Noncyclic geometric phase for neutrino oscillation

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We provide explicit formulae for the noncyclic geometric phases or Pancharatnam phases of neutrino oscillations. Since Pancharatnam phase is a generalization of the Berry phase, our results generalize the previous findings for Berry phase in a recent paper [Phys. Lett. B, 466 (1999) 262]. Unlike the Berry phase, the noncyclic geometric phase offers distinctive advantage in terms of measurement and prediction. In particular, for three-flavor mixing, our explicit formula offers an alternative means of determining the CP-violating phase. Our results can also be extended easily to explore geometric phase associated with neutron-antineutron oscillations.

I. INTRODUCTION

Pontecorvo's suggestion [1] nearly half a century ago that neutrinos had finite masses implied that neutrino mass eigenstates need not be identical with the weak eigenstates and thus may give rise to neutrino oscillations. Indeed, recent experiments from atmospheric neutrino data in the Super-Kamiokande experiments [2], IMB collaboration [3], Soudan II [4] and MACRO [5] experiments have provided strong confirmation of such oscillations.

In a recent paper [6], it was found that the geometric phase appears naturally in the standard Pontecorvo formulation of neutrino oscillations. The Berry phase [7] for oscillating neutrinos was calculated and found to be a functional of the mixing angle for the two-flavor neutrinos. Since it is possible in principle to observe the geometric phase, it was suggested that the mixing angle could then be deduced through the observation of the Berry phase. However, the measurement of the Berry phase is only applicable for cyclic adiabatic evolution. Thus one can only measure a state after it has undergone a closed circuit with some period, T. For neutrinos, this period is relatively long. Thus in order to measure the Berry phase for neutrinos, we need to place the detector sufficiently distant from the source so that the neutrinos traverse exact distance corresponding to a complete cycle. Experimentally, this technique can be difficult.

Three-flavor neutrino oscillations are also particularly interesting due to its physical implications in CP violation. Nevertheless, based on the formula in ref [6], it may not be easy to determine the CP-violating phase. In this paper, we generalize the idea of the Berry phase to a non-cyclic geometric phase and discuss how some of the above difficulties could be circumvented through the generalization. Naturally, our explicit formula for the non-cyclic geometric phase reduces to the Berry phase formula in ref [6] when the time of measurement is set to the oscillating period of the neutrino.

The generalization of geometric phase to noncyclic evolution [8, 9] dates back to an important seminal paper by Pancharatnam [10]. Experimental results for non-cyclic geometric phase or Pancharatnam phase have been demonstrated recently in experiments [11, 12]. Following the idea raised in ref. [6], one can in principle extract information concerning states of the neutrinos by observing the noncyclic geometric phase at different times.

This paper is organized as follows. In section II, we briefly describe the notion of non-cyclic phase and consider two-flavor neutrino oscillation. In section III, we extend the same calculation to the three-flavor case and show how the CP-violating phase can in principle be deduced from the non-cyclic phase. Finally, we summarizes the results in section IV.

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II. NONCYCLIC GEOMETRIC PHASE

We first explain how we can compute the non-cyclic geometric phase [13, 14]. Suppose state $|\chi(0)\rangle$ evolves to a state $|\chi(t)\rangle$ after a certain time t. If the scalar product

$$\langle\chi(0)| \exp\left[\frac{i}{\hbar}\int_0^t < E > (t')dt'\right] |\chi(t)\rangle$$

can be written as $r\exp[i\beta]$, where r is a real number, then we say that the non-cyclic geometric phase due to the evolution from $|\chi(0)\rangle$ to $|\chi(t)\rangle$ is the β. This non-cyclic geometric phase generalizes the cyclic geometric phase since the latter can be regarded as a special case of the former for which $r = 1$.

