The Mathematics Educator
2014, Vol. 15, No. 2, 1-30

Generalisation of Linear Figural Patterns in Secondary School Mathematics

Boon Liang Chua
National Institute of Education
Nanyang Technological University
Singapore

Celia Hoyles
London Knowledge Lab
University of London
UK

Abstract: This paper reports on the performance of 104 Grade 8 Singapore students in pattern generalising tasks to determine the success rates, the rules formulated and the ways the rules were represented. A written test comprising four linear figural generalising tasks was used and the students had to answer all the four tasks in 45 min. About 70% of the students were successful in rule construction in each task, producing a variety of functional rules expressed prevalently in symbolic notations. Suggestions for practice based on these findings are suggested.

Keywords: pattern generalisation, linear generalising task, Singapore secondary school students

Introduction

Generalising is a fundamental and valuable skill in mathematics learning, with wide applications in many topics. For instance, stating that the sum of two odd numbers is always even can be considered a manifestation of the generalising skill in arithmetic. This skill is developed normally through the topic of number patterns in the mathematics curricula around the world. This topic offers students an opportunity to work with pattern generalising tasks requiring them to look for an underlying pattern structure, to use the structure to extend the pattern and to establish a rule that can be used to predict any term of the pattern. Research studies on pattern generalisation undertaken in different countries have suggested that recognising a valid underlying pattern structure is often not hard for students, but what poses a substantial challenge for many of them is articulating the rule in words or in algebraic notation (English & Warren, 1995; Rivera & Becker, 2007; Stacey & MacGregor, 2001; Ursini, 1991). Indeed, in Singapore, evidence has frequently emerged in the GCE "O" level[1] examiners' reports (Cambridge International Examinations, 2003, 2005, 2010), indicating that secondary school students in the Express course experience difficulties in formulating a rule for the immediate calculation of any term (i.e., output) in a pattern given its position (i.e., input).

For over a decade, pattern generalising tasks have been a feature in the Singapore school mathematics curriculum. The fact that fresh evidence about student difficulty in expressing generality keeps on emerging in the examiners' reports, even after all these years, is of grave concern. Yet there has been little discussion on secondary school students' abilities in generalising patterns, and this issue has not been systematically examined in depth. Thus, this paper seeks to answer the following research questions with respect to linear figural patterns:

(1) What were the success rates of the secondary school students in linear figural patterns?
(2) What rules did the secondary school students establish for a linear figural pattern?
(3) What was the modality of the rules that the secondary school students established?

Literature Review

This section describes what the generalisation process entails and why it is important for mathematics learning, followed by what linear figural generalising tasks are, as well as the different types of rules formulated and their modalities.

1 The General Certificate Examination at the Ordinary Level (GCE "O" Level) is a national examination conducted by the Cambridge International Examinations syndicate in collaboration with the Ministry of Education, Singapore. It is taken by students in the Express course at the end of their secondary education.

Generalisation

Generalisation has been widely acknowledged as a process involving at least one of the following activities:

(i) to examine a few particular cases to identify a commonality (Dreyfus, 1991; Ellis, 2007; Mason, 1996; Radford, 2008),
(ii) to extend one's reasoning beyond those particular cases (Ellis, 2007; Harel & Tall, 1991; Radford, 2008), and
(iii) to establish a broader result for those particular cases (Dreyfus, 1991; Dubinsky, 1991; Ellis, 2007).

Of crucial importance is the last activity wherein some researchers, such as Stacey and MacGregor (1997), regard the expression of a functional rule for describing the particular cases as evidence of generalisation.

The importance of generalisation in the field of mathematics is well established in the literature with some researchers going as far as to declare it as the heart of mathematics (Kaput, 1999; Mason, 1996). Such a claim is not surprising given the many applications of the generalising skill in the learning of mathematics, in particular, the topic of algebra, where laws and theorems are considered as generalisations (Mason, 1996). Taking, for instance, the commutative law for addition over integers, secondary school students should be able to express numerical identities such as $2 + 3 = 3 + 2$, $5 + 4 = 4 + 5$ and $6 + 9 = 9 + 6$ symbolically as $a + b = b + a.$ When this happens, they have developed a sense of the numerical identities and abstracted a relationship between the two integers (that is, the sum of the two integers is the same regardless of the order in which the integers are added). Articulating the symbolic identity is a manifestation of students' attempt in applying the generalising skill. One type of mathematical task that supports students' development of generalising skill is pattern generalising tasks.

Pattern generalising tasks

Pattern generalising tasks are often used in class by mathematics teachers to engage students in identifying a pattern, extending the pattern to find the value of a near or distant term, and articulating the functional relationship underpinning the pattern using symbols. They are classified broadly as *numerical* when the pattern is listed as a sequence of numbers, or as *figural* when the pattern is set in a pictorial context showing one or more configurations. Numerical tasks tend to list the first four or five terms of a pattern in a sequential order (see Hargreaves, Threlfall, Frobisher, &

Shorrocks-Taylor, 1999; Stacey, 1989). Figural tasks tend to show more variations. There are two widely used approaches: first, to provide three or more successive configurations (see Radford, 2008; Rivera, 2010), and second, to show just a single configuration to represent a generic case of the figural pattern (see Hoyles & Küchemann, 2001; Lannin, Barker, & Townsend, 2006a; Steele, 2008). Other less common approaches include providing two or three non-successive configurations (see Healy & Hoyles, 1999; Warren & Cooper, 2008).

The underpinning patterns are labelled *linear* or *quadratic* because their general terms take the form of $an + b$ or $an^2 + bn + c$ respectively. So the set of numbers [2, 4, 7, 11, 16, …] used in a study by Hargreaves, Threfall, Frobisher and Shorrocks-Taylor (1999) forms the first five terms of a quadratic numerical sequence because they can be generated by the expression $\frac{1}{2}n^2 + \frac{1}{2}n + 1$. On the other hand, the classic matchstick task (see Rivera & Becker, 2007) showing a square made of four matchsticks in Figure 1, a row of two squares made of seven matchsticks in Figure 2, and a row of three squares made of 10 matchsticks in Figure 3 is an example of a linear figural task in a successive format because the number of matchsticks needed to form *n* squares is given by $3n + 1$.

Pattern generalising tasks are viewed as a powerful and useful vehicle for promoting and supporting algebraic thinking. For instance, they can be used to introduce the notion of a variable (Mason, 1996; Warren & Cooper, 2008), to develop two core aspects of algebraic thinking: (i) the emphasis on relationships among quantities like the inputs and outputs (Radford, 2008), and (ii) the idea of expressing an explicit rule using letters to represent numerical values of the outputs (Kaput, 2008), and to develop the notion of equivalence of algebraic expressions (Warren & Cooper, 2008). With such merits pivotal to fostering algebraic thinking, it is not difficult to understand why pattern generalisation is normally placed under algebra in many countries including the US and Singapore.

