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Connecting the Dots in Task Design

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Mathematical connections are true relationships and are important for deep mathematical understanding. Connections are also important components of successful problem solving. At the heart of effective teaching and learning of mathematics is task design. This chapter unpacks the different types of mathematical connections through concrete examples to address aspects of task design that draw attention to connections. The relationships between task design, anticipated pedagogies and student learning are also illustrated for teachers to make links and connections explicit in the primary mathematics classrooms.

Keywords: Connections, task design, primary mathematics, journal writing

Introduction

Educational standards from various countries recommend that connections are made in the mathematics discipline. In Singapore, connection is one of the processes in the Singapore Mathematics Pentagon Framework and connections refers to “the ability to see and make linkages among mathematical ideas, between mathematics, and other subjects, and between mathematics and the real world.” (MOE, 2012, p. 15). The *Principles and Standards for School Mathematics (PSSM)* published by the National Council of Teachers of Mathematics (NCTM) highlights the importance of problem solving and establishing connections: “when students connect mathematical ideas, their understanding is deeper and more lasting” (NCTM, 2000, p. 64) and they come to view mathematics as a coherent whole. Department for Education (2013, p. 3) states that “Mathematics is an interconnected subject in which pupils need to be able to move fluently between representations of mathematical ideas”. The document also recommends that “pupils should make rich connections across mathematical ideas to develop fluency, mathematical reasoning and competence in solving increasingly sophisticated problems”. The Australian Curriculum, Assessment and Reporting Authority (ACARA, n.d.) states that one of the mathematics aims of the Australian curriculum is “to ensure that students recognise connections between the areas of mathematics and other disciplines and appreciate mathematics as an accessible and enjoyable discipline to study”. These curriculum materials draw our attention to the importance of making connections in mathematics. Indeed, the mathematical connections that students make are important tools for problems solving. Connections are also integral components of successful problem solving (Hodgson, 1995). Lawson and Chinnapan (2000) investigated the relationship between problem-solving performance and the quality of the organisation of Year 10 students’ knowledge between high-achieving (HA) and low-achieving (LA) students. They reported that connectedness indicators used in their study were more influential than content indicators in differentiating the groups on the basis of their success in problem solving.

Understanding and making connections are strongly linked. One of the features of mathematical understanding is the realization of connections and a result of making connections (Cai & Ding, 2017; Hiebert & Carpenter, 1992). In other words, "connections are the result of understanding, but also that understanding can be an action of making connections" (García-García & Dolores-Flores, 2018, p. 229). Viewed from this perspective, making connections play a fundamental part in achieving mathematical understanding especially when making connections among mathematical ideas becomes an important indicator of understanding (García-García & Dolores-Flores, 2018). That is, we can observe the connections that a person can demonstrate and infer their understanding from these observations (Barmby, Harries, Higgins, Suggate, 2009). However, Cai and Ding caution that mathematical understanding is a dynamic and continuous process with different levels and types of understanding. For example, three increasingly sophisticated levels of understanding of mathematical principles were highlighted by Greeno and Riley (1987), namely (i) conformity (procedural knowledge) (ii) implicit knowledge (the ability to tell the difference between examples and counterexample of a principle and analogical use of structure to map procedures (e.g. from concrete materials to algorithms) transfer and solve application problems) (iii) explicit knowledge (conceptual knowledge). Therefore, "a student who has some understanding will be able to make connections between ideas, concepts, procedures, representations and meanings" (García-García & Dolores-Flores, 2018, p. 229). To determine someone's understanding is not just a case of "looking at the number of connections that a person makes but the quality or strength of the connections as well" (Barmby et al., 2009, p. 221).

