proximal development* (Goos et al., 2002, p. 196). This collaborative learning process entails the role of social means of meaning making and communication that take place in the course of solving problems and in individual students' intellectual development. Hence, the problem-solving space is mediated by resources that involve student thinking, dialogue, social interactions and tools employed, such as diagrams or various representations. In this context, a model that is used to represent quantities and quantitative relationships also serves as a means that mediates collaborative problem solving. Wells (1999) described three kinds of transformation of such resources throughout a problem-solving activity. "First, a transformation of the individual's intellectual functioning and of his or her capacity for effective participation in the activity; second, a transformation of the situation brought about by the participants' actions; and, third, a transformation of the tools and practices as they are creatively adapted to suit the particular situation and activity in which they are used" (p. 295). Therefore, how the group members interact in co-constructing solution pathways, as well as identifying the challenges that they faced and how they respond to overcome the challenges, can be studied in terms of these transformations and the evolution of meditational tools as manifested in the problem-solving discourse, behavior, and representations.

These transformations take place through various negotiations among the members. The outcome of the negotiations in the co-construction process is achieved as a result of building intersubjective knowledge where "the individuals reach a state of mutual agreement about the meaning of the results of their interactions" (Steffe & Thompson, 2000, p. 193). Therefore, the negotiation segments in which intersubjectivity is reached can become a potential unit of analysis to reveal the successes as well as the challenges encountered by each member in solving the problems.

However, in a natural group setting, students are found to be less likely to effectively engage with peers in solving problems without a carefully built classroom culture. For example, Artzt & Armour-Thomas (1992) showed two common scenarios of unfavourable collaboration: one is called 'one-man-show' in which a predominant group member makes most decisions without sufficiently negotiating with other group members; the other is working independently in group situations where the members are disengaged from collaboration and focused on individual problem solving. The two scenarios are not conducive to study enabling and disenabling factors of peer interaction in collaborative problem solving process. Thus, to induce students' high-engagement with peers, additional measure has to be taken such as implementing thinking-aloud protocol (Ericsson & Simon, 1993) to encourage every group member to verbalize their thoughts and hence increase the chance of generating negotiation with other members. Nevertheless, the thinking aloud instruction used by the researchers from time to time may also interfere students' cognitive processes and hence needs to be minimized. One possible way to minimize it is to reduce the group size, such as the pair-work settings employed by Schoenfeld (1985, 1992). In Schoenfeld's studies, by working together in a pair as the *smallest* unit of group work, the undergraduate students were able to engage in effective interactions throughout the course of problem solving activities. Considering students' age group, that is, 11-12 years old for Fifth Graders and 13-14 years old for Seventh Graders, who might be less mature than undergraduate students to work effectively and collaboratively in pairs, we combined the thinking-aloud procedure (Ericsson & Simon, 1993) with Schoenfeld's (1985) pair-work setting to make it conducive for students to clarify ideas and assumptions, discuss each other's interpretations, and make collaborative decisions.

Methods and Data

As part of a larger research project, we conducted intensive observations on 7 Fifth Grade and 8 Seventh Grade student pairs to understand their challenges in problem solving, the negotiation processes in dealing with these challenges, and compare and contrast their progression and the lack of it. Using thinking-aloud approach, all student pairs were asked to solve the same set of five mathematical problems. For the purpose of this study, we chose to focus our analysis on two of the tasks, the Tourist task and the Marriage task.

Data sources

The seven Grade 5 pairs from two primary schools (Primary A and B) and eight Grade 7 pairs from two secondary schools (Secondary D and E) became the data sources for this study. The selection of the schools followed the larger project in which the sampling covered schools from a wide range of banding – upper, middle, and lower band. Primary A and Secondary E belong to upper band, while Primary B and Secondary D belong to the middle band. Table 1 shows some details of the fifteen pairs (pseudonyms) who participated in this study.

Table 1: The seven primary pairs and eight secondary pairs in the study

No	**Primary Pairs (PP)**	**Schools**
PP1	Francisca and Zoe	Primary A
PP2	Gerry and Kelvin	Primary A
PP3	Jack and Ester	Primary A
PP4	Abraham and Robin	Primary B
PP5	Joanne and Yvonne	Primary B
PP6	Billy and Leo	Primary B
PP7	Johnson and Chad	Primary B
No	**Secondary Pairs (SP)**	**Schools**
SP1	Jerry and Richard	Secondary D
SP2	Christine and Mary	Secondary D

SP3	David and Ray	Secondary D
SP4	George and Julie	Secondary D
SP5	Nelly and Susan	Secondary E
SP6	James and William	Secondary E
SP7	Ron and Neville	Secondary E
SP8	Alvin and Samuel	Secondary E

In Table 1, Fifth Grade pairs were indexed as PP and Seventh Grade pairs as SP. The student pairs were listed according to the order of the schools according to the period of observation (Primary A to B, Secondary D to E) and the pairs who came from the same class were clustered together.

Task

The two tasks that become the focus of our analysis are presented below:

> The Tourist task: A group of tourists paid $200 for admission to a theme park. Adults paid $8 each and children $4 each. If there were 7 more adults than children, how many adults and children were there in the group?
>
> The Marriage task: In a certain town, two-thirds of the adult men are married to three-fifths of the adult women. What fraction of the adults in the town are married?

Table 2 presents a simple task analysis of the two word problems that include the lists all the quantities – both known and unknown quantities –, all the quantitative relationships, and all possible problem-solving heuristics (according to the MOE's list of heuristics) that Singaporean students might use to solve each of the two problems.

Table 2: The list of quantities, quantitative relationships, and possible problem-solving heuristics in the two items

Tourist item	Marriage item
A group of tourists paid $200 for admission to a theme park. Adults paid $8 each and children $4 each. If there were 7 more adults than children, how many adults and children were there in the group?	*In a certain town, two-thirds of the adult men are married to three-fifths of the adult women. What fraction of the adults in the town are married?*
Known quantities: 1) Amount paid by each adult = $8 2) Amount paid by each child = $4 3) Amount paid by all tourists = $200 Unknown quantities: 1) # of adults 2) # of children Quantitative relationships: 1) # of adults = # of children + 7 2) # of tourists = # of adults + # of children 3) Amount paid by all adults = # of adults × $8 4) Amount paid by all children = # of adults × $4 5) Amount paid by all tourists = Amount paid by all adults + Amount paid by all children.	Known quantities: Nil. Unknown quantities: 1) # of men 2) # of married men 3) # of women 4) # of married women 5) # of adults 6) # of married adults. Quantitative relationships: 1) # of married men = $\frac{2}{3}$ # of men 2) # of married women = $\frac{3}{5}$ # of women 3) # of married men = # of married women 4) # of married adults = # of married men + # of married women

	5) # of adults = # of men + # of women Hidden quantitative relationship: The relationship between # of men and # of women.
Possible heuristics: 1) Draw a model 2) Guess and check 3) Use an equation	Possible heuristics: 1) Draw a model 2) Guess and check 3) Put in a list 4) Make a supposition 5) Use an equation

Table 2 shows that the Marriage task is more complex than the Tourist task both conceptually and quantitatively. The Tourist task deals witha Whole Number topic in which students would generally have acquired the necessary knowledge to solve such word problems by Grade 4 in Singaporean schools. On the other hand, the Marriage task deals with Fractions which is one of the most difficult topics in elementary schools. As compared to the Whole Number concepts in the Tourist task, we could expect students to have more challenges in dealing with Fraction concepts. Quantitatively, the Marriage task has higher complexities than the Tourist tasks. First of all, the magnitudes of all quantities and all the quantitative relationships in the Tourist task are made explicit. There are five quantities with three known and two unknown quantities. Using guess and check may be the *easiest* choice as the guesses are the magnitudes of quantities to be found with # of children and # of adults although children may find it tedious. Model method might not be one-and-all strategy to solve this item since the solution does not quite *fit* into the steps in 'model method' which involves proportional reasoning among the units. However, it may serve as a powerful aid to represent the network of quantities. The Marriage task involves six quantities – men, married men, women, married women, adults, married adults – and six quantitative relationships. Unlike the Tourist task, in this task all quantities are unknown quantities and one quantitative relationship is hidden, which is the relationship between the number of men and women.