Types of rules formulated

Students are often asked to construct a rule to describe the pattern structure that they see in a generalising task. As the review of a number of studies has shown, the rules constructed by the students take on mainly two forms: *recursive* and *functional* rules (see Lannin, Barker, & Townsend, 2006b; Rivera & Becker, 2007; Warren, 2005). The *recursive* rule allows the computation of the next term of a sequence using the immediate term preceding it whereas the *functional* rule refers to the rule expressed as a function that computes the term directly using its position in the sequence. For instance, a recursive rule for the matchstick task mentioned above could be expressed as "*add three to get the next term*" and $3n + 1$ is its functional rule.

Research has demonstrated that students tended to produce a recursive rule when the generalising tasks they were dealing with "provided a clear connection to incremental change" (Lannin et al., 2006a, p.12). The connection was often established when "the input values were relatively close" (Lannin et al., 2006a, p.12). This finding seems to be sensible because by building on the previous terms in a pattern, the recursive approach allows subsequent terms to be determined effortlessly. As a result, such an approach is particularly popular amongst many students, especially the younger ones (see Hargreaves et al., 1999). Unfortunately, while the recursive rule is useful for finding subsequent terms quickly, one of its serious drawbacks is that it is not efficient for the immediate calculation of any term whose position number is a large value or when the pattern is presented in a non-successive manner. Further, it also does not promote the ability to examine the functional relationship between the terms and their positions, a viewpoint which many researchers have argued is key to algebraic thinking (Kaput, 2008; Mason, Graham, & Johnston-Wilder, 2005; Radford, 2008). This is why formulating a functional rule is deemed so crucial and helpful to students.

Modality of written rules

The external representation of a mathematical idea can be conveyed in many ways: concrete materials, graph, words, and symbols. These different modes of representation are referred to as the *modality* of the mathematical idea. For a pattern generalising task, three modes are often used by students: purely symbolic, purely in words, and in alphanumeric form. Consider

Rivera and Becker's (2007) matchstick task mentioned previously again. A functional rule expressed entirely in symbols for the number of matchsticks needed to build a row of n squares is given by $3n + 1$. This rule can be stated wholly in words as: *add one to three times the number of squares*. Written alphanumerically, the rule can also take the form: $3 \times number\ of\ squares + 1$. Similarly, a recursive rule can also be represented wholly in words and in symbols: *add three to the previous term to get the next* and $T_n = T_{n-1} + 3$ respectively.

Some researchers specify very clearly how they want students to express the rule by introducing letters into the question. For instance, Rivera and Becker (2007) asked students to find the number of matchsticks needed to make *n* squares and such a question elicits a symbolic representation. Stacey and MacGregor (2001) required 2000 Australian students in Years 7 to 10 to produce symbolic functional rules but observed that only a small proportion of them could do so in two linear tasks.

However, not all pattern generalising tasks state explicitly that a symbolic rule is expected. This is especially so at the lower level of study when algebra has not been taught. Even if algebra has been taught, some students might not know how to use letters to describe the pattern structure. Thus for these students, a rule written wholly in words is equally acceptable. Stacey and MacGregor (2001) reported that nearly half of their sample of Years 7 to 10 students described the functional rules in words. Mavrikis, Noss, Hoyles and Geraniou (2012) noted a student using the alphanumeric form such as $2 \times model\ number + 4$, a combination of words and symbols.

Singapore students' performance in pattern generalisation

Due to a lack of research on pattern generalisation in Singapore, much of our knowledge about the performance of Express students in pattern generalisation is gleaned from the GCE "O" level examiners' reports and the TIMSS reports. A striking feature of the number pattern questions in the "O" level examinations from 1995 to 2009 is that they are primarily of the numerical type, with the exception of only two figural generalising tasks. The numerical questions tested students on two skills: *finding a particular term when its position is known* and *deriving an expression for the general term of the sequence*. These questions comprised both linear and quadratic sequences.

The examiners’ reports indicated that those questions dealing with finding a particular term when given its position in the sequence were consistently well answered. For instance, most Singapore students had no trouble stating the next two terms of the quadratic sequence [12, 11, 9, 6, …] in the 1996 examination (University of Cambridge Local Examinations Syndicate, 1997), and the 12th term of the linear sequence [25, 22, 19, 16, …] in the 2007 examination (Cambridge International Examinations, 2008). Even if the examination question took a different format, many students were still equally successful. Take, for instance, the 2009 examination question: *the first term in a sequence is 38 and each following term is found by subtracting 7 from the previous term.* Despite this atypical way of presenting the pattern in prose, the majority of students were still successful in finding the second and third terms of the sequence correctly. Only a small number of students interpreted the phrase “second and third terms” mistakenly to mean the second and third terms after 38 and so produced 24 and 17 as a result (Cambridge International Examinations, 2010).

Singapore students appeared to face very much the same difficulty as students in other countries did in examination questions asking for an expression for the general term of a linear sequence. The examiners’ reports commented consistently about students failing to formulate the correct algebraic expression. Taking, for instance, the 2002 examination question involving the linear sequence [5, 9, 13, 17, 21, …], many students gave the incorrect expression $n + 4$ for the n^{th} term (Cambridge International Examinations, 2003). Similar findings had also been found in the examiners’ reports for the 2004 and 2009 examinations (Cambridge International Examinations, 2005, 2010). Clearly, many Singapore students found the general term of linear sequences far from being straightforward to develop. Not surprisingly, then, the students also had limited success in developing quadratic functional rules correctly (Cambridge International Examinations, 2008; University of Cambridge Local Examinations Syndicate, 1997).

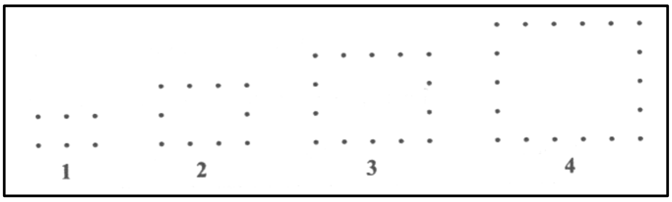
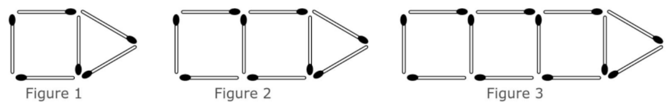


Figure 1. 1998 “O” level figural pattern

Notably, not all “O” level examination questions testing rule construction were poorly done. One question that the students had done well was the figural task from the 1998 examination in Figure 1 above. The linear algebraic rule for determining the number of dots around the perimeter of any configuration was derived correctly by the majority of students (University of Cambridge Local Examinations Syndicate, 1999).