We first consider the two-flavor oscillating neutrino states as

$$|\nu_e(0)\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle \tag{1}$$
$$|\nu_\mu(0)\rangle = -\sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle \tag{2}$$

respectively. At time t, the states $|\nu_e\rangle$ and $|\nu_\mu\rangle$ evolve to the states

$$|\nu_e(t)\rangle = e^{-iHt}|\nu_e(0)\rangle = e^{-i\omega_1 t}\cos\theta|\nu_1\rangle + e^{-i\omega_2 t}\sin\theta|\nu_2\rangle \tag{3}$$
$$|\nu_\mu(t)\rangle = e^{-iHt}|\nu_\mu(0)\rangle = -e^{-i\omega_1 t}\sin\theta|\nu_1\rangle + e^{-i\omega_2 t}\cos\theta|\nu_2\rangle \tag{4}$$

To calculate the noncyclic geometric phase for the evolution from $|\nu_e(0)\rangle$ to $|\nu_e(t)\rangle$, we define a new state, $|\tilde{\nu}_e(t)\rangle$, given by

$$|\tilde{\nu}_e(t)\rangle = \exp\left[i\int_0^t < E > (t')dt'\right] |\nu_e(t)\rangle \tag{5}$$
$$= \exp[i(\omega_1\cos^2\theta + \omega_2\sin^2\theta)t]|\nu_e(t)\rangle \tag{6}$$

so that

$$\langle\nu_e(0)|\tilde{\nu}_e(t)\rangle = \exp[i(\omega_1\cos^2\theta + \omega_2\sin^2\theta)t]\left[\cos^2\theta e^{-i\omega_1 t} + \sin^2\theta e^{-i\omega_2 t}\right] \tag{7}$$
$$\equiv r\exp[i\beta] \tag{8}$$

Denoting $\Omega = -\dfrac{\omega_1+\omega_2}{2}$ and $\phi = -\dfrac{\omega_1-\omega_2}{2}$, we have

$$\langle\nu_e(0)|\tilde{\nu}_e(t)\rangle = \exp[i(\omega_1\cos^2\theta + \omega_2\sin^2\theta + \Omega)t]\left[\cos^2\theta e^{i\phi t} + \sin^2\theta e^{-i\phi t}\right] \tag{9}$$

This can also be written as

$$\langle\nu_e(0)|\tilde{\nu}_e(t)\rangle = \exp[-i\phi t\cos 2\theta]\left[\cos^2\theta e^{i\phi t} + \sin^2\theta e^{-i\phi t}\right] \equiv re^{i\beta} \tag{10}$$

where the explicit expressions for r and β are then given by

$$r = \sqrt{1 - \sin^2 2\theta\sin^2\phi t} \tag{11}$$

and

$$\beta = -\phi t\cos 2\theta + \tan^{-1}[\cos 2\theta\tan(\phi t)]. \tag{12}$$

We can see that there is indeed a nonzero geometric phase under non-cyclic evolution. In particular, this phase reduces to the value of $2\pi\sin^2\theta$ if the time t is set to the period of the oscillating neutrinos, that is $t = \dfrac{2\pi}{\omega_2-\omega_1}$, upon choosing the appropriate branch. Thus one recovers the result in ref [6] for the Berry phase. However, since the noncyclic geometric phase can be measured at arbitrary time t, there is no need to restrict the time, t, of the measurement to exactly one period, $T = \frac{2\pi}{\omega_2-\omega_1}$. From the experimental point of view, such relaxation would facilitate the measurement of the geometric phase. Hence, from eq(12), one can see that it is possible in principle to deduce the mixing angle either by measuring the value of r (which can be done by counting the neutrino flux) or detecting the geometric phase at two different times and then solving the resulting simultaneous equations for θ and ϕt.