Analysis

We adopted an interpretive approach in examining the pair discourse and actions as well as the mediational means and artefacts in their overcoming the challenges. Video methods (Ratcliff, 2004) were used to *capture* student' pairs' problem solving processes in ways that can be viewed and analysed repeatedly for thorough interpretations. Constant comparative method (Glaser & Strauss, 1967) was used across all the student pairs to compare and contrast their performances in each of the two word problems, and generate the cases in which the pair interactions fostered or hindered students' overcoming their challenges. The video analysis software *StudioCode 2.5.45* was used to organise the data sources and support thorough analysis within and across the student pairs. The software is especially capable in linking the transcriptions with the video-clips' timeline for thorough analysis on students' actions and discourse, coding significant instances in the clips and transcripts, and comparing and contrasting the instances across all student pairs by extracting those instances in a single movie.

Results

Our analysis identified four major cases of students' successful outcome when the pairs collaborated in ways that enabled them to 1) reduce their cognitive workload of having to keep in mind all quantities and quantitative relationships during problem solving; 2) co-construct knowledge when a *weaker* and a *stronger* student build upon each other's ideas; 3) re-orient the pair work focus onto quantitative relationships, thus leading to effective quantitative reasoning; and 4) utilize or develop students' positive attitudes toward problem solving as a conducive environment for the other factors to surface. In all the above cases, crucial communication features, such as the willingness to understand each other's ideas and the ability to communicate one's reasoning, are also identified. All four cases show important aspects of learning in peer's assisted zone of proximal development (Goos et al., 2002) that could not be found in individual work settings.

Case 1: Sharing of cognitive workload of having to keep in mind all quantities and quantitative relationships during problem solving

In solving word problems, students often have challenges in keeping in mind all quantities and quantitative relationships in solving a problem (Lioe, 2009; Thompson, 1995). This ability is crucial to assist students' effectiveness in reasoning quantitatively. In the following discussion, a Fifth Grade pair, PP2 (Gerry and Kelvin), shows the potential of pair interaction that fostered effective quantitative reasoning when one of the members had challenges in keeping in mind those quantities and quantitative relationships throughout the course of their reasoning. In contrast, another Fifth Grade pair, PP5 (Joanne & Yvonne), shows the failure to overcome such a challenge when both members were hindered by this difficulty.

Sharing cognitive workload in drawing a model

PP2 (Gerry & Kelvin) when they solved the Marriage task shows a case of overcoming challenges by sharing cognitive workload. The solution began with the mediation of a model that the pair used to help them represent and understand the quantities and quantitative relationships in the task. However, the challenges occurred in representing the quantitative relationship itself, from the text to the model. Gerry stored *wrong* quantitative relationship in his mind when he unpacked the quantitative relationships from the text. He then translated the *wrong* quantitative relationship in his drawn model, as shown in the figure below.

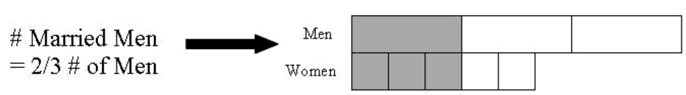


Figure 5: Gerry's (PP2) mistake by perceiving wrong quantitative relationship

Figure 5 shows that instead of conceiving "# of married men = 2/3 # of men", he perceived it as "# of married men = 1/3 # of men" (by drawing 1/3 of # of men to be equal to 3/5 of # of women). Kelvin noticed Gerry's mistake and responded to it by taking over the work of model drawing from Gerry. Kelvin then fixed the model into the one that contains the desired quantitative relationship as shown in the right-hand model of Figure 6.

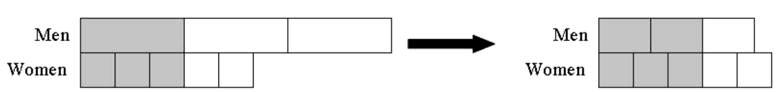


Figure 6: Amending the model by Kelvin from PP2

Let us take a closer look at this part of their exchange (Turn 19-23).

Turn	Speaker	Utterances
19.	Kelvin	((assessing the model)) Married…. 2/3… 2/3 of the men… hah? Oops. ((Erased some parts of the model and about rectified it))
20.	Gerry	((Kelvin about to amend)) The thing is equal. ((refers to the same number of married men and married women))
21.	Kelvin	Ah?
22.	Gerry	Equal. Should be equal.
23.	Kelvin	((ignored Gerry and continued drawing)) 2/3 of the adult men are married.

Gerry did not object Kelvin's amendment which shows that he could have realised the *wrong* quantitative relationship that he earlier conceived. On top of that, Gerry's reaction to Kelvin's attempt to amend the model in Turn 20 was reminding Kelvin of the equal number of married men and married women. From Kelvin's response to Gerry's reminder in Turn 21 and 23, it was not clear whether Kelvin remembered the quantitative relationship alluded by Gerry. However, Gerry's reminder clearly shows the benefit of sharing cognitive workload. If Kelvin remembered this quantitative relationship, Gerry's reminder served as reinforcement to his mental image to be translated in the model. If not, Gerry's reminder served as filling up the *hole*, in case this quantitative relationship is "missing" in Kelvin's mind.

The interaction between Gerry and Kelvin thus shows that sharing cognitive workload is helpful in building upon each other a solid representation of the network of quantities and quantitative relationships, and the drawn models mediated the process of understanding and the pairs' building upon each other's mental representation of the quantitative relationships. This benefit cannot be found in the case of students work individually. For example, if Gerry worked on his own, there was a chance that he ended up reasoning with a *wrong* model that reflected *wrong* mental representation of some quantitative relationships. Although he might still be able to spot this mistake and clarify his own mental representation, the cognitive demand of having to do it himself was higher than that of sharing cognitive workload with a peer.

Failure in sharing cognitive workload – hindrances to overcome challenges related to mental representation

As shown in the successful case, working collaboratively in pairs helped reduce student's cognitive workload in keeping in mind all the quantities and quantitative relationships in the course of their reasoning. In particular, the role of the drawn model mediated both students in checking each other's conception of quantitative relationships in their mind. In contrary, for those partners whose interactions were unproductive, they failed to enjoy such benefit. For instance, when both students were overtly driven by *visually-perceived relationships* from the model (instead of quantities and quantitative relationships), they were doomed to fail. Joanne & Yvonne (PP5), in the Tourist task, is an illustrative example.

Joanne first drew a model to mediate their co-construction of the network of quantities and quantitative relationships representation, and based upon it to obtain the amount paid by all pairs, by calculating “\$8 × 7 = \$56”, followed by subtracting it from the amount paid by all tourists (\$200), and divided the result (\$144) by two through referring the divisor to 2 units. Figure 7 shows the above reasoning.

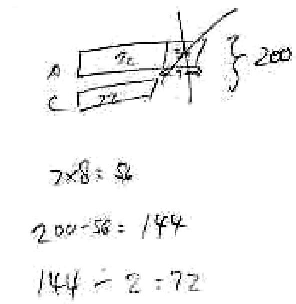


Figure 7: PP5 (Joanne & Yvonne) who were distracted by visually-perceived relationship of “two units” that led them to divide “144 by 2” to evaluate the value for each unit

Note that prior to performing “144 ÷ 2”, they were successfully mediated by the model in getting the total amount paid by the same number of adults and children (\$144). However, when Joanne assigned the value 72 (result of “144 ÷ 2”) to each unknown unit in the model, it shows that she was *distracted* by the relationships that she perceived visually from the model and had focus on evaluating the two unknown units. Since Yvonne shares the same perception, this visually-perceived relationship disenabled them from proceeding to further steps in the model method reasoning. This has led them to switch the strategy to guess-and-check approach, in order to find the number of adults and children that constitute the total amount \$144.

In performing guess and check, the students used a table as a meditational mean in sequencing a series of guessing the number of adults and children and checking against the total amount \$144. The table consists of four columns in the following order: number of pairs of adults and children (labelled as “No.”), the amount paid by children (labelled as “No C.”), the amount paid by adults (labelled as “No A”), and the total amount paid (labelled by “T”). Figure 8 shows their working on the first three guesses: 30 pairs, 20 pairs, and 10 pairs shown in each row respectively.