The TIMSS findings similarly reveal some student weaknesses and misconceptions in pattern generalisation. The TIMSS–2003 matchstick task in Figure 2a asked Year 8 students (14 years old) to choose from five options the number of matchsticks needed to make Figure 10. This question was thought to be a straightforward item because the answer could be verified easily by drawing out the configuration in Figure 10. Yet only 73% of Singapore students chose the correct answer (B) (Martin, 2005). Although they outperformed the Year 8 students internationally (49%), it was still rather astonishing to discover that more than a quarter of the participating Singapore students failed to do it correctly. What is more surprising is that of the four wrong answers, (A) was the most popular response, selected by 11% of the Singapore students. The answer can be obtained using what Stacey (1989) called the *difference* strategy: that is, take the product of the figure number and the common difference. This students’ choice of answer (A) highlights the kind of misconception they have when making such a generalisation.

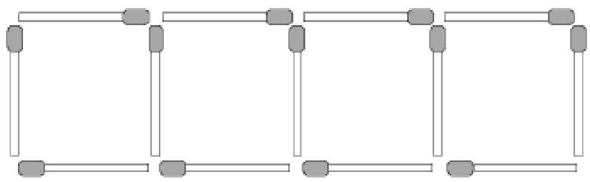
Matchsticks are arranged as shown in the figures.



If the pattern is continued, how many matchsticks would be used to make Figure 10?

Ⓐ 30
Ⓑ 33
Ⓒ 36
Ⓓ 39
Ⓔ 42

Figure 2a. TIMSS-2003 matchstick task (ID: M012017)



In the figure, 13 matches were used to make 4 squares in a row. What is the number of squares in a row that can be made in this way using 73 matches? Show the calculations that lead to your answer.

Figure 2b. TIMSS-2007 matchstick task (ID: M032640)

The TIMSS–2007 task in Figure 2b provided a single configuration showing a row of four squares made of 13 matchsticks and students were asked about the number of squares in a row that could be made using 73 matchsticks. To restate this task, it is asking students to find the figure number of a given term (i.e., Figure 4 is made using 13 matchsticks. Which figure is made using 73 matchsticks?). Such a task can be fairly tough for many students if they cannot make a link between the number of squares and the number of matchsticks. So it is not a surprise at all to realise that the task defeated the majority of students: 59% of Singapore students compared to 91% of Year 8 students internationally (Foy & Olson, 2009).

Figure 3 presents a TIMSS–2007 multiple-choice item involving rule construction. The item, resembling a typical GCE "O" level examination question, provided the first four terms of a sequence and students had to choose the rule that would generate each of these terms. It is encouraging to note that a vast majority of Singapore students (80%) picked the correct answer (B) (Foy & Olson, 2009). What is disappointing, however, is seeing a sizeable number of students failing to identify the correct rule especially when options were provided and could be verified easily. Of the four wrong answers, (C) was the top choice of Singapore students. The finding points to a worrying misunderstanding some students might have. That is, it would suffice to test the validity of a rule using just one or two cases. For answer (C), the fact that multiplying the first term 2 by 3 and then subtracting 1 to yield the second term 5 was enough to convince nearly 10% of the Singapore students to believe that the rule will also be valid for the remaining terms.

2, 5, 11, 23, ...

Starting the pattern at 2, which of these rules would give each of the terms in the number pattern above?

(A) Add 1 to the previous term and then multiply by 2.

(B) Multiply the previous term by 2 and then add 1.

(C) Multiply the previous term by 3 and then subtract 1.

(D) Subtract 1 from the previous term and then multiply by 3.

Figure 3. TIMSS-2007 item involving rule construction (ID: M032273)

Summary of literature review

From the review of research literature, we concur with many researchers that being able to generalise is very crucial for learning mathematics. This is because the generalising skill applies not only to algebra but also to several mathematical topics. More importantly, it is a key part of algebraic thinking. Yet several past studies undertaken in different countries have shown that expressing generality is notoriously elusive for many students. While many of them can often spot the underlying pattern structure in a generalising task, their formulation of the functional rule is not always guaranteed. Thus rule construction remains a persistent obstacle to many students around the world, including top-performing Singapore students as well.

Different generalisations emerge as a result of different student reasoning, structure interpretation and discernment in the task, thereby producing different equivalent forms of the rule. The literature review has indicated three ways of expressing a functional rule: purely in words, purely in algebraic notations or in alphanumeric form.

In Singapore, the topic of number patterns is not new in the secondary mathematics curriculum and has been taught since the 1990s. Despite this circumstance, recent evidence in the GCE "O" level examiners' reports and TIMSS studies highlight that student difficulties in pattern generalisation, particularly at the stage of formulating a functional rule, still remain very much in evidence. Due to limited research, the current state of Singapore secondary school students' generalisations of figural patterns is still not well understood and studied. As a result, it has not yet been established what rule they formulate and in what mode of representation they express the rule when they deal with those tasks. In this context, we conceptualised our research study to investigate the students' generalisations. Our study aimed not only to further broaden current knowledge of how Singapore secondary school students generalise figural patterns but also to complement the body of work on pattern generalisation that has been largely undertaken in the west.

Methods

This research study involved the collection of empirical data through the *Strategies and Justifications in Mathematical Generalisation* (*JuStraGen*) test. This section details the profiles of the participating students, the test instrument and the data analysis plan.

Participants

104 Secondary Two students (Grade 8, aged 14 years) from a secondary school in Singapore, selected through convenience sampling, were involved. They comprised 55 girls and 49 boys from three intact classes selected by the school and were taught by different mathematics teachers. Their mean PSLE (Primary School Leaving Examination, a national examination taken by 12-year-old students at the end of their primary education) aggregate score was 222, indicating below average ability.

The students had learnt the topic of number patterns before participating in this study. They should be able to continue for a few more terms any pattern presented as a sequence of numbers or figures, find near and distant terms in the sequence and establish the functional rule in the form of an algebraic expression for directly predicting any term. They should be familiar in dealing with linear patterns, which are commonly featured in their mathematics textbook.

Research instrument

The *JuStraGen* test was designed to assess students' ability to generalise figural pattern tasks. It was a paper-and-pencil test consisting of eight generalising tasks of which four involved linear patterns and four quadratic. The eight tasks were divided into two sets of four tasks, administered on two separate days to reduce student fatigue. Each set comprised two linear and two quadratic tasks. Students had to attempt all four generalising tasks in each set in 45 minutes. In this paper, only the four linear tasks as shown in Figure 4 are reported.