It is also possible to compute the other components, namely $\langle\nu_e(0)|\tilde{\nu}_\mu(t)\rangle$, $\langle\nu_\mu(0)|\tilde{\nu}_e(t)\rangle$ and $\langle\nu_\mu(0)|\tilde{\nu}_\mu(t)\rangle$ and their associated noncyclic geometric phases. The results are summarized in the following tables.

component	expression
$\langle\nu_e(0)\|\tilde{\nu}_\mu(t)\rangle$	$(\exp[-2i\phi t\sin^2\theta] - \exp[2i\phi t\cos^2\theta])\cos\theta\sin\theta$
$\langle\nu_\mu(0)\|\tilde{\nu}_e(t)\rangle$	$(\exp[-2i\phi t\cos^2\theta] - \exp[2i\phi t\sin^2\theta])\cos\theta\sin\theta$
$\langle\nu_\mu(0)\|\tilde{\nu}_\mu(t)\rangle$	$\exp[i\phi t\cos 2\theta]\left[\cos^2\theta e^{-i\phi t} + \sin^2 e^{i\phi t}\right]$

component	r	β
$\langle\nu_e(0)\|\tilde{\nu}_\mu(t)\rangle$	$\sin 2\theta\sin\phi t$	$\phi t\cos 2\theta - \frac{\pi}{2}$
$\langle\nu_\mu(0)\|\tilde{\nu}_e(t)\rangle$	$\sin 2\theta\sin\phi t$	$-\phi t\cos 2\theta - \frac{\pi}{2}$
$\langle\nu_\mu(0)\|\tilde{\nu}_\mu(t)\rangle$	$r = \sqrt{1-\sin^2 2\theta\sin^2\phi t}$	$\phi t\cos 2\theta - \tan^{-1}[\cos 2\theta\tan(\phi t)]$

III. THREE-FLAVOR OSCILLATION

The noncyclic geometric phase for the case of three-flavor mixing can also be computed using the same method. In the case of three-flavor mixing, the electron neutrino state at time t is

$$|\nu_e(t)\rangle = e^{-i\omega_1 t}\cos\theta_{12}\cos\theta_{13}|\nu_1\rangle + e^{-i\omega_2 t}\sin\theta_{12}\cos\theta_{13}|\nu_2\rangle \\ +e^{-i\omega_3 t}e^{i\delta}\sin\theta_{13}|\nu_3\rangle \tag{13}$$

where θ_{12} and θ_{13} are the appropriate mixing angles. More generally, one can consider the mixing in terms of the Cabibbo-Maskawa-Kobayashi (CKM) matrix,U, using the parametrization

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & e^{i\delta}s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{-i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{-i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{-i\delta} & c_{23}c_{13} \end{pmatrix} \tag{14}$$

where $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$ and δ is the CP violating phase of the CKM matrix.

Moreover, as in the two-flavor case, for the electron neutrino,

$$|\tilde{\nu}_e(t)\rangle = \exp[i\int_0^t < E > (t')dt']|\nu_e(t)\rangle \tag{15}$$

with

$$< E > (t) = \omega_1\cos^2\theta_{12}\cos^2\theta_{13} + \omega_2\sin^2\theta_{12}\cos^2\theta_{13} + \omega_3\sin^2\theta_{13}.$$

A straightforward calculation yields

$$\begin{aligned}\langle\nu_e(0)|\tilde{\nu}_e(t)\rangle &= \exp[i(\omega_1\cos^2\theta_{12}\cos^2\theta_{13} + \omega_2\sin^2\theta_{12}\cos^2\theta_{13} + e^{i\delta}\omega_3\sin^2\theta_{13})t] \\ &\quad \left[\cos^2\theta_{12}\cos^2\theta_{13}e^{-i\omega_1 t} + \sin^2\theta_{12}\cos^2\theta_{13}e^{-i\omega_2 t} + e^{2i\delta}\sin^2\theta_{13}e^{-i\omega_3 t}\right] \\ &\equiv r_{ee}e^{i\beta_{ee}}\end{aligned} \tag{16}$$

where r_{ee} and β_{ee} are the modulus and phase of the inner product in eq(16) between $\nu_e - \nu_e$ states respectively. The left hand side of eq(16) can be written as