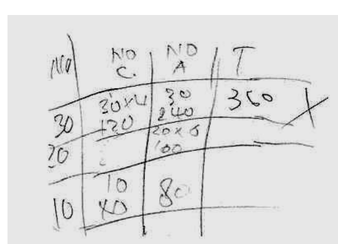


Figure 8: The first part of the guess-and-check procedure done by the PP5

Note that behind these three guesses, both students had conceived “1 pair = 1 adult + 1 child”. However, they had not found the desired guesses yet. This made them lose focus in making sense of a guess. They no longer kept in mind that “# of adults = # of children” in the pairs, instead they shifted to getting the right combination of numbers to make up the correct guesses. Their exchange from Turn 114 – 124 below shows the interaction when they managed to *hit upon* the right combination

Turns	**Speaker**	**Utterances**
114.	Yvonne	HEY!! I got the answer.
115.	Joanne	What?
116.	Yvonne	If ah… here, right? Got 10. 10 times 4, will be?
117.	Joanne	40
118.	Yvonne	Then, here got 13. 13 times 8, will be? 104. 104 plus 40?
119.	Joanne	Okay. Okay, but, let's see first, huh.
120.	Yvonne	It's correct.
121.	Joanne	So... you say that the number of children is ... 10, and the number of adults is...
122.	Yvonne	13
123.	Joanne	13. Okay, so, this one, how many? 40. This one? 104. Add up, 144.
124.	Yvonne	Yea, we got it, right? How many adults, there correct, *lah* you. So, will be

From Turn 114 – 118, Yvonne managed to strike the *right* combination "10 children and 13 adults" to make up the total cost $144. Recall that a child paid $4 and an adult paid $8, which led Yvonne to calculate "10 × 4" and "13 × 8". In this interaction, both students were equally happy in obtaining the right combination and none of them realised that they had ignored the quantitative relationship "1 pair = 1 adult + 1 child" and that they were guessing the number of pairs. Since neither of the students focused on the quantity to be guessed (# of pairs) and quantitative relationship (1 pair = 1 adult + 1 child), they failed altogether in sharing cognitive the workload to keep the network of quantities and quantitative relationships in mind.

The interactions that lead to successful sharing of cognitive workload in PP2 and the case of failure in PP5 suggest that examining peer member's images of the network of quantities is essential for the other pair member to spot and point out any "missing quantitative relationships" or "wrong quantitative relationships" as shown in the case of PP2 (Gerry & Kelvin). Mediated by verbalised statements and visual representations such as models, student pairs could be enabled to examine their peers' images of quantities and quantitative relationships. This is a metacognitive skill that students need to acquire. One way to develop this skill is by assigning specific roles such as Thinker & Listener (Whimbey & Lochhead, 1999) where the Thinker's role is to solve the task and think aloud and the Listener's role is to continuously demand the Thinker's verbalisation, understand the Thinker's reasoning, seek for clarification, and point out the mistake or loophole in the Thinker's reasoning. My small experiment (Lioe, Ho, & Hedberg, 2005) suggests that fifth graders could adapt such roles if they were given sufficient time and space for training for effective role play.

Case 2: Knowledge co-construction – overcoming challenges related to mathematical understanding

A seventh grade student pair, SP5 consists of Susan (the *stronger* student) and Nelly (the *weaker* student), can be an illustrative example of such successful knowledge co-

construction. Figure 9 shows the worksheet with their written solution to the Marriage task.

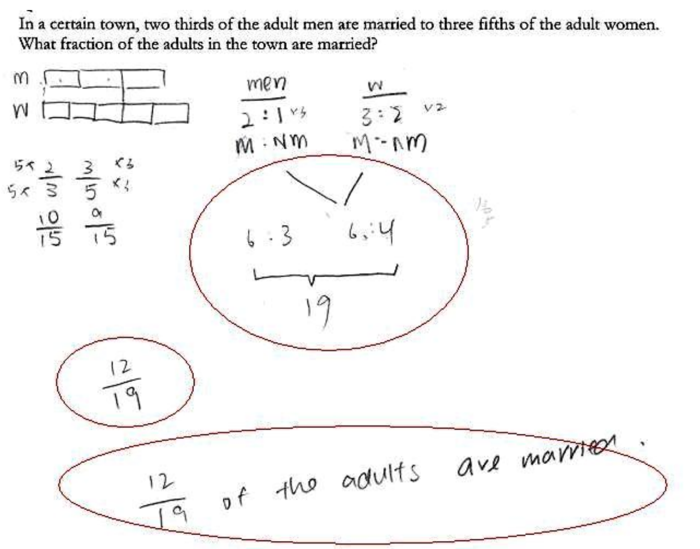


Figure 9: Susan & Nelly's solution to the Marriage task

The solution began with the mediation of a model (the top-left corner) that the pair used to help them represent and understand the quantities and quantitative relationships in the task. However, the visual nature of this model also disenabled them to draw quantitative operations, shown by their numerical exploration with the fractions "2/3" and "3/5" below the model. This led Susan, who had *strong* knowledge of ratio, to shift the strategy to using ratio (found at the right side of the model) and her reasoning is circled in red. Nelly who lacked understanding of ratio could not understand Susan's reasoning and demanded her explanation. See the following discourse:

Turn	Speaker	Utterances
44.	Nelly:	Can add altogether huh? ((refers to "6 : 3 6 : 4"))
45.	Susan:	They are the same units already what.
46.	Nelly:	Yea.
47.	Susan:	Yea. They are the same units so you can add them together what.
48.	Nelly:	4 over *n*? ((pause)) Then is like... how many fraction of total?
49.	Susan:	Adults in town
50.	Nelly:	12, this is 7 ((added up "3" in "6 : 3" and "4" in "6 : 4")). 7 ...
51.	Susan:	Because this one ((6 in "6 : 3")) is married this ((6 in "6 : 4")) is married. So we can just add them, simply add them together.
52.	Nelly:	So it's 12... 6, 12. 12 over ...
53.	Susan:	No you should get the total, plus total units. 12, 4, 16, 16 plus 3, 19
54.	Nelly:	So 12 over 19
55.	Susan:	Yea

In the above segment, Susan built her understanding on Nelly's questions and Nelly willingly listened, tried to understand, and implemented Susan's explanation that eventually led her to understand Susan's reasoning. The construct of unit from the model method (used three times in Turn 45, 47, and 53) became the mediation tool for them to establish the common language for understanding ratio. This dynamic process brought a *weaker* student to a higher-level understanding of the mathematical idea involved and at the same time also benefited the *stronger* student from understanding her peer's questions and helping her to find the answer. This short interaction segment described a rich interaction in co-constructing knowledge by building upon each other's understanding.

In addition, when the above crucial communication features in the pair dynamics were absent, the pair interactions were found to hinder students' overcoming challenges. For example, if Susan's understanding of ratio was not strong, the co-construction process would less likely be fruitful. Or, if the *stronger* student (and sometime the *weaker* student) overly dominated the decision making, the challenges might not be overcome and the stronger student's reasoning would diminish as well. These hindrances are illustrated in the following failure case.

Failure in the knowledge co-construction – hindrances to overcome challenges related to the lack of mathematical understanding

Contrary to the case of Susan and Nelly, if both students were equally *weak*, they were more likely to fail the task. Furthermore, the pair interaction might also lead them to further *misunderstanding* of mathematical ideas. The case of a fifth grade pair, PP1 (Francisca and Zoe), when they solved the Marriage task illustrates this type of interaction.

In the beginning, their progression was similar to that of Susan and Nelly. The solution began with the mediation of a *model* that the pair used to help them represent and understand the quantities and quantitative relationships in the task. They also faced the same challenges where the visual nature of this model also disenabled them to draw quantitative operations. Unlike Susan & Nelly who abandoned their *insensible* numerical exploration, this pair added the two fractions, 2/3 + 3/5, which resulted in an insensible answer, which is a fraction bigger than a whole. By doing so, clearly both students did not carry sufficient knowledge of fraction. Nevertheless, Francisca had a strong sense that the calculation was insensible, and hence rejected the calculation, and tried to hold on to partitioning the model. Francisca eventually "managed" to partition the model by relying solely on her visually-perceived relationships. Figure 10 shows her attempt to partition the model (on the left hand side) into the one having common units (on the right hand side).