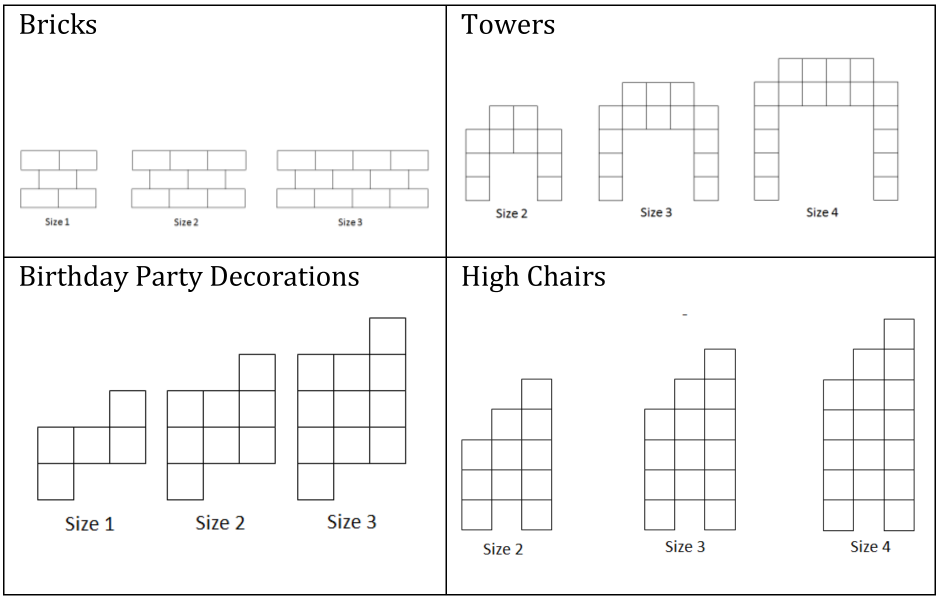


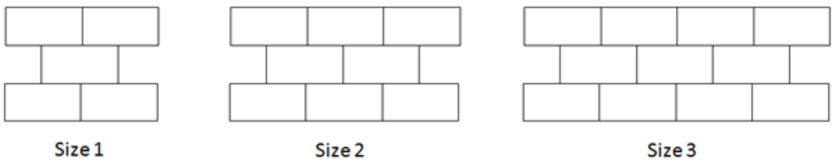
Figure 4. Linear generalising tasks

The development of the *JuStraGen* test was guided by numerous considerations. One of them is the number of generalising tasks to set. We decided to set eight tasks after pre-piloting a task to gauge the amount of time students needed to complete it. We believed that this number was also a reasonable figure for covering a range of linear and quadratic patterns. Another consideration is the structure of the task. All the eight tasks were deliberately unstructured in order to allow students scope for exploration so that they could come up with their own interpretations of the pattern. So there were no part questions guiding students to detect and construct the functional rule underpinning the pattern. This then permits us to see how the students came to recognise and perceive the pattern without scaffolding. The *Bricks* task in Figure 5 offers an example of the linear generalising tasks administered to the students.

The prototype version of the *JuStraGen* test was shown to over 15 secondary school mathematics teachers attending an in-service workshop on pattern generalisation conducted by the first author in September 2009, and later to two experts in mathematics education from the National Institute of Education in January 2010 before the pilot study began. The teachers and experts were asked to work out the functional rules for all the generalising tasks and all their rules matched our intended rules. Additionally, they also had to check that the generalising tasks were written with clear instructions and sufficient details. Some suggestions for improving the task instructions were provided by them and the test instrument was modified based on their feedback.

John used identical bricks to make several designs of different sizes on a long wall.

The diagrams below show three designs he made.



As the size number became larger, more bricks were used.

John wanted to find the number of bricks he had to use to make any size.

He used a rule to find this number.

(a) Write down the rule John might have used in terms of the size number.
(b) Describe clearly how you obtained the rule. You may use diagrams and words.

Figure 5. The Bricks task

The revised *JuStraGen* test instrument was piloted in a secondary school with 45 students. Based on the analysis of students' responses, the queries some students had, and observations made during the invigilation of the pilot test, none of the generalising tasks needed to be clarified or rewritten. The images of the configurations in every task were replaced with larger ones. The test duration of one hour seemed slightly long and was revised to 45 minutes.

Data analysis

The revised *JuStraGen* test was administered to the 104 participating students on two separate days in July 2011. The test scripts were collected and before coding the written student responses, each script was coded according to the student number in the register. The written student responses were first examined for correctness then followed by the rule formulated and the modality of the rule. Task by task, coding of each written response was carried out using the respective coding schemes for the type of rules formulated and the modality of the rule.

The coding scheme for the type of rules formulated was initially developed using *a priori* ideas drawn from the pilot study results, the solutions of the in-service teachers who checked the *JuStraGen* test instrument, and the first author's solutions using the various generalising strategies described in the research literature. Each different rule formed a category, which was assigned a three-digit code. The leftmost digit of the code indicates the task number in the *JuStraGen* test instrument. Taking the *Birthday Party Decorations* task for instance, the two correct but different-looking functional rules, $3n + 2$ and $3(n + 1) - 1$, were coded as 301 and 305 respectively. A correct recursive rule was coded as 320. As the coding process progressed, if a correct functional rule not matching any of the available codes was encountered, a new code was created. However, new but similar rules were subsumed under the same category. For instance, in the *Bricks* task, the new rule, $2n + (n + 1) + 1$, was regarded as similar to $2n + (n + 2)$, which had been assigned Code 105. So Code 105 was expanded to comprise the two rules.

The coding scheme for the modality of the rule was simpler and more straightforward to develop. From the pilot study, we observed that the recursive rule was typically expressed in words. On the other hand, the functional rules were stated in words, in algebraic notations or in alphanumeric form. Other than producing a recursive or functional rule, some student responses were partially correct, some described only particular cases, some were totally incorrect or irrelevant whilst some were left blank. The *description-of-particular-cases* category was created originally to account for those student responses that showed how particular cases were obtained. However, due to a low occurrence of such responses for each task, we decided to merge this category with the *partially correct rule* category, which accounted for responses that were incomplete but could possibly lead to a correct functional rule if done fully. The modality of the rule was finally narrowed to one of the following six categories: written *in words*, *written in notations*, *written in alphanumeric form*, *partially correct rule or description of particular cases*, *incorrect or irrelevant rule* or *blank*.

To illustrate how the coding was carried out, consider the responses of two students, Students 37M1 and 52M1, as provided in Figure 6. Figure 6a shows Student 37M1 moving the topmost row of tiles to the third row to form a $(n+1)$ by $(n+2)$ rectangle first, then followed by removing $(n-1)n$ tiles from the rectangle. The symbolic expression for the number of tiles in any configuration was, therefore, $(n+1)(n+2)-(n-1)n$, as seen in the rightmost diagram. Although this student simplified that initial rule to its closed form $4n+2$ later, it was the initial rule that reflected how the pattern structure was perceived. Hence, $(n+1)(n+2)-(n-1)n$, and not $4n+2$, was coded 606 for the type of rule formulated and 2 for the modality of rule.