$$\begin{aligned}\langle\nu_e(0)|\tilde{\nu}_e(t)\rangle &= \exp[i(\omega_1\cos^2\theta_{12}\cos^2\theta_{13} + \omega_2\sin^2\theta_{12}\cos^2\theta_{13} + e^{i\delta}\omega_3\sin^2\theta_{13} - \frac{\omega_1+\omega_2}{2})t] \\ &\quad\times\left[\cos^2\theta_{12}\cos^2\theta_{13}e^{i\phi t} + \sin^2\theta_{12}\cos^2\theta_{13}e^{-i\phi t} + e^{2i\delta}\sin^2\theta_{13}e^{i(2q-1)\phi t}\right] \\ &= \exp[\omega_1\sin\theta_{13}^2(e^{2i\delta}-1) + i(2\phi\sin^2\theta_{12}\cos^2\theta_{13} + 2q\phi e^{i\delta}\sin^2\theta_{13} - \phi)t] \\ &\quad\times\left[\cos^2\theta_{12}\cos^2\theta_{13}e^{i\phi t} + \sin^2\theta_{12}\cos^2\theta_{13}e^{-i\phi t} + \sin^2\theta_{13}e^{2i\delta}e^{i(2q-1)\phi t}\right]\end{aligned} \tag{17}$$

where $q = \dfrac{\omega_3-\omega_1}{\omega_2-\omega_1}$ and $\phi = -\dfrac{\omega_1-\omega_2}{2}$ as defined previously for the two-flavor case. After some algebraic manipulations, the geometric phase can be found to be

$$
\begin{aligned}
\beta_{ee} \equiv \beta = {} & \omega_1 \sin\theta_{13}^2(\cos(2\delta)-1) + (2\phi\sin^2\theta_{12}\cos^2\theta_{13} + 2q\phi\cos(2\delta)\sin^2\theta_{13} - \phi)t \\
& + \tan^{-1}\frac{\cos 2\theta_{12}\cos^2\theta_{13}\sin\phi t - \sin^2\theta_{13}\sin[(2q-1)\phi t - 2\delta]}{\cos^2\theta_{13}\cos\phi t + \sin^2\theta_{13}\cos[(2q-1)\phi t - 2\delta]}
\end{aligned} \tag{18}
$$

In general, we do not expect the CP-violating phase, δ, to be zero. However if we take the CP violating phase to be zero, as in ref [6], then we get

$$
\begin{aligned}
\beta = {} & (2\phi\sin^2\theta_{12}\cos^2\theta_{13} + 2q\phi\sin^2\theta_{13} - \phi)t \\
& + \tan^{-1}\frac{\cos 2\theta_{12}\cos^2\theta_{13}\sin\phi t - \sin^2\theta_{13}\sin(2q-1)\phi t}{\cos^2\theta_{13}\cos\phi t + \sin^2\theta_{13}\cos(2q-1)\phi t}
\end{aligned} \tag{19}
$$

If we take q to be a rational number and t to be the cyclic period, namely $t = \dfrac{2\pi}{\omega_1 - \omega_2}$, we recover the result in ref [6]. However, from the formula in ref [6], it is difficult to deduce the mixing angles even if we can measure the Berry phase because the formula involves too many unknowns. Clearly, our explicit formula in eq(18) provides in principle a better means of deducing the mixing angles and, more importantly, the CP-violating phase through the measurement of the noncyclic geometric phases at several different times and then solving the resulting simultaneous equations. If necessary, errors in the measurement can also be reduced by using some form of least square fit. Since three flavor mixing is very important in CP violation, our formula offers an invaluable tool for resolving the issue through the measurement of geometric phase.