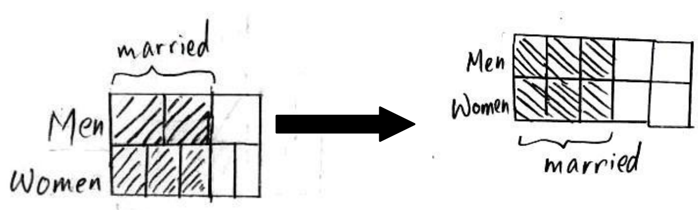


Figure 10: Francisca's way to represent "men" and "women" having the same unit size.

In partitioning the model, Francisca conceived an idea that any two groups of identical amounts, married couples (shaded parts) and all couples (equal number of men and women), could be partitioned into the same number of fractional parts. In implementing this operation, Francisca used the number of fractional parts of women population to represent the men population. Zoe did not agree with this partitioning. She had some reason that the men could not be represented as five fractional parts in which three of them constituted married men. She then asked for a justification from Francisca and they went through a negotiation the following excerpt (Turn 173 – 183).

Turn	**Speaker**	**Utterances**
173.	Zoe	But that means you are saying this one ((number of fractional parts of married men)) is also 3 and this one ((number of fractional parts of married women)) is 3.
174.	Francisca	Uh-huh. Can or not?
175.	Zoe	Okay, then 2/3, how to over over over 5?
176.	Francisca	Huh?
177.	Zoe	((laughs))
178.	Francisca	That means …
179.	Zoe	10/15 cannot divide by 3, what?
180.	Francisca	3/5 lor. Because it's the same one, what! You see this, this 3…((pointed at the married women then the married men))
181.	Zoe	Yea hor!
182.	Francisca	Yea ah!
183.	Zoe	So 3/5. Three. This one also 3. And then this one married *lah*. That means my method wrong *lah*.

In Turns 173 – 179, Zoe addressed her concern that the fraction 2/3 (or 10/15) could not be converted to 3/5 since they were not equivalent fractions. Zoe's reason was mathematically correct and had a potential to challenge Francisca's *logic* behind her partitioning. However, in Turn 180, Francisca managed to *justify* her visually-based reasoning to Zoe. Turn 181 – 183 shows that Zoe was convinced with this reasoning and rejected the counter argument that she initially had.

Zoe could be convinced by Francisca's reasoning since she might not have other resources for counter arguing Francisca's reasoning other than procedural correctness of equivalent fractions. In other words, Zoe did not counter argue against Francisca from a quantitative orientation. Therefore, when Francisca acted out how the two amounts were *visually* identical, this argument seemed to make sense to Zoe. Since there was no further reason for Zoe to disagree with Francisca, she then accepted the partitioning and rejected her own idea.

Thus the negotiation that took place between Zoe and Francisca was no different from the metaphor, "the blind leading the blind". Out of this interaction, Zoe might construct (mis)understanding of a particular way of partitioning. On the one hand, this could be seen as a negative learning effect of working collaboratively with a peer member who is equally *weak*. On the other hand, this might reflect the reality in the daily practices in which students' (mis)understanding might influence their peers' (mis)construction of knowledge. From the radical constructivist perspective, such (mis)construction might not be *bad* either if it is regarded as part of students' construction of knowledge. The challenge for teachers is then how to become aware of a student's progress and assist to clarify the *loophole* in his or her reasoning at the right time. As shown in PP1's case, by observing the two students who were *weak* in a certain area of knowledge, a teacher could understand the students' *logic* behind their reasoning and acts to respond to such *logic*. This might open up a window which serves to clarify which part of the *logic* could be clarified so students' development process could be mediated to a higher level.

Case 3: Re-orienting the pair work focus onto quantitative relationships

One of the common challenges in reasoning quantitatively is the shift of focus from quantities and quantitative relationships to numbers and numerical relationships (Lioe, 2009; Thompson, 1995). Such a displacement influence students' loss of sense making of the numerical operations performed, their numerical results, and the representation of problem situations. While such tendency in the course of reasoning is prevalent among students (Foong & Koay, 1997; Nesher, 1980; Verschaffel, Greer, & De Corte, 2000), we are interested to unravel how pair interactions can contribute to overcoming this challenge or hinder the process. We shall discuss a successful case of PP5 (Joanne and Yvonne) and the failure case of PP6 (Billy and Leo) in Marriage task.

Externalising mentally-perceived relationships

PP5 (Joanne & Yvonne) started their solution with a *wrong* model, as a meditational mean of their subsequent reasoning, where both men and women were represented in the same unit size (see the following figure), which was inconsistent with monogamous marriage assumption.

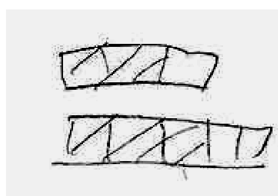


Figure 11: Joanne's (PP5) first drawn model in the Marriage task

By drawing a *wrong* model, it was inevitable that all kind of reasoning drawn from this meditational tool led to a *wrong* answer, unless the mistakes were spotted and fixed. However, for the purpose of this study, this meditational tool is valuable since it reflects the student's conception of the quantities and quantitative relationships in the problem situation. Furthermore, the wrong answer drawn from the wrong model can also be "mathematically correct" if the procedure being implemented was consistent with the model. In this case, Joanne made further mistake since her reasoning followed on the model was disconnected with the quantitative relationships being represented in the *wrong* model. Figure 12 shows that Joanne deduced the fraction of adults who were married by taking part-whole ratio between the number of shaded units and the number of the total units. However, the part-whole ratio was not taken directly from the number of units in the drawn model. Instead, Joanne conceived a certain mental image (see the model in the middle) and drew her reasoning from the mental image.

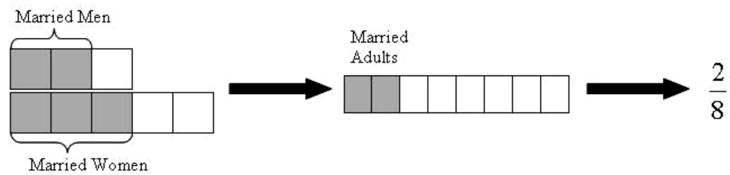


Figure 12: Inconsistent quantitative relationships conceived by PP5

It was clear that Joanne's mental image was disconnected from the drawn model. There were five shaded units (constituted by two units of men and three units of women) in the drawn model, while there were only two shaded units in the mental model. Joanne's result "2/8" shows that she conceived "2 units of married adults" out of the total 8 units of adults. In other words, she took the number of married men for the number of married couples and disregarded the number of married women which indicated her lack of understanding in differentiating the number of "couples" and the number of "people".

In the following excerpt, we shall see how Yvonne responded to Joanne's reasoning and helped Joanne to spot and fix the disconnection between her mental image and the drawn model.

Turn	Speaker	Utterances
23.	Yvonne	Wait, are you sure? … Never ask you eh, adults means adult men and adult women leh!
24.	Joanne	Ya. Adult men and adult women lah.
25.	Yvonne	Ya. Then how the… how the adults be 1/4? 1/4 … 2/8 is only the… the women, no… the men.

In the excerpt above, Yvonne noted the disconnection between 2/8 and the model drawn. Yvonne managed to relate the number "1/4" with the quantitative operation that 1/4 was the fraction that described married men of the total adult population. She attempted to point out the inconsistency of quantitative relationships to Joanne. Joanne responded by drawing another model to justify her reasoning. See the figure below.

Figure 13: Joanne's (PP5) second drawn model in the Marriage task

The following excerpt shows Joanne's justification while at the same time drawing the model above.

Turn	Speaker	Utterances
26.	Joanne	Okay, let's make it like this lah ((starts drawing)). Make it into 8 boxes. ((finished drawing)) 1, 2, 3, 4, 5, 6, 7, 8. So ah? 2 of the, of the … that one the adult men are married. Then erm… three fifth of the adult women are married also with these 2, right? So ah? Then if like this, it's 5/8 is married. Then ah, the adult, the adults, oh yea ah! 5/8 ah. So, the answer is 5/8, I think.

Joanne's externalization of her mental model offered opportunity for her to examine the connection between quantitative relationships represented in the first model and the quantitative relationships conceived in her mental model. In other words, in justifying her reasoning, Joanne had re-oriented herself to the quantitative orientation and managed to overcome her inconsistency. In the end of her statement, Joanne replaced her answer 2/8 with 5/8, which was consistent with the first drawn model.