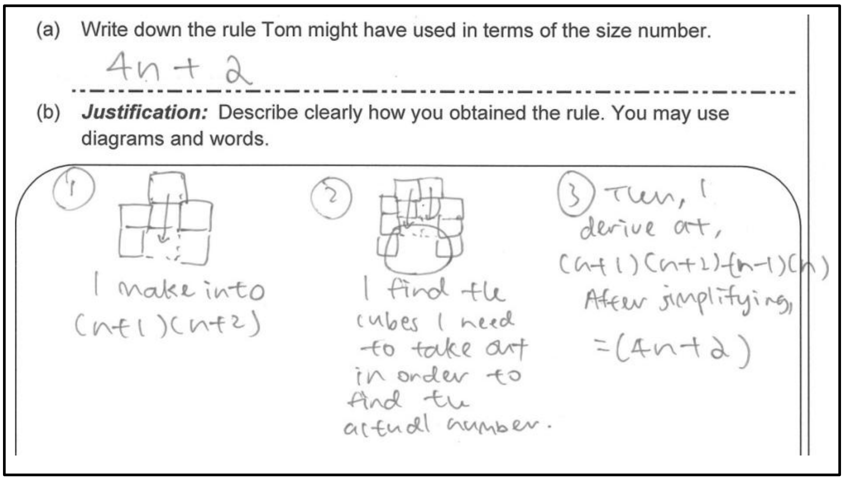
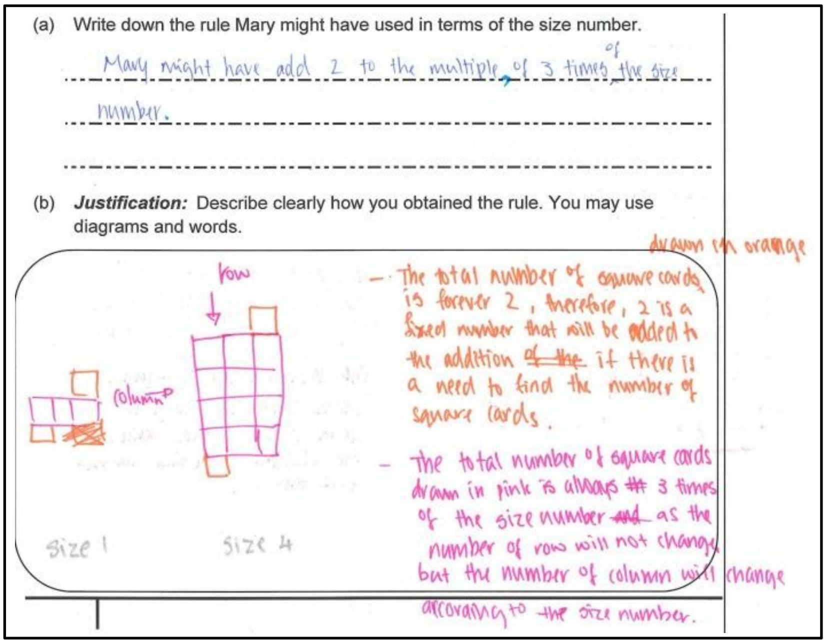


Figure 6a. Coding of student responses: Student 37M1


(a) Write down the rule Mary might have used in terms of the size number.

Mary might have add 2 to the multiple, of 3 times of the size number.

(b) *Justification:* Describe clearly how you obtained the rule. You may use diagrams and words.

drawn in orange

row

column

- The total number of square cards is forever 2, therefore, 2 is a fixed number that will be added to the addition ~~of the~~ if there is a need to find the number of square cards.

- The total number of square cards drawn in pink is always ~~##~~ 3 times of the size number ~~and~~ as the number of row will not change but the number of column will change according to the size number.

Size 1 Size 4

Figure 6b. Coding of student responses: Student 52M1

In Figure 6b, Student 52M1 viewed the configurations as being composed of three blocks. The top and bottom blocks each always had one card. The number of cards in the middle block was *always 3 times of the size number*. So the rule, *add 2 to...3 times of the size number*, was established by summing up the number of cards in the three blocks. This rule was entirely expressed in words, which translated to $3n + 2$ if written in notations. Thus it was coded 301 for the type of rule formulated and 1 for its modality.

Given the establishment of the respective codes for the type of rule formulated and the modality of the rule for all student responses, a small sample of the test scripts were selected and then passed to an experienced and retired mathematics teacher for coding to safeguard consistency in the coding process. Before starting to code, the mathematics teacher was trained by the first author to use the two coding schemes. The agreement level in coding between the first author and the mathematics teacher was over 90%, which we believed to be adequate for the present study.

Results

The analyses of data yielded a rich source of information about students' rules and the modality of their rules when dealing with linear figural generalising tasks. This section addresses the three research questions by reporting what has been found from the analyses.

(1) What were the success rates of the secondary school students in linear figural patterns?

All the generalising tasks in the *JuStraGen* test required students to express the rule specifically in terms of the size number. So a functional, and not recursive, rule was expected in the answer. Tables 1, 2 and 3 present the different functional rules that the students developed in each task and the frequencies of occurrence.

There was a predominance of functional rules. The success rates were 71%, 72%, 73% and 69% for *Bricks*, *Birthday Party Decorations*, *Towers* and *High Chairs* respectively.

(2) What rules did the secondary school students establish for a linear figural pattern?

Following an analysis of student responses for the type of rules generated, a remarkable result to emerge from the analysis is that approximately 70% of the student sample established a correct functional rule in each of the four linear generalising tasks. Under 15% of them produced a correct recursive rule. As explained previously, when two or more equivalent expressions of the functional rule were seen in a student response, the one that captured how the pattern structure was visualised was coded. So the rules did not have to be simplified and expressed in the closed form.

As Table 1 shows, six categories of different-looking but equivalent expressions of the functional rules were observed in *Birthday Party Decorations*. The student response presented in Figure 6b reveals a category of rules and the remaining five categories are $5 + 3(n - 1)$, $2(n + 1) + n$, $3(n + 2) - 4$, $3(n + 1) - 1$ and $2n + (n + 2)$. In the other three generalising tasks, five categories of functional rules were observed in

Bricks (refer to Table 1), nine in *Towers* (refer to Table 2) and seven in *High Chairs* (refer to Table 3).