Mathematical Connections

Hiebert and Carpenter (1992) stated that "a mathematical idea or procedure or fact is understood if it is part of an internal network". That is, students would have understood the mathematics "if its mental representation is part of a network of representations" (p. 67). Translating what this means to a diagram would be part of a network structured like a spider's web (Eli, et al., 2013). The nodes or dots of the spider web stands for pieces of information of knowledge knots. The lines joining each dot are the connections or relationships (See Figure 1) - Just like how mathematics is understood by some experts as a complex system of relationships. This imagery is analogous to how Piaget perceived the schema to "the basic unit necessary for mental organization and mental functioning" (Tan et al., 2017, p. 66). Piaget (1952; also Flavell, 1963) defined a schema as "a cohesive, repeatable action sequence possessing component actions that are highly interconnected and governed by a core meaning" (As cited in Tan et al., p. 66). The development of a person's mental processes according to Piaget refers to the "increase in the number and complexity of the schemata that a person had learned" (Tan et al., p. 67). That is, the strength and cohesiveness of a schema is dependent on connectivity of components within the schema or between groups of schemata. Thus, in the Piagetian framework, making connections is a natural activity and through the processes of assimilation and accommodation, adaptation occurs - where the organism interacts with the environment and "develops schemata that enable it to continue to function in that environment" (Tan et al., p. 67).

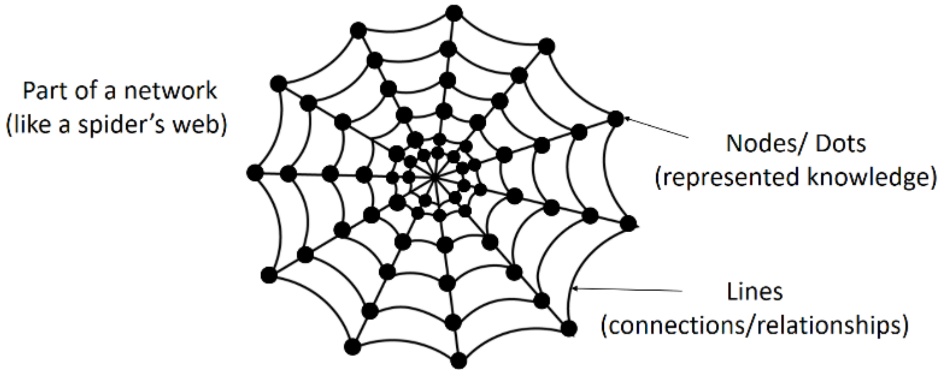


Figure 1. Mental modes.

A review of the literature revealed that several aspects were considered in the definitions of mathematical connections. Blum, Galbraith, Henn, and Niss (2007) suggested two major types of mathematical connections: (1) recognising and applying mathematics to contexts outside of mathematics (the links between mathematics, other disciplines or the real world); and (2) interconnections between ideas in mathematics. Eli et al. (2013) defined mathematical connection as "a link (or bridge) in which prior or new knowledge is used to establish or strengthen an understanding of relationship(s) between or among mathematical ideas, concepts, strands, or representations." (p. 122) Ekdahl et al. (2018) examined connection both within and between examples for Grade 3 mathematics. Montes et al. (2016) presented a characterization of connections as having six dimensions: (1) outside mathematics connections (e.g. modelling real-world situations); (2) intra-conceptual connections (various representation systems); (3) transverse connections (mathematical objects that can be displayed in different forms and context through different grade levels); (4) auxiliary connections (mathematics concepts not directly linked to the situation being discussed); (5) K-connections (simplification and complexification); and (6) Syntactic connections (for the generation of new knowledge e,g, reasoning, heuristic). Businskas (2008, p.18-19) proposed seven categories as a framework for thinking about what are connected when making mathematical connections: (1) alternate representations; (2) equivalent representations; (3) common features; (4) inclusion e.g. A includes or contains B such as a parabola contains a vertex; (5) generalization e.g. $ax^2 + bx + c = 0$ is a generalization of $2x^2 - 7x + 3 = 0$; (6) A implies B, logical relationships; and (7) procedure. García-García and Dolores-Flores (2018, p. 230) identified the following features of mathematical connections:

- Mathematical connections are true relationships and should be useful in the improvement of mathematical understanding
- A correct answer does not imply that the student makes mathematical connection, but the use of mathematical connections leads to consistent answers from the mathematical point of view
- Use of different representations is an important part of making connections
- Logical relationships are part of making mathematical connections
- The modelling of non-mathematical problems is also a type of mathematical connection
- The connections are a product of the belief system attributed to the student. Therefore, each student will make mathematical connections at a different level