For completeness, we have also computed the other eight possible components and their noncyclic geometric phase. These results are summarized as follows.

$$
\begin{aligned}
\beta_{e\mu} = {} & \omega_1 t\sin^2\theta_{13}\sin^2\theta_{23}(\cos(2\delta)-1) - \phi t - 2q\phi t \\
& + 2q\phi t\cos^2\theta_{13}\sin^2\theta_{23} + 2\phi t(\cos\theta_{12}\cos\theta_{23} - \cos\delta\sin\theta_{12}\sin\theta_{13}\sin\theta_{23})^2 \\
& \qquad - \sin^2\delta\sin^2\theta_{12}\sin^2\theta_{13}\sin^2\theta_{23} \\
& + \tan^{-1}\frac{\left\{\begin{array}{l} \sin\theta_{13}\sin\theta_{23}(\sin(\phi t+\delta) - \sin^2\theta_{12}\sin\zeta_- \\ \qquad - \cos^2\theta_{12}\sin\zeta_+) \\ \qquad - \cos\theta_{23}\cos(2q\phi t)\sin(\phi t)\sin(2\theta_{12}) \end{array}\right\}}{\left\{\begin{array}{l} \sin\theta_{13}\sin\theta_{23}(\cos(\phi t+\delta) - \sin^2\theta_{12}\sin\zeta_- \\ \qquad - \cos^2\theta_{12}\sin\zeta_+) \\ \qquad - \cos\theta_{23}\sin(2q\phi t)\sin(\phi t)\sin(2\theta_{12}) \end{array}\right\}}
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
\beta_{e\tau} = {} & \omega_1 t\sin^2\theta_{13}\sin^2\theta_{23}(\cos(2\delta)-1) - \phi t - 2q\phi t \\
& + 2q\phi t\cos^2\theta_{13}\sin^2\theta_{23} + 2\phi t(\cos\theta_{12}\cos\theta_{23} - \cos\delta\sin\theta_{12}\sin\theta_{13}\sin\theta_{23})^2 \\
& \qquad - \sin^2\delta\sin^2\theta_{12}\sin^2\theta_{13}\sin^2\theta_{23} \\
& + \tan^{-1}\frac{\left\{\begin{array}{l} \sin\theta_{13}\cos\theta_{23}(\sin(\phi t+\delta) - \sin^2\theta_{12}\sin\zeta_- \\ \qquad - \cos^2\theta_{12}\sin\zeta_+) \\ \qquad - \sin\theta_{23}\cos(2q\phi t)\sin(\phi t)\sin(2\theta_{12}) \end{array}\right\}}{\left\{\begin{array}{l} \sin\theta_{13}\cos\theta_{23}(\cos(\phi t+\delta) - \sin^2\theta_{12}\sin\zeta_- \\ \qquad - \cos^2\theta_{12}\sin\zeta_+) \\ \qquad - \sin\theta_{23}\sin(2q\phi t)\sin(\phi t)\sin(2\theta_{12}) \end{array}\right\}}
\end{aligned} \tag{21}
$$

$$
\beta_{\mu e} = \omega_1 t\sin^2\theta_{13}(\cos(2\delta)-1) - \phi t + 2q\phi t\sin^2\theta_{13}\cos(2\delta) + 2\sin^2\theta_{12}\cos^2\theta_{12}\phi t
$$

$$+\tan^{-1}\frac{\left\{\begin{array}{l}\sin\theta_{13}\sin\theta_{23}\left(\begin{array}{l}\sin^2\theta_{12}\sin(\phi t+\delta)\\ -\cos^2\theta_{12}\sin(\phi t-\delta)-\sin\zeta_-\end{array}\right)\\ \qquad -\cos\theta_{23}\sin(\phi t)\sin(2\theta_{12})\end{array}\right\}}{\left\{\begin{array}{l}\sin\theta_{13}\sin\theta_{23}\left(\begin{array}{l}\sin^2\theta_{12}\sin(\phi t+\delta)\\ +\cos^2\theta_{12}\sin(\phi t-\delta)\\ +\cos\zeta_-\end{array}\right)\end{array}\right\}} \tag{22}$$