Table 1
Functional rules for Bricks and Birthday Party Decorations by rule modality (n = 104)

Bricks		Modality				Birthday Party Decorations		Modality			
Code	Rule	W	N	WN	Total	Code	Rule	W	N	WN	Total
101	$3n+2$	1	39	3	43	301	$3n+2$	3	56	3	62
102	$5+3(n-1)$		6	1	7	302	$5+3(n-1)$		2	1	3
103	$2(n+1)+n$	1	19	1	21	303	$2(n+1)+n$		3		3
105	$2n+(n+2)$		2		2	304	$3(n+2)-4$		3		3
106	$3(n+1)-1$		1		1	305	$3(n+1)-1$		1		1
						306	$2n+(n+2)$		3		3
	Functional	2	67	5	74		Functional	3	68	4	75
	%	2	64	5	71		%	3	65	4	72
120	Recursive	8				120	Recursive	10			
	%	8					%	10			

Table 2
Functional rules for Towers by rule modality (n = 104)

Towers			Modality		
Code	Rule	W	N	WN	Total
601	$4n + 2$	1	27	1	29
602	$6 + 4(n - 1)$		1		1
603	$2(n + 1) + 2n$	1	33	2	36
604	$2n + (n + 2) + n$		1		1
606	$(n + 1)(n + 2) - n(n - 1)$		1		1
607	$10 + 4(n - 2)$		2		2
608	$2(2n + 1)$		1		1
609	$2(2n) + 2$		3		3
614	$3n + (n + 2)$		2		2
	Functional	2	71	3	76
	%	2	68	3	73
120	Recursive	14			
	%	13			

Table 3

Functional rules for High Chairs by rule modality (n = 104)

High Chairs			Modality			
Code	Rule		W	N	WN	Total
801	$3n + 5$		1	32	1	34
803	$2(n + 1) + (n + 3)$			14		14
804	$3(n + 1) + 2$			12	1	13
805	$3(n + 2) - 1$			6		6
807	$11 + 3(n - 2)$			2		2
812	$(3n - 1) + 6$				1	1
813	$4n + 4 - (n - 1)$			1		1
	Functional		1	67	3	71
		%	1	65	3	69
120	Recursive		14			
		%	13			

In three of the four generalising tasks, the majority of the correct functional rules were expressed in the closed form: $3n + 2$ in *Bricks* (43 students) and in *Birthday Party Decorations* (62 students), and $3n + 5$ in *High Chairs* (34 students). The only task whose closed form was not the modal rule was *Towers*. Of the 76 successful students, 36 of them constructed the rule, $2(n + 1) + 2n$, whilst another 29 derived the closed form, $4n + 2$.

A few functional rules in some of the tasks were particularly worth mentioning because of the thinking and reasoning that students engaged in when they formulated the rules. One prime example was the rule $2n + (n + 2)$ detected in *Birthday Party Decorations*, as shown in Figure 7. The rightmost generic configuration, drawn by the student and labelled by the first author, portrayed clearly how the student discerned and reasoned about the pattern structure in an intriguing manner. First, two identical rectangles, A and D, each comprising *n* square cards, were cut out from the first and third columns. The remaining portion of the configuration was further divided into two parts, B and C. B was a 7-shaped figure, containing n square cards and C was a two-square horizontal rectangle. Adding up the $2n$ square cards in A and D and the $(n+2)$ square cards in B and C produced the rule $2n+(n+2)$. This way of visualising the pattern structure is somewhat unconventional, hence it is worth highlighting.

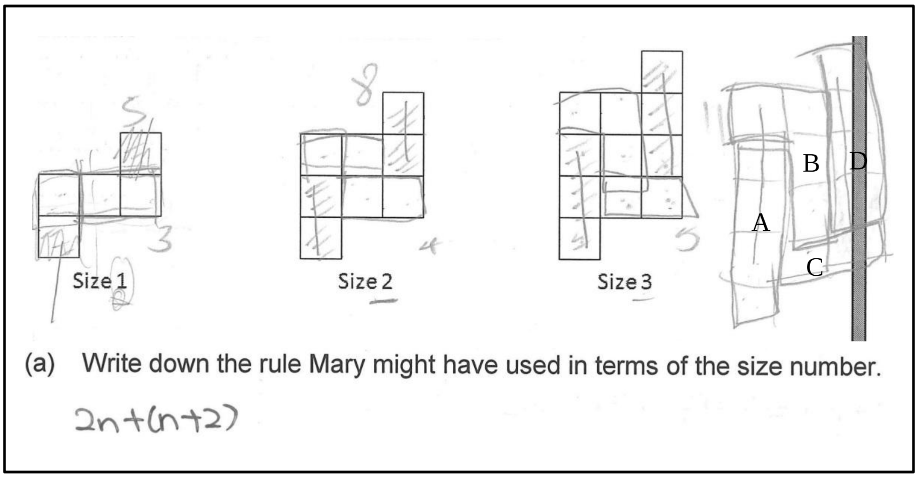


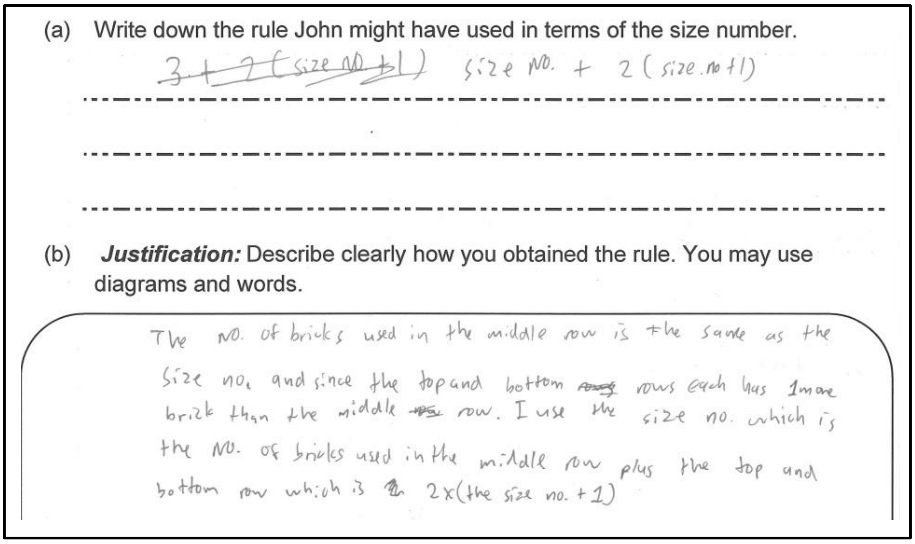
Figure 7. Visual representation of $2n+(n+2)$

Another example was the rule $(n+1)(n+2)-(n-1)n$ shown in Figure 6a above. By rearranging the original configuration into a rectangle with $(n+1)(n+2)$ tiles, Student 37M1 knew that this expression was not the correct formula for finding the actual number of tiles in any configuration. The rule needed adjustment, which was to remove $(n-1)n$ tiles from the rectangle. Hence, the rule for finding the number of tiles in any configuration was $(n+1)(n+2)-(n-1)n$. This rule takes on an interesting form, which might be easily mistaken for a quadratic function. However, it is actually a linear function.

(3) What was the modality of the rules that the secondary school students established?

The three ways of expressing a functional rule are *in words*, *in notations* and *in alphanumeric form*. Tables 1 to 3 present a detailed breakdown of the modality of the various functional rules that were established by the student sample. The most prevalent mode of representation was the *written in notations* category and the other two modes, on the other hand, were relatively infrequent.

The tables indicate that a substantial majority of the correct functional rules (almost 65%) were expressed in notations: 67 out of 74 in *Bricks*, 68 out of 75 in *Birthday Party Decorations*, 71 out of 76 in *Towers*, and 67 out of 71 in *High Chairs*. The student responses in Figures 6a and 7 are examples of two functional rules expressed in notations.