This case illustrates the importance of externalizing the mental representation, especially when the primary students were still at the developmental stage of pictorial-based reasoning so that any *loophole* in the mental representation can be checked and rectified against the network of quantities. In this case, pair interaction is shown to be useful and effective in mediating such externalisation. Yvonne's attempt to seek for clarification triggered the whole re-examination process, which prevented Joanne from making another mistake. Should Joanne work individually, there would have been a high chance that this mistake could be left unnoticed given her strong tendency to conclude the first result as the final answer.

Failure in clarifying the base of quantitative operations – hindrances in overcoming challenges related to students' orientation

The case of PP6 (Billy & Leo) in the Marriage task illustrates how the attempt of externalising a mentally-perceived relationship was hindered by the *weaker* student. Similar to the starting model drawn by Joanne from PP5, Leo drew the *wrong* model as shown in Figure 14.

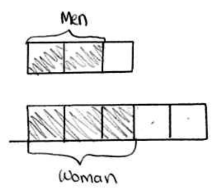


Figure 14: The model drawn by Leo from Primary Pair 6.

Unlike in the case of PP5 where the mistake in drawing this model was not spotted, in the case of PP6, Billy spotted the mistake but he failed to communicate this mistake to Leo and hence his attempt to modify the model was hindered. First, we see Leo's reaction to his model upon completion drawing the model that shows he was hindered by his visually-perceived relationship as shown in his statements below.

Turn	Speaker	Utterances
36.	Leo	Then how? That means there are concubines.
37.	Billy	((laughs))
38.	Leo	If you take 2 plus 3 equals to 5, *wah lau.*

From Leo's statement, he believed more on visually-perceived relationships that "# of married women (3 units) > # of married men (2 units)" than on the quantitative relationship: "# of married men = # of married women". On the other hand, Billy still kept this quantitative relationship in his consideration and addressed this issue to Leo as shown in the following excerpt.

Turn	Speaker	Utterances
41.	Billy	2/3 of the adult men are married to 3/5. That means 2/3 of the adult men is equal

		to 3/5 of the adult women.
42.	Leo	2/3 equals 3/5?
43.	Billy	Yea.
44.	Leo	So, write down there. ((about to start writing))
45.	Billy	No. Must change the model. Because if it is 2/3 to 3/5 that means 2....
46.	Leo	1 man have more er... 2 more... er…
47.	Billy	No wait! That means 2/3 hor, that means 2 parts equals to 3 parts of the women.
48.	Leo	Okay, so 2/3... ((starts writing))
49.	Billy	Equals
50.	Leo	is the same ... which one is bigger?
51.	Billy	As...
52.	Leo	As
53.	Billy	the 3/5 of the adult women
54.	Leo	3/5... 3/5. ((finished writing))
55.	Billy	Yea.

Figure 15 shows the statement written by Leo in the above conversation.

Figure 15: Leo's (PP6) way of representing the equal number of married men and married women

It could be seen that Billy's intention was sounding to Leo that the drawn model had to be changed to reflect the equal number of married men and married women. In Turn 45 and 47, Billy in fact stated clearly that the "2 parts of men" had to be drawn in the same length as "3 parts of women". However, instead of changing the model, Leo expressed the quantitative relationship "# of married men = # of married women" in a sentence (in Figure 15) which did not help much in assisting their reasoning. In this case, the sentence became the next meditational tool to represent this crucial quantitative relationship. Instead of being supported, Billy's attention was distracted by this sentence. He did not disagree with this statement as he also perceived this statement as the right representation of the quantitative relationship that he mentioned, but it did not help them change the model to reflect the *correct* network of quantities as he intended earlier. Therefore, Billy's attempt to bridge Leo's visually-perceived relationships to the quantitative relationships failed.

Case 4: Utilize or develop students' positive attitudes toward problem solving as a conducive environment for the other factors to surface

The previous three cases show that the cognitive and metacognitive aspects of pair dynamics become the enabling or disenabling factors in the success of overcoming challenges. In this last case, we found that the affective aspects from one student in a pair were also shown to be the enabling and disenabling factors. The case of SP6 (James and William) in the Marriage task shows that one student's persistence in examining quantitative relationships is more likely to lead the other one to re-orient him- or herself onto quantitative relationships. Such influence is useful to strengthen the foundations for reasoning quantitatively for both students in the pair work.

Similar to the previous three cases, James and William used model method, with the *correct* model serves as meditational tool, to solve the Marriage task (see Figure 16).

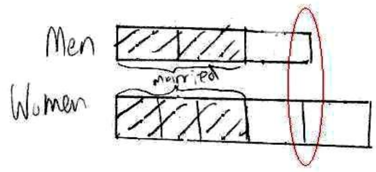


Figure 16: The model drawn by the Secondary Pair 6.

They encountered the same difficulty as the other student pairs, i.e. the inability to partition the model, and they were also hindered by visually-perceived relationships by perceiving the same unit size of “the single man” and “the single woman” as circled in red. This led them to explored various other ways, including trying out different representations and using nonsensical calculations, to overcome their roadblock, resulting in spending significantly longer time (about 18.5 minutes) to solve this problem as compared to other pairs (on average spent about 7 minutes). Eventually, the model became supportive meditational tool for them to partition and overcome the road block (with some minor mistake). Thus this case makes an outstanding illustration of how students’ persistence can become a single important factor for success, which helps to bring the required mathematical understanding to surface and eventually overcome the difficulty.

An example of nonsensical calculation that they performed is shown in Figure 17. At this stage, they have decided to “give up” on this problem and simply write any numerical procedures for the sake of completing the task.

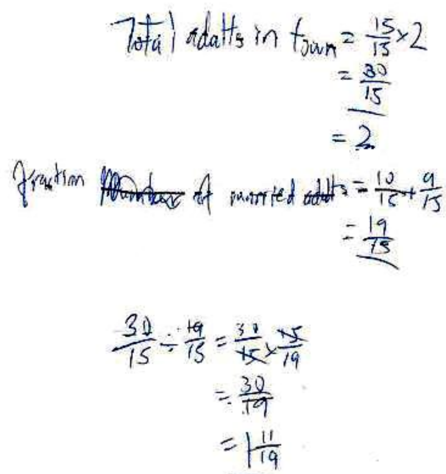


Figure 17: The calculation performed by the Secondary Pair 8

Despite their purpose of writing a *wrong* solution, William’s strong orientation of quantities *saved* them from the ‘trap’ and in fact this moment became the turning point for them to overcome their challenges. William did not *see* any *sound* image of relationship in the third calculation, “$\frac{30}{15} \div \frac{19}{15}$”.

Turn	Speaker	Utterances
173.	William	30 over 9 lah...What you doing!!! What you doing? ((laughs))
174.	James	((laughs)) What am I doing?? Cancel....... ((cancelling all the calculation)) This question hor... later check with Ms Lee ((their Maths teacher). Should be all wrong. All wrong right?
175.	William	Quick lah … 20 minutes. Wrong already lah like that.
176.	James	Our aim, now our aim is to get a wrong answer and quickly go back.

Although they wanted to simply write a procedure for the sake of completing the task, they still wanted the procedure to make some sense. This then led them back to examine the quantitative relationships in the problem situation.

Turn	Speaker	Utterances
177.	William	Okay the adults in town. Total adults in town.... Aiyah...
178.	James	This kind I do before leh.
179.	William	Ya.
180.	James	Remember this number eh.
181.	William	Then....then.... then... What 2? *Siao*! 2/3.
182.	James	That means there's more than. That means there's more men than women but that does matter. Because means the men and women not equal so the fraction, can't be 30 over 15
183.	William	Yea lah. 2/3 of the adult men
184.	James	If 1/3 is equal to 2/5
185.	William	Wah....
186.	James	Miss out. Eh no 1/3 is not equal 2/5 for, 1/3 men not equal 2/5 women
187.	William	1/3 of men is ...
188.	James	But 2/3 men is...
189.	William	3/10 of the women
190.	James	3/10. so it's
191.	William	Oh, if we find number of men is equal to 9/10 of the women, that means
192.	James	Eh? If we change this 6 units here ((looks at the model))
193.	William	How to change?

Their persistence to focus on quantitative orientation was paid off. In Turn 192, James *suddenly* conceived the idea of finding the common unit when he looked at the model. He conceived the idea of dividing the "married men" and "married women" into 6 units. With this insight, James had overcome his challenges in finding the common unit size of the two quantities that he experienced earlier. James then partitioned the "married adults" part first (see Figure 18) and followed by extending it to the "single adults" part (see Figure 19).