$$\begin{aligned}\beta_{\mu\mu} &= \omega_1 t\sin^2\theta_{13}\sin^2\theta_{23}(\cos(2\delta)-1)+\phi t\\ &+2q\phi t\cos^2\theta_{13}\sin^2\theta_{23}+2\phi t(\cos\theta_{12}\cos\theta_{23}-\cos\delta\sin\theta_{12}\sin\theta_{13}\sin\theta_{23})^2\\ &\qquad -\sin^2\delta\sin^2\theta_{12}\sin^2\theta_{13}\sin^2\theta_{23}\\ &+\tan^{-1}\frac{\left\{\begin{array}{l}\sin^2\theta_{13}\sin^2\theta_{23}\left(\begin{array}{l}\cos^2\theta_{12}\sin(\phi t-2\delta)\\ -\sin^2\theta_{12}\sin(\phi t+2\delta)\end{array}\right)\\ \qquad -\cos^2\theta_{23}\cos(2\theta_{12})\sin(\phi t)\\ \qquad -\cos^2\theta_{13}\sin^2\theta_{23}\sin(2q-1)\phi t\\ \qquad +\cos\delta\sin\theta_{13}\sin(2\theta_{12})\sin(2\theta_{23})\sin(\phi t)\end{array}\right\}}{\left\{\begin{array}{l}\sin^2\theta_{13}\sin^2\theta_{23}\left(\begin{array}{l}\cos^2\theta_{12}\sin(\phi t-2\delta)\\ +\sin^2\theta_{12}\sin(\phi t+2\delta)\end{array}\right)\\ \qquad -\cos^2\theta_{23}\cos(\phi t)\\ \qquad -\cos^2\theta_{13}\sin^2\theta_{23}\cos(2q-1)\phi t\\ \qquad +\sin\delta\sin\theta_{13}\sin(2\theta_{12})\sin(2\theta_{23})\sin(\phi t)\end{array}\right\}}\end{aligned} \tag{23}$$

$$\begin{aligned}\beta_{\mu\tau} &= \omega_1 t\sin^2\theta_{13}\sin^2\theta_{23}(\cos(2\delta)-1)-\phi t\\ &+2q\phi t\cos^2\theta_{13}\sin^2\theta_{23}+2\phi t(\cos\theta_{12}\cos\theta_{23}-\cos\delta\sin\theta_{12}\sin\theta_{13}\sin\theta_{23})^2\\ &\qquad -\sin^2\delta\sin^2\theta_{12}\sin^2\theta_{13}\sin^2\theta_{23}\\ &+\tan^{-1}\frac{\left\{\begin{array}{l}\frac{1}{2}\sin(2\theta_{23})\left(\begin{array}{l}\cos^2\theta_{13}\sin(2q-1)\phi t+\cos(2\theta_{12})\sin(\phi t)\\ +\sin^2\theta_{13}\cos^2\theta_{12}\sin(\phi t-2\delta)\\ -\sin^2\theta_{13}\sin^2\theta_{12}\sin(\phi t+2\delta)\end{array}\right)\\ \qquad +\cos\delta\sin\theta_{13}\sin(2\theta_{12})\cos(2\theta_{23})\sin(\phi t)\end{array}\right\}}{\left\{\begin{array}{l}\frac{1}{2}\sin(2\theta_{23})\left(\begin{array}{l}\cos^2\theta_{13}\cos(2q-1)\phi t-\cos(\phi t)\\ +\cos^2\theta_{13}\cos^2\theta_{12}\cos(\phi t-2\delta)\\ -\cos^2\theta_{13}\sin^2\theta_{12}\cos(\phi t+2\delta)\end{array}\right)\\ \qquad +\sin\delta\sin\theta_{13}\sin(2\theta_{12})\cos(2\theta_{23})\sin(\phi t)\end{array}\right\}}\end{aligned} \tag{24}$$

$$\beta_{\tau e} = \omega_1 t\sin^2\theta_{13}(\cos(2\delta)-1)-\phi t+2q\phi t\sin^2\theta_{13}\cos(2\delta)+2\sin^2\theta_{12}\cos^2\theta_{12}\phi t$$