(a) Write down the rule John might have used in terms of the size number.

~~3 + 2 (size no +1)~~ size no. + 2 (size. no +1)

(b) ***Justification:*** Describe clearly how you obtained the rule. You may use diagrams and words.

The no. of bricks used in the middle row is the same as the size no. and since the top and bottom rows each has 1 more brick than the middle row. I use the size no. which is the no. of bricks used in the middle row plus the top and bottom row which is 2x(the size no. + 1)

Figure 8. Rule expressed in alphanumeric form

Very few functional rules in each task were expressed in a combination of words and notations. There were three cases each in *Towers* and *High Chairs*, four in *Birthday Party Decorations*, and five in *Bricks*. One of the five rules in *Bricks*, presented in Figure 8, showed a student using words instead of a letter such as n for the input variable in the rule: $size\ no + 2(size\ no + 1)$. Equally uncommon were functional rules written in words. There were just a few cases in each task: one in *High Chairs*, two each in *Bricks* and *Towers*, and three in *Birthday Party* Decorations. An example from *Birthday Party Decorations* has been provided in Figure 6b above.

Discussion

As detailed above, the present study had produced some noteworthy results from the data analyses. Given that the data were drawn from students in the Express course in a secondary school, the results should thus be treated with considerable caution.

As is widely known in the research literature, the construction of a functional rule is extremely challenging for many students, including those from Singapore. Many Singapore students had failed consistently to produce a correct algebraic expression for predicting any term of a linear numerical sequence in GCE "O" level examinations (Cambridge International Examinations, 2003, 2005, 2010). In contrast to these earlier findings, our study found that a significant number of students were capable of developing functional rules for linear figural patterns. This finding suggests that most students were familiar with the task requirement of formulating a functional, and not recursive, rule. Our finding also corroborates the results of a figural generalising task in the 1998 GCE "O" level examination (University of Cambridge Local Examinations Syndicate, 1999). How might one explain why students performed better in figural tasks than in numerical tasks?

A typical figural generalising task comprises the configurations as well as the input variables (i.e., figure number). To construct a functional rule for the inherent pattern in the task, students have to use the input variable as a *generator* of relationship (Chua, 2009) to establish a link with certain component of the configuration. When this relationship is established, it helps the students to better understand how the pattern grows with the input variable. Taking *Bricks* for instance, students will use the input value to establish the number of bricks in the top, middle and bottom rows in each configuration. This is where they will discover that each middle row always has the same number of bricks as the input value, and the top and bottom rows each have one more brick than the input value. Putting all the three rows together, the number of bricks in any configuration is three times the input value plus two. In this illustration, the link between the generator and the number of bricks in the configurations is said to be *explicit* (Chua, 2009).

In the case of a numerical generalising task, the input variables are not given explicitly, unlike in a figural task. What students will see are just the terms of a numerical sequence, but their position numbers can be inferred easily. This missing feature might have contributed to poor student performance in such a task because the generator-term link is not immediately conspicuous. Thus it is not surprising that students focus on the term-to-term relationship instead of the term-to-position relationship.

Adding further challenge to the numerical task is another latent difficulty inherent in the generator-term relationship. To illustrate this aspect of the relationship, consider the fourth term of a sequence whose first five terms are [1, 4, 7, 10, 13]. The manifestation of the *fourness* of the generator is not very obvious in the term 10 if this number stays in its present form. Unless students can deconstruct the number 10 in terms of the common difference between two consecutive terms into $1 + 3(4 - 1)$ or $4(3) - 2$ using the *repeated substitution* strategy, the relationship between the generator 4 and the term 10 will remain in oblivion. This generator-term obscurity can hamper students' recognition of the inherent pattern structure. This illustration thus highlights just how important the noticing of the generator-term relationship is to making a generalisation.

Another finding that broadens our knowledge of Singapore students' ability in pattern generalisation is that they were capable of formulating a variety of equivalent functional rules. The form of the functional rule often offered considerable insight into how they visualised, thought and reasoned about the pattern structure. Most of the functional rules are *algebraically meaningful* but some are probably not. By algebraically meaningful, we mean that a rule makes sense and can be explained using the numerical or figural cues established from the pattern. So the rules shown in Figures 6 to 8 are algebraically meaningful since each term of the rule has a geometrical interpretation. On the other hand, consider the functional rule $4n + [4 - (n - 1)]$ in *High Chairs*. No description was provided by the student to explain how it was derived. After much deliberation, we still could not figure out how each term of the rule was related to the configurations and so we came to believe that this rule was probably constructed through mere guessing. Given a lack of any geometrical interpretation, the rule is, therefore, deemed to be not algebraically meaningful.

Finally, the functional rules in the present study were predominantly expressed in symbolic notations. This finding affirms the level of competence of most students in one particularly important aspect of algebraic thinking: that is, using letters to represent numerical values. This is, in fact, a crucial skill in algebra learning which Kaput (2008) had hoped generalising tasks will help students to develop. Although the students expressed the symbolic rules very competently, a problem that showed up in many of their responses is their failure to explain what the letter used in their responses actually represented.

Implications for Instruction

In teaching pattern generalisation, it is often useful for teachers to begin the discussion of a generalising task by spending some time to have students explore and talk about the pattern that they see in the task and then search for the commonality. As the students are doing this, the teachers can advise them to focus on the structures and the relationship between terms, as emphasised by Radford (2008). To encourage students to articulate what they see in the pattern, they can be asked to predict some terms that are both near to and far from the last given term in the pattern. This is also an attempt to make them realise two things: (1) determining a near term will require them to know the term immediately preceding it and the differences between consecutive terms, and (2) the approach in (1) is not an efficient method for determining a far term because the term immediately preceding it may not be available. Crucially, the main intention here is eventually to have students recognise a need to devise a rule to compute the far term directly.

Before demonstrating the formulation of a functional rule, it is helpful if teachers can begin by explaining and emphasising the functional relationship between the input and output variables, which is key to successful rule formulation as the present study has established. For instance, in *Bricks*, there are five bricks in Size 1, eight bricks in Size 2, 11 bricks in Size 3, and so on. Next, students can be asked to identify the input and output values in the relationship. Once this is done, it is important for the teachers to make clear to the students about using the size number as a generator of relationship to connect it with the output variable. When establishing this relationship using the size number, we prefer to let the students decide how the size number is linked to the pattern. This learning experience not only provides them with a meaningful opportunity to explore the pattern structure but also convinces them that there are often multiple ways of explaining and visualising the same pattern structure. Furthermore, when asking the students to express the functional rule in symbols, the teachers can encourage them to choose their own letters for representing the size number if the question does not specify what letter to use. Importantly, have them to define their letters as well. Doing this helps students to "assign meaning to the formal letters and to appreciate the variable nature of these symbols" (English & Warren, 1995, p. 6).