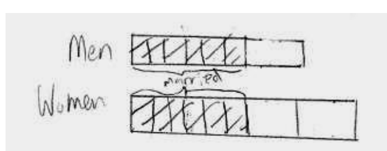


Figure 18: The first part of the model partition done by the Secondary Pair 6

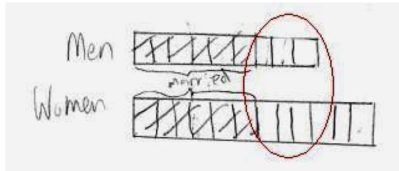


Figure 19: The final version of the model partition done by the Secondary Pair 6

Figure 19 shows that both students recognised the same unit size between single men and single women since they were drawn *nearly* the same. This was a distraction from visually-perceived relationship that was not overcome and led them to make a minor mistake in the part-whole reasoning (Figure 20). Instead of 19 units, the students deduced that there were 21 units for the whole adults.

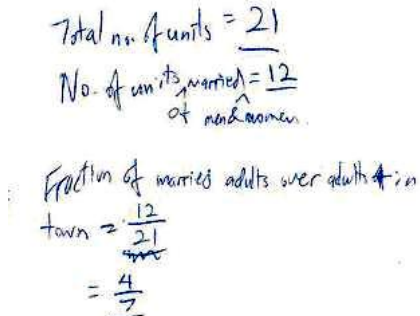


Figure 20: The final computational procedure done by the Secondary Pair 6

In all, there were no less than three times when James fell into reasoning numerically. William's persistence in having quantitative orientation had *saved* them from reasoning numerically. We could see although William was not the one who overcame the challenge behind partitioning the model, his positive attitude had helped James to re-orient their attentions onto quantitative orientations. The findings show that a peer member was more likely to listen to someone who seems to make sense. Should James have to solve this task individually, there was high chance that he would fall into numerical reasoning in the first time when he proposed numerical calculation.

Influence of peer's negative attitude

In contrast with the above case, a pair member's negative attitude becomes disenabling factor in the process of overcoming their challenges. An example of this attitude is the persistence of a *weaker* student who had a stronger voice in the decision making as shown by a fifth grade pair, PP1 (Francisca & Zoe), in solving the Marriage task.

The conflict occurred towards the end of their progression when Zoe tended to accept an unreasonable answer while Francisca who had better conceptual knowledge of fractions than Zoe wanted to reject it. Their negotiation shows how Zoe dominated the decision making process and Francisca was put in a "defender" position despite of Francisca's attempt to clarify the mistake in Zoe's reasoning. Figure 21 shows their solution.

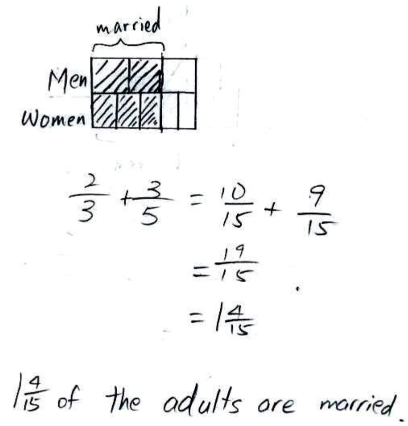


Figure 21: PP1's model and calculation led to fraction bigger than one whole in the Marriage task

The working solution shows that they both calculated "2/3 + 3/5" and obtained a fraction bigger than one whole, $1\frac{4}{15}$, as the result. The argument between Francisca and Zoe is found in the following excerpt

Turn	Speaker	Utterances
109.	Zoe	Ah? Same what! Ah never mind *lah*, can make it.
110.	Francisca	Cannot be more than one whole *leh*!
111.	Zoe	Can!
112.	Francisca	Cannot! You see ah, this one ((the whole model)) is one whole ah
113.	Zoe	Uh huh.
114.	Francisca	Then this one ((the shaded portion)) is less than one whole *leh*!
115.	Zoe	Add up together, what!
116.	Francisca	Cannot be *leh*!
117.	Zoe	Because you see ah, this one the model
118.	Francisca	This whole thing is one whole. Then after that how can it be more than one whole?
119.	Zoe	Maybe it's two whole, what!
120.	Francisca	I don't know
121.	Zoe	Because there'll be two persons eh
122.	Francisca	But everything came in one whole, what!
123.	Zoe	So, you want to do what method?
124.	Francisca	I don't know. Adults. ((reading the problem))What fractions of the adults in the town are married
125.	Zoe	But I think the answer is this one *leh*.
126.	Francisca	I don't think so.
127.	Zoe	Okay, then you do your method here ((empty space in the worksheet))
128.	Francisca	I don't know what method, you see!
129.	Zoe	Then I write first ah ((writing final statement to conclude the solution))

From the above interaction, it is obvious that Zoe dominated the decision making process. When Zoe conceived the relationship as a fraction of two wholes, Francisca disagreed since she has the knowledge that the whole adults is one whole. Zoe's tone was "if you can't show an alternative solution that fits in what you said, then I will conclude this answer". Zoe decided to conclude the solving process by writing the final statement using $1\frac{4}{15}$ as the answer while Francisca was still attempting to find an alternative procedure.

Eventually, their several rounds of trials were guided by the orientation to find the right combinations of numbers to fit into the models they drew.

Educational or Scientific importance of the study

The above four cases presented enabling and disenabling aspects of pair interactions in overcoming problem solving challenges have highlighted contrast in the following four areas: students' knowledge, students' orientation, communication ability, and attitudes towards problem solving. These four areas manifested in students' speech, actions, and representations that mediate the collaborative problem solving space. Students' knowledge is indeed crucial since lack of subject domain matter understanding is found invariably leading to wrong solutions. However, if one of the students possesses higher level of knowledge than the other student in the pair, chances are that they will build upon each other's knowledge to co-construct problem solving solution and hence higher knowledge to both of them. For knowledge co-construction to happen, the other two areas – students' orientation to quantitative reasoning and their communication ability – are no less important. If the student who has better knowledge lacks communication ability, the process of overcoming challenges tend to become ineffective. Secondly, if students' orientation is not based on the quantities and quantitative relationships, they could also be distracted by their meditational means, such as diagram that creates visually-perceived relationships or numbers that limits students' sense making, that lead to ineffective quantitative reasoning. The influence of students' attitudes is also a strong factor since it was shown to strongly affect the other pair member.

These features have important implications for designing effective collaborative work in classrooms. The elements embodied in both successful and unsuccessful cases will inform teachers about the steps and cautions to take when employing group work or pair work, and how to anticipate and facilitate more productive collaborative work. Besides, the observer's voice in giving thinking-aloud instructions could also inform teachers on how to guide students in working collaboratively especially in the instances when the student do not notice his own or peer's difficulties or when both experience the same difficulties. The findings have informed professional development programs on how to build teachers' knowledge around students' thinking and reasoning and help teachers in teaching *via* problem solving. We have also drawn our analysis and findings to build video cases to familiarise teachers with analyzing students' mental activities behind their reasoning and help them form pedagogical habit in their interactions with students. These cases will also be used for teacher development in pre-service and in-service courses and even as on-line resources for teacher learning[1].

Collaborative problem-solving activities in classrooms

By considering the elements leading to successful cases in the pair work, teachers could select students to form a group or a pair to expect a productive interaction and learning opportunity in classrooms. For example, pairing up a student who is used to paying

[1] This idea has been conceptualized as part of our ongoing funded project in developing web-based video cases for teacher professional development.

attention to quantities and quantitative relationships with a student who is not might increase the chance of overcoming difficulties that they experience as compared to pairing students who both lack in orientation to quantities and quantitative relationships. The cases of failure also inform teachers about the cautions they might need to take when employing group work or pair work and how to anticipate them.