Tasks Design from the Connected School Mathematics Curriculum Perspective

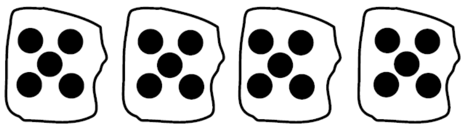
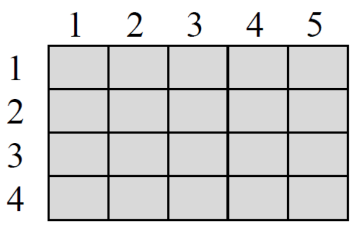
The intended School Mathematics Curriculum is a multidimensional connected curriculum that promotes seven types of connections (Lee et al., 2019) and encompasses the various definitions discussed in the preceding section. They are holistic connections, inter and intra conceptual-connections, conceptual-procedural connections, transitional learning connections, executive control connections and real-life connections. For the rest of this section, examples to illustrate how the seven connections contributes towards the design of meaningful tasks for the primary mathematics classrooms are provided.

Holistic connections

Holistic connections seek to address affective aspects of learning. One of the key features of the Singapore mathematics curriculum document is the explication of learning experiences which provides guidance to "teachers on the opportunities that students should be given part of their learning" (MOE. 2012, p. 6). Furthermore, "students attitude towards mathematics are shaped by their learning experience. Making the learning of mathematics fun, meaningful, and relevant goes a long way to inculcating positive attitudes towards the subject" (MOE, 2012, p. 17). Through careful design of the learning activities, teachers can provide positive learning experiences to develop a more positive attitude towards and in the process of learning mathematics. The following example focuses on the positive experience in learning mathematics and therefore encourages positive feeling towards the subject 'What have I learnt? What do I like about learning this topic?'

Inter-conceptual & intra-conceptual connections

Inter-conceptual connections promote connection to extend existing knowledge and skills (e.g., spiral curriculum) while intra-conceptual connections encourage connection to make sense of learning through multiple representations (e.g., Concrete-Pictorial-Abstract pedagogical approach). Through inter-conceptual and intra-conceptual connections, students develop conceptual understanding as they make sense of various mathematical ideas, their connections and applications. Comparing students' responses to "What do you know about 4 ×5?" in Figure 2, we observe that Student B's responses show connections between various mathematical ideas and representations (multiplication as repeated addition and grouping, areas and arrays, continuous and discrete models, multiplicative comparison, combinations).

<table>
<tr><td colspan="2">What do you know about 4×5?
Student A: 20</td></tr>
<tr><td>What do you know about 4×5?
Student B:
4 groups of 5
There are 4 groups but inside have 5
$4 \times 5 = 20$
$5 + 5 + 5 + 5 = 20$ is the same as $4 \times 5 = 20$

Bala has $5. Caili has 4 times as much as Bala. How much does Caili have?</td><td>Area of rectangle 4 unit × 5 unit = 20 unit^2

Ron has 4 different coloured shirts and 5 different coloured pants. There are $4 \times 5 = 20$ different outfits.</td></tr>
</table>

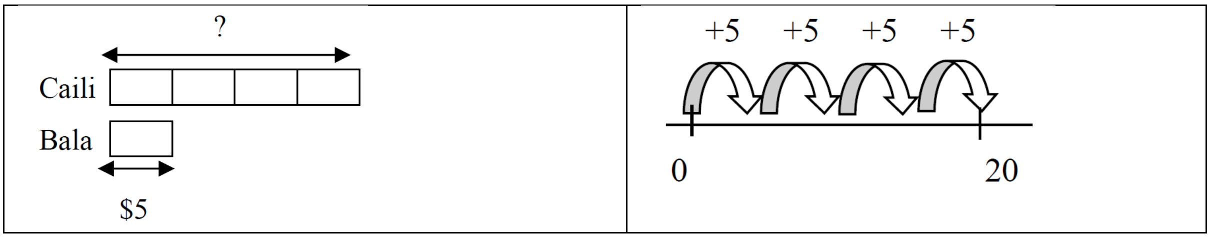


Figure 2. Students' responses to 4 × 5

In the Singapore primary mathematics curriculum, there are many instances where connections are explicitly spelt out in the learning experience. For example, students should have the opportunities to "recognise that a decimal is made up of a whole-number part and a fractional part, represent the decimal on a number line, and make connections between decimals, fractions and measurement" (MOE, 2012, p. 49). Figure 3 illustrates how a worksheet can be designed to facilitate these connections during the enactment of the lessons. Some cells are left intentionally blank for the mathematics teacher to unpack and make the connections explicit during the enactment of the lesson. We drew this task design move from Leong et al. (2019) and Leong et al. (2018) where several recommendations were made for mathematics teachers to "make explicit" several key ideas in the design of their instructional materials before and during enactment of a mathematics lesson.