With evidence from the present study demonstrating that students hold different interpretations of the pattern structure, together with considerable emphasis on making justification in Mathematics learning in Singapore in recent years, teachers can encourage students with a remarkable way of visualising the pattern structure to share their thinking and reasoning with others. Teachers can also present an algebraically meaningful functional rule and invite students to explain how it comes about. In doing so, students will hopefully realise that the rule does not occur by chance but it follows a certain pattern consistently.

Conclusion

This paper has given an account of the performance of Secondary Two students in pattern generalisation involving four linear figural generalising tasks. We arrived at the conclusion that the majority of students produced correctly a variety of functional rules that were predominantly expressed in algebraic notations. Our work clearly has some limitations: for instance, the student sample was small and was derived from one secondary school. Therefore, our findings may not be generalised to all secondary school students. Despite these limitations, we believe our work offers the first step towards enhancing our understanding of Singapore secondary school students' generalisations of figural patterns. In addition, findings from our study also contribute to the field of pattern generalisation as rich data to deepen one's understanding of how Southeast Asian secondary school students visualise, think and reason about the pattern structure, as well as to facilitate comparisons with previous findings already reported in the literature.

References

Cambridge International Examinations. (2003). *Report on the November 2002 Examination.* Cambridge: UK.

Cambridge International Examinations. (2005). *Report on the November 2004 Examination.* Cambridge: UK.

Cambridge International Examinations. (2008). *Report on the November 2007 Examination.* Cambridge: UK.

Cambridge International Examinations. (2010). *Report on the November 2009 Examination.* Cambridge: UK.

Chua, B. L. (2009). Features of generalising task: Help or hurdle to expressing generality? *Australian Mathematics Teacher, 65*(2), 18 - 24.

Chua, B. L., & Hoyles, C. (2012). The effect of different pattern formats on secondary two students' ability to generalise. In T. Y. Tso (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 155 - 162). Taipei, Taiwan.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25 - 41). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95 - 123). Dordrecht, The Netherlands: Dordrecht, The Netherlands: Kluwer Academic Publishers.

Ellis, A. B. (2007). Connections between generalising and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics Education, 38*(3), 194 - 229.

English, L. D., & Warren, E. A. (1995). General reasoning processes and elementary algebraic understanding: Implications for initial instruction. *Focus on Learning Problems in Mathematics, 17*(4), 1 - 19.

Foy, P., & Olson, J. F. (2009). *TIMSS 2007 International Database and User Guide.* Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.

Harel, G., & Tall, D. (1991). The general, the abstract, and the generic in advanced mathematics. *For the learning of mathematics, 11*(1), 38 - 42.

Hargreaves, M., Threlfall, J., Frobisher, L., & Shorrocks-Taylor, D. (1999). Children's strategies with linear and quadratic sequences. In A. Orton (Ed.), *Patterns in the teaching and learning of mathematics* (pp. 67 – 83). London: Cassell Publishers.

Healy, L., & Hoyles, C. (1999). Visual and symbolic reasoning in mathematics: Making connections with computers? *Thinking and Learning, 1*(1), 59 - 84.

Hoyles, C., & Küchemann, D. (2001). Tracing development of students' algebraic reasoning over time. In K. Stacey, H. Chick & M. Kendal (Eds.), *Proceedings of* *the 12th International Commission on Mathematical Instruction Study Conference: The Future of Teaching and Learning Algebra* (Vol. 2, pp. 320 - 327). Melbourne, Australia: University of Melbourne.

Kaput, J. (1999). Teaching and learning a new algebra with understanding. In E. Fennema & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133 – 155). Mahwah, NJ: Erlbaum.

Kaput, J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5 - 17). New York: Lawrence Erlbaum Associates.

Lannin, J. K., Barker, D. D., & Townsend, B. E. (2006a). Algebraic generalisation strategies: Factors influencing student strategy selection. *Mathematics Education Research Journal*, *18*(3), 3 - 28.

Lannin, J. K., Barker, D. D., & Townsend, B. E. (2006b). Recursive and explicit rules: How can we build student algebraic understanding? *Journal of Mathematical Behaviour*, *25*(4), 299 - 317.

Martin, M. O. (2005). *TIMSS 2003 User guide for the international database.* Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65 - 86). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Mason, J., Graham, A., & Johnston-Wilder, S. (2005). *Developing thinking in algebra*. London: Paul Chapman Publishing.

Mavrikis, M., Noss, R., Hoyles, C., & Geraniou, E. (2012). Sowing the seeds of algebraic generalization: Designing epistemic affordances for an intelligent microworld. *Journal of Computer Assisted Learning*, 1 - 17. doi: 10.1111/j.1365-2729.2011.00469.x

Radford, L. (2008). Iconicity and contraction: A semiotic investigation of forms of algebraic generalisations of patterns in different contexts. *ZDM: International Journal in Mathematics Education, 40*, 83 - 96.

Rivera, F. D. (2010). Visual templates in pattern generalisation activity. *Educational Studies in Mathematics, 7*(3), 297 - 328.

Rivera, F. D., & Becker, J. R. (2007). Abduction-induction (generalisation) processes of elementary majors on figural patterns in algebra. *Journal of Mathematical Behaviour, 26*(2), 140 - 155.

Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics, 20*, 147 - 164.

Stacey, K., & MacGregor, M. (1997). Building foundations for algebra. *Mathematics Teaching in the Middle School, 2*, 253 - 260.

Stacey, K., & MacGregor, M. (2001). Curriculum reform and approaches to algebra. In R. Sutherland, T. Rojano, A. Bell & R. Lins (Eds.), *Perspectives on school algebra* (pp. 141 - 154). Dordrecht, Netherlands: Kluwer Academic Publishers.

Steele, D. (2008). Seventh-grade students' representations for pictorial growth and change problems. *ZDM: International Journal in Mathematics Education, 40*, 97 - 110.

University of Cambridge Local Examinations Syndicate. (1997). *Report on the November 1996 Examination*. Cambridge: UK.

University of Cambridge Local Examinations Syndicate. (1999). *Report on the November 1998 Examination*. Cambridge: UK.

Ursini, S. (1991). First steps in generalisation processes in algebra. In F. Furinghetti (Ed.), *Proceedings of the 15th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 316 - 323). Assisi, Italy.

Warren, E. (2005). Young children's ability to generalise the pattern rule for growing patterns. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 305 - 312). Melbourne, Australia.

Warren, E., & Cooper, T. (2008). Generalising the pattern rule for visual growth patterns: Actions that support 8 year olds' thinking. *Educational Studies in Mathematics, 67*(2), 171 - 185.

Authors:

Boon Liang Chua (corresponding author), National Institute of Education, Nanyang Technological University, Singapore, boonliang.chua@nie.edu.sg;
Celia Hoyles, London Knowledge Lab, University of London, UK.