The pair context has also identified at least three types of successful pairs learning scenarios such as taking over peer's work (Kelvin in PP2), peer coaching (Yvonne in PP5), and peer-assisted knowledge co-construction (Susan in SP5). It has been noted that the third type has the highest learning potential, as a lot of exchange involved understanding, justifying and clarifying one's reasoning. This kind of interaction was termed by Kruger (1993) as *transactive* reasoning, which also includes actions of clarification, elaboration, justification, and critique of one's own or a peer's reasoning. Such transactive reasoning was also noted by Goos, Galbraith, and Renshaw (2002) as one key element that defines students' collaborative zone of proximal development. Teachers can expect to have such learning processes when employing group work by designing teaching *via* problem solving (Schroeder & Lester, 1989) that involves the elements of productive peer-interaction, coupled with supportive instructional strategies such as specification of role sharing (Gillies, 2004; King, 1994; Whimbey & Lochhead, 1999) or specific metacognitive instructions (Teong, 2003). By combining aspects in pair context and designing instructions on orienting students in reasoning quantitatively, the learning process can be maximised.

Bridging 'model method' and 'symbolic-algebraic method'

Students' use of meditational tool such as models and speech in the collaborative problem solving space offers insight in addressing the issue of the gap between students' iconic and symbolic reasoning in Singapore education (Fang et al., 2009; Ng, Lee, Ang, & Khng, 2006). The design of Singapore curriculum follows Bruner's (1960) developmental theory, where students move through three stages of representation: enactive, iconic, and symbolic in their cognitive development. However, there is a tension in bridging the iconic stage to symbolic stage at the secondary levels, especially when students start to learn algebraic method in lower secondary schools. Anecdotal evidence suggests that secondary teachers often discourage students from using model method in order to teach and familiarise students with algebraic method. For example, when Ng et al. (2006) interviewed in-service secondary teachers, they viewed model method as childish, non-algebraic (hence having *less* rigor), and took it as a hindrance to the teaching of symbolic algebra.

As compared to the fifth graders, the findings on the seventh graders' approach in solving both tasks offer insight in how to bridge students' reasoning between using model method and symbolic-algebraic method, which is hoped to contribute in *easing the tension* between model method and algebraic method in teaching secondary students. As Thompson (1996) stated that "mathematical reasoning at all levels is firmly grounded in imagery (p. 267)", students' meditational means such as speech, representations, and numbers often reveal the kind of imagery that the seventh graders had was based on the model that they learned in elementary schools. In the Marriage task, the majority of secondary students drew models to unpack the network of quantities and quantitative relationships. This suggests that the problem situation was not *easy* to comprehend and they needed the model as a 'map of the territory' to build their solution method. In the students' reasoning, quite often what appears to be *symbolic* was actually based on a mental image of a model. For example, when Susan from SP5 used proportional reasoning based on a ratio representation: "6:3" and "6:4" to get "12/19", the mental image that she based on was a model, shown in the excerpt when she justified this reasoning to Nelly as posted in the case one. Susan's *mental* model consists of "married men" and "married women" constituted by 6 fractional parts, "single men" by 3 fractional parts, and "single women" by 4 fractional parts as shown in Figure 22.

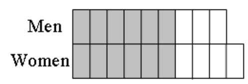


Figure 22: The mental model conceived by Susan from the Secondary Pair 5

Anecdotal evidence suggests that teachers might have misunderstandings about "model method" as problem-solving strategy and its application to learning algebra in secondary schools. Teachers might be afraid that students would stay in the 'comfort zone' after having used model methods for more than 4 years and would not want to come out of their 'comfort zone' since algebraic reasoning is naturally *abstract* and deemed *harder* to learn. The findings seem to suggest otherwise, that in fact model method has facilitated their algebraic reasoning. With the evidence that in reasoning symbolically, the image of the network of quantities and quantitative relationships in which the student based their reasoning took the form of rectangular models that they have learned in elementary schools. We could see that the form of elementary school's model had been somewhat embedded in students' cognition. On the one hand, the fact that SP5 had difficulties in partitioning the model and only became successful after they used the ratio representation shows that the students was not 'attached' to model method as the single tool. They managed to overcome their difficulties since they had a wider repertoire of representations and strategies, such as the knowledge of ratio and proportions. Such repertoire was lacking in the elementary subjects since they had not learned ratio and algebraic approaches at the time of data collection. As they constructed higher mathematical ideas in secondary schools, they will have *better* choices in dealing with different situations especially when one method does not work. This is in line with Ng (2007) who advocated the importance of students' having multiple representations as a case of students' multiple *literacies* of representations. Ng refers the *literacies* to students' flexibility in applying their repertoire of representation, be it iconic or symbolic, to solve problems.

Building teachers' knowledge on students' thinking and reasoning

Building teachers' knowledge on students' thinking and reasoning is also an important aspect to help teachers in teaching *via* problem solving, and the collaborative learning processes in these cases offer opportunity for teachers to study students' thinking. In Singapore schools with big classroom size of around 30-40 students, the opportunities to *see* a student's articulating his or her thinking tend to be limited. Hence, one possible way to build teachers' knowledge in this area is by turning the four cases above into *video cases* for teacher development in pre-service and in-service courses and even as on-line resources for teacher learning[2].

There are several advantages of developing cases from the pair work data in this study. First, the study was designed and implemented in ways that *conditioned* the subjects to verbalise their thinking through researchers' monitoring, and hence students' verbalized reasoning offers a rare but important opportunity for teachers to examine the thinking, reasoning, and cognitive obstacles that students experienced in the problem solving processes. As they watch the video cases and examine the students' interactions presented in the cases, teachers could have a natural tendency to associate them with what they observe from their own students, and hence develop their awareness and sensitivity to students' thinking and find a way to access and evaluate their mental activities. An example of such video cases can draw on student pair interactions when they overcome challenges. The results have suggested three learning opportunities that took place among successful cases. For example, the cases can be made to contrast the different learning opportunities in the pair interactions, or between successful and unsuccessful cases. These cases can consist of extracted clips of 5-10 minutes in length from the two pairs with commentaries to set up the context that facilitates teachers to focus their attention on the students' interaction and reasoning as well as some guided questions for teachers to analyse the students' thinking. Examples of questions are "what Gerry might be thinking when Kelvin took over the drawing from him and amended his mistake?", "what went through Joanne's mind since she started justifying her reasoning to Yvonne until she noticed the inconsistency that she conceived? What was the *turning point* of her reasoning? What made her spot her inconsistency?" Such questions will lead teachers to compare the two successful instances and notice the learning opportunities that are otherwise usually invisible if we only focus on the *result*, which is, whether or not they eventually get the correct answer.

Other possible questions like, for example, "what might the students do when they faced similar obstacles in future?" might also trigger teachers' reflection and learning. Such questions will extend the scenario beyond what is shown in the video clips to the issue of how students transfer what they have learnt from the interactions with peers that lead them to overcome difficulties in their future experience in solving another problem. Through this reflection, teachers' insight on the students' trajectories of learning can be developed, and such insight is an important aspect in uncovering the *black box* of students' learning *via* problem solving. Also a question can be like "what would you do as a teacher when you saw for yourselves the argument between Yvonne and Joanne?" This may stimulate teachers' thoughts on how they could facilitate or improve the students' learning to a higher level. Such guidance can then become part of teachers' strategies in teaching *via* problem solving.

[2] The following discussion is from ongoing CRPP funded project in developing web-based video cases for teacher professional development.

Reference

Artzt, A., & Armour-Thomas, E. (1992). Development of a cognitive-metacognitive framework for protocol analysis of mathematical problem solving in small groups. *Cognition and Instruction, 9*, 137-175.

Bednarz, N., & Janvier, B. (1996). A problem-solving perspective on the introduction of algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for learning and teaching* (pp. 115-136). The Netherlands: Kluwer Academic Publishers.

Board of Studies NSW. (2002). *Mathematics K-6 syllabus 2002*. New South Wales, Australia: Board of Studies, NSWo. Document Number)

Carr, M., & Biddlecomb, B. D. (1998). Metacognition in mathematics from a constructivist perspective. In D. J. Hacker, J. Dunlosky & A. C. Graesser (Eds.), *Metacognition in educational theory and practice* (pp. 69-92). Hillsdale, NJ: Erlbaum.

Confrey, J. (1991). Learning to listen: A student's understanding of powers of ten. In E. v. Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. pp. 111-138). Netherlands: Kluwer Academic Publishers.

Curriculum Planning and Development Division. (2000). *Primary mathematics syllabus 2001*. Singapore: Ministry of Educationo. Document Number)

Curriculum Planning and Development Division. (2005). *Primary mathematics syllabus 2007*. Singapore: Ministry of Educationo. Document Number)

De Corte, E., Verschaffel, L., & De Win, L. (1985). Influence of rewording verbal problems on children's representations and solutions. *Journal of Educational Psychology, 77*, 460-470.