$$
+\tan^{-1}\frac{\left\{\begin{array}{l}\sin\theta_{23}\sin(2\theta_{12})\sin(\phi t)\\ \qquad +\sin\theta_{13}\cos\theta_{23}\Big(\cos^2\theta_{12}\sin(\phi t-\delta)\\ \qquad +\sin^2\theta_{12}\sin(\phi t+\delta)+\sin\zeta_-\Big)\end{array}\right\}}{\left\{\begin{array}{l}\sin\theta_{13}\cos\theta_{23}\Big(\cos^2\theta_{12}\cos(\phi t-\delta)\\ \qquad +\sin^2\theta_{12}\cos(\phi t+\delta)+\cos\zeta_-\Big)\end{array}\right\}} \tag{25}
$$

$$
\begin{aligned}
\beta_{\tau\mu} = {} & \omega_1 t\sin^2\theta_{13}\sin^2\theta_{23}(\cos(2\delta)-1)-\phi t\\
& +2q\phi t\cos^2\theta_{13}\sin^2\theta_{23}+2\phi t(\cos\theta_{12}\cos\theta_{23}-\cos\delta\sin\theta_{12}\sin\theta_{13}\sin\theta_{23})^2\\
& \qquad -\sin^2\delta\sin^2\theta_{12}\sin^2\theta_{13}\sin^2\theta_{23}\\
& +\tan^{-1}\frac{\left\{\begin{array}{l}\frac{1}{2}\sin(2\theta_{23})\Big(-\cos^2\theta_{13}\sin(2q-1)\phi t+\cos(2\theta_{12})\sin(\phi t)\\ \qquad +\sin^2\theta_{13}\cos^2\theta_{12}\sin(\phi t-2\delta)\\ \qquad +\sin^2\theta_{13}\sin^2\theta_{12}\sin(\phi t+2\delta)\Big)\\ \qquad +\cos\delta\sin\theta_{13}\sin(2\theta_{12})\cos(2\theta_{23})\sin(\phi t)\end{array}\right\}}{\left\{\begin{array}{l}\frac{1}{2}\sin(2\theta_{23})\Big(\cos^2\theta_{13}\cos(2q-1)\phi t-\cos(\phi t)\\ \qquad +\cos^2\theta_{12}\sin^2\theta_{13}\cos(\phi t-2\delta)\\ \qquad +\sin^2\theta_{12}\sin^2\theta_{13}\cos(\phi t+2\delta)\Big)\\ \qquad +\sin\delta\sin\theta_{13}\sin(2\theta_{12})\cos(2\theta_{23})\sin(\phi t)\end{array}\right\}}
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\beta_{\tau\tau} = {} & \omega_1 t\sin^2\theta_{13}\sin^2\theta_{23}(\cos(2\delta)-1)-\phi t\\
& +2q\phi t\cos^2\theta_{13}\sin^2\theta_{23}+2\phi t(\cos\theta_{12}\cos\theta_{23}-\cos\delta\sin\theta_{12}\sin\theta_{13}\sin\theta_{23})^2\\
& \qquad -\sin^2\delta\sin^2\theta_{12}\sin^2\theta_{13}\sin^2\theta_{23}\\
& -\tan^{-1}\frac{\left\{\begin{array}{l}\sin^2\theta_{13}\sin^2\theta_{23}\Big(\cos^2\theta_{12}\sin(\phi t-2\delta)\\ \qquad -\sin^2\theta_{12}\sin(\phi t+2\delta)\Big)\\ \qquad +\cos^2\theta_{23}\cos(2\theta_{12})\sin(\phi t)\\ \qquad +\cos^2\theta_{13}\cos^2\theta_{23}\sin(2q-1)\phi t\\ \qquad +\cos\delta\sin\theta_{13}\sin(2\theta_{12})\sin(2\theta_{23})\sin(\phi t)\end{array}\right\}}{\left\{\begin{array}{l}\sin^2\theta_{13}\sin^2\theta_{23}\Big(\cos^2\theta_{12}\sin(\phi t-2\delta)\\ \qquad +\sin^2\theta_{12}\sin(\phi t+2\delta)\Big)\\ \qquad +\sin^2\theta_{23}\cos(\phi t)\\ \qquad +\cos^2\theta_{13}\cos^2\theta_{23}\cos(2q-1)\phi t\\ \qquad -\sin\delta\sin\theta_{13}\sin(2\theta_{12})\sin(2\theta_{23})\sin(\phi t)\end{array}\right\}}
\end{aligned} \tag{27}
$$