Ericsson, K. A., & Simon, H. A. (1993). *Protocol analysis: Verbal report as data*. Massachusetts: The MIT Press.

Fan, L. H., Wong, N. Y., Cai, J., & Li, S. (2004). *How Chinese learn mathematics: Perspectives from insiders*. Singapore: World Scientific.

Fang, Y., Ho, K. F., Lioe, L. T., Wong, K. Y., & Tiong, Y. S. J. (2009). *Developing the repertoire of heuristics for mathematical problem solving: Technical Report for Project CRP1/04 TSK/JH.* Singapore: Centre for Research in Pedagogy and Practice, National Institute of Education, Nanyang Technological University.o. Document Number)

Foong, P. Y. (1993). Development of a framework for analysing mathematical problem solving behaviours. *Singapore Journal of Education, 13*(1), 61-75.

Foong, P. Y., & Koay, P. L. (1997). School word problems and stereotyped thinking. *Teaching and Learning, 18*(1), 73-82.

Foong, P. Y., Yap, S. F., & Koay, P. L. (1996). Teachers' concerns about the revised mathematics curriculum. *The Mathematics Educator,, 1*(1), 99-110.

Gillies, R. M. (2004). The effects of cooperative learning on junior high school students during small group learning. *Learning and Instruction, 14*, 197-213.

Glaser, B., & Strauss, A. (1967). *The discovery of grounded theory*. Chicago: Aldine.

Glasersfeld, E. v. (1995). *Radical constructivism: A way of knowing and learning*. London: Falmer Press.

Goos, M., Galbraith, P., & Renshaw, P. (2002). Socially mediated metacognition: creating collaborative zones of proximal development in small group problem solving. *Educational Studies in Mathematics*, *49*, 193-223.

Greer, B., Verschaffel, L., & De Corte, E. (2002). "The answer is really 4.5": Beliefs about word problems. In G. C. Leder, E. Pehkonen & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 271-292). The Netherlands: Kluwer Academic Publishers.

Hegarty, M., Mayer, R., & Monk, C. A. (1995). Comprehension of arithmetic word problems: A comparison of successful and unsuccessful problem solvers. *Journal of Educational Psychology, 87*, 18-32.

Johnson, D., Johnson, R., & Holubec, E. (1998). *Cooperation in the classroom*. Boston.

Kho, T. H. (1987). *Mathematical models for solving arithmetic problems.* Paper presented at the The Fourth SEAC on Mathematics (ICMI-SEAMS).

Kho, T. H., Yeo, S. M., & Lim, J. (2009). *The Singapore model method for learning Mathematics*. Singapore: EPB Pan Pacific.

Kieran, C. (2004). Algebraic thinking in the early grades: what is it? *The Mathematics Educator, 8*, 139-151.

Kilpatrick, J. (1985). A restropective account of the past 25 years of research on teaching mathematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving* (pp. 1-15). Hillsdale, NJ: Lawrence Erlbaum.

King, A. (1994). Guiding knowledge construction in the classroom: Effects of teaching children how to question and how to explain. *American Educational Research Journal, 31*(2), 338-368.

Kruger, A. C. (1993). Peer collaboration: Conflict, cooperation or both? *Social Development, 2,* 165-182.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27*(1), 29-63.

Lioe, L. T. (2009). *Challenges in reasoning quantitatively and ways of overcoming them in pair work: Cases of Primary five and Secondary one students.* Unpublished Masters thesis, National Institute of Education, Nanyang Technological University, Singapore.

Lioe, L. T., Ho, K. F., & Hedberg, J. G. (2005, August). *Thinker-listener pair interactions to develop students' metacognitive strategies in mathematical problem solving.* Paper presented at the ICMI Regional Conference: The Third East Asia Regional Conference on Mathematics Education, Shanghai, Nanjing, and Hangzhou, China.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: author.

National Council of Teachers of Mathematics. (2000). *Professional standards for teaching mathematics*. Reston, VA: author.

Nesher, P. (1980). The stereotyped nature of school word problems. *For the Learning of Mathematics, 1*(1), 41–48.

Ng, S. F. (2004). Developing algebraic thinking in early grades: Case study of the Singapore primary mathematics curriculum. *The Mathematics Educator, 8*(1), 39-59.

Ng, S. F. (2007). *Multiple "literacies" of representations.* Paper presented at the Keynote lecture addressed at on 1 June 2007 at the Mathematics Teacher Conference, National Institute of Education, Singapore.

Ng, S. F., & Lee, K. (2004). *Word problems - Can the model method help?* Paper presented at the the Mathematics Education into 21st Century Project, The Future of Mathematics Education.

Ng, S. F., & Lee, K. (2009). Model Method: A visual tool to support algebra word problem solving at the primary level. In K. Y. Wong, P. Y. Lee, B. Kaur, P. Y. Foong & S. F. Ng (Eds.), *Mathematics Education: The Singapore Journey* (pp. 169-203). Singapore: World Scientific.

Ng, S. F., Lee, K., Ang, S. Y., & Khng, F. (2006). Model method: Obstacle or bridge to learning symbolic algebra. In W. D. Bokhorst-Heng, M. D. Osborne & K. Lee (Eds.), *Redesigning pedagogy: Reflections on theory and praxis* (pp. 227-242). Rotterdam, The Netherlands: Sense Publishers.

Ratcliff, D. (2004). Video methods in qualitative research. In P. M. Camic, J. E. Rhodes & L. Yardley (Eds.), *Qualitative research in psychology: Expanding perspectives in methodology and design*. Washington, DC: American Psychological Association.

Schmittau, J., & Morris, A. (2004). The development of algebra in Davydov’s elementary curriculum. *The Mathematics Educator, 8*(1), 60-87.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: problem solving, metacognition and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning, Chapter 15* (pp. 334-370). New York: Macmillan.

Schroeder, T. L., & Lester, F. K. (1989). Developing understanding in mathematics via problem solving. In P. R. Trafton & A. P. Shulte (Eds.), *New directions for elementary school mathematics* (pp. 31-42). Reston, VA: National Council of Teachers of Mathematics.

Smith, J. P., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. Kaput, D. Carraher & M. Blanton (Eds.), *Algebra in the early grades* (pp. 95-132). New York: Erlbaum.

Stacey, K. (1992). Mathematical problem solving in groups: Are two heads better than one? *Journal of Mathematical Behavior, 11*, 261-275.

Stacey, K. (2005). The place of problem solving in contemporary mathematics curriculum documents. *Journal of Mathematical Behavior, 24*, 341-350.

Steffe, L. P., & Thompson, P. W. (2000). Interaction or intersubjectivity? A reply to Lerman. *Journal for Research in Mathematics Education, 31*(2), 191-209.

Teong, S. K. (2000). *The effect of metacognitive training on the mathematical word problem solving of Singapore 11-12 year olds in a computer environment.* Unpublished Unpublished PhD dissertation, University of Leeds, United Kingdom.

Teong, S. K. (2003). The effect of metacognitive training on mathematical word-problem solving. *Journal of Computer Assisted Learning, 19*, 46-55.

Thompson, P. W. (1995). Notation, convention, and quantity in elementary mathematics. In J. Sowder & B. Schapelle (Eds.), *Providing a foundation for teaching middle school mathematics* (pp. 199-211). Albany, NY: SUNY Press.
Verschaffel, L., Greer, B., & De Corte, E. (2000). *Making sense of word problems*. Lisse: Swets & Zeitlinger.
Vygotsky, L. S. (1978). *Mind in society: The developmental of higher psychological processes*. Cambridge, MA: Harvard University Press.
Wells, G. (1999). *Dialogic inquiry: Toward a sociocultural practice and theory of education*. US: Cambridge University Press.
Whimbey, A., & Lochhead, J. (1999). *Problem solving and comprehension*. NJ: Erlbaum.
Wood, T., Cobb, P., & Yackel, E. (1995). Reflections on learning and teaching mathematics in elementary school. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 401-422). Hillsdale, NJ, US: Erlbaum.
Zbiek, R. M. (1998). Prospective teachers' use of computing tools to develop and validate functions as mathematical models. *Journal for Research in Mathematics Education, 29*(2), 184-201.