where $\zeta_\pm = (2q\pm 1)\phi t-\delta$.

IV. DISCUSSION AND CONCLUSION

Although we have restricted our computation to neutrino oscillations, the results can be extended easily to the case of neutron-anti-neutron oscillation [15]. Under certain circumstances, the measurement of geometric phase can be obtained more robustly in experiments and this idea of extending the noncyclic geometric phases of neutrino oscillation to $n - \bar{n}$ oscillation may provide an alternative experimental basis for detecting baryon number violation. Moreover, our results holds for oscillations of any mixed state bosons, for example Kaons, η' and so forth.

It is noteworthy to remark that Berry phase has recently been shown to exhibit essentially fault-tolerant behavior in quantum computation through NMR experiments [16]. In general, this fault-tolerant behavior holds for any geometric phase, be it adiabatic or non-adiabatic, cyclic or non-cylic. In a similar context, it has also been shown to be suitable for analyzing entangled quantum states [17].

In summary, we have calculated the non-cyclic geometric phases with both two-flavor and three-flavor mixing for the neutrino oscillations. If we set the time of measurement to the period of the oscillation, we recover the previous results found in ref [6]. Thus, our formulae naturally generalize the results for the Berry phase [6]. Finally, our formulae could have a potential application for determining the mixing angles of oscillating neutrinos and the CP violating phase.

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[1] B. Pontecorvo, Soviet Phys. JETP, **6**, 429 (1958).

[2] Y. Fukuda et al., Kamiokande collaboration, Phys. Lett. B, **335**, 237 (1994); Y. Fukuda et al, Phys. Rev. Lett., **81**, 1562 (1998).

[3] R. Becker-Szendy et al., IMB collaboration, Nucl. Phys. B, **38** (proc. suppl.) 331 (1995).

[4] W.W. M. Allison et al., Soudan collaboration, Phys. Lett. B, **335** 237 (1994).

[5] M. Ambrosio et al., Phys. Lett. B, **434**, 451 (1998).

[6] M. Blasone, P.A. Henning and G. Vitiello, Phys. Lett. B, **466**, 262 (1999)

[7] M V Berry, J. Mod. Optics, **34**, 1401 (1987)

[8] A G Wagh and V C Rakhecha, Phys. Lett. A, **197**, 107 (1995)

[9] Y.S. Wu and H.Z. Li, Phys. Rev. B **38**, 11907 (1988)

[10] S. Pancharatnam, see in *Geometric phase in Physics* (World Scientific, Singapore, 1989).

[11] A G Wagh and V C Rakhecha, Phys. Rev. Lett., **81**, (1998)

[12] H Weinfuter and G Badurek, Phy. Rev. Lett., **64**, 1318 (1990).

[13] I. Mendas and D.B. Popovic, J. Phys. A: Math. Gen., **26**, 3313 (1993).

[14] G.G. de Polavieja, Phys. Rev. Lett., **81** 1 (1998); G.G. de Polavieja and E. Sjöqvist, Am. J. Phys., **66**, 431 (1998).

[15] R.E. Marshak and R.N. Mohapatra, Phys. Lett. B, **91**, 222, (1980); R.N. Mohapatra, hep-ph/9604414.

[16] J. A. Jones, V. Vedral, A. Ekert and G. Castagnoli, Nature, **403**, 869 (2000).

[17] G.S. Agarwal, Opt. Comm., **87**, 193 (1992).