Fractions		Decimals	
	$\frac{1}{10}$	1 tenth	0.1
	$\frac{2}{10}$	2 tenths	0.2
	$\frac{3}{10}$	3 tenths	0.3
	$\frac{4}{10}$		
	$\frac{10}{10}$	10 tenths	1.0
	$\frac{11}{10}$		1.1
		12 tenths	

Ones	Tenths	Hundredths
0	1	
0	2	
0	3	
1	0	
1	1	

Figure 3. Connections between decimals and fractions

Another learning experience in the Singapore primary mathematics curriculum is "represent equivalent decimals such as 0.2, 0.20 and 0.200, and explain that they are the same numbers" (MOE, 2012, p. 49). This can easily be proven using equivalent fractions $\frac{2}{10} = \frac{20}{100} = \frac{200}{1000}$. The use of percentage scale was suggested to "illustrate the part-whole concept of percentage, and to show the relationship between percentage and fraction" (MOE, 2012, p. 55) while linear scale was suggested to show the relationship between percentage and decimal" (MOE, 2012, p. 55). Aligning these two scales and pointing out the equivalence between the scales and representations can help students make sense of the connections across mathematical ideas. For example, in Figure 4 students can reason that $\frac{3}{10}$ is 3 out of 10 equal parts. $\frac{3}{10}$ can be expressed as 0.3 on the linear scale. $\frac{3}{10}$ is equivalent to 30 out of 100 equal parts and can be expressed as

30% and represented on the percentage scale. Once the connections between fractions, decimals and percentages are established, other mathematical ideas can be extended such as percentage of percentage and connections to fraction of fraction. For example, to solve the word problem *John spent 20% of his money on a T-shirt and 50% of the remaining money on a pair of shorts. What percentage of his money did he spend on the pair of shorts?,* students can use the percentage scale and part-whole model in Figure 5 to reason that 50% of the remaining can be represented by 50% of 8 remaining parts. This gives 4 out of 10 equal parts where 10 equal parts represent the total amount of money John has. This reasoning can be expressed as 50% of 80% and is equivalent to the expression $\frac{1}{2} \times \frac{4}{5}$.

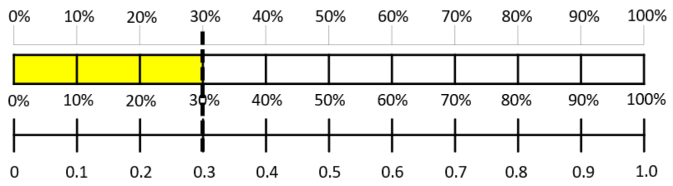
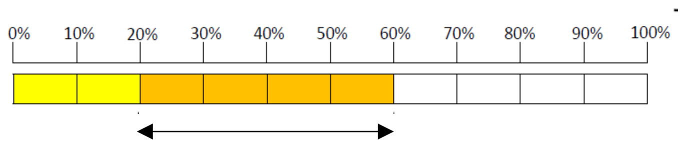


Figure 4. Connections between fractions, decimals and percentages.



$$50\% \times 80\% = 40\%$$
$$\frac{1}{2} \times \frac{4}{5} = \frac{4}{10} = \frac{40}{100} = 40\%$$

Figure 5. Connections between percentage of percentage and fraction of fraction.

In measurement topic, unit iteration is cornerstone to measurement reasoning. Several cognitive milestones were highlighted by Battista (2007, p. 903) such as maintaining, coordinating and locating the units, units properly organised into composites, iterations properly structured, operating on iterations, iteration schemes generalised into fractional schemes reasoning. This math idea is crucial to the reading of time, angles using protractor and various scales, for example, weighing scale, scales in measuring cylinders, beakers, picture graphs, bar graphs and line graphs.

Within graphical representation, several connections can also be made as diSessa, Hammer, Sherin, and Kolpakowski (1991) pointed out "one of the difficulties with conventional instruction ... is that students' meta-knowledge is often not engaged, and so they come to know how to graph without understanding what graphs are for or why the conventions" (1991, p. 157). Thus, students can be encouraged to "make connections between bar and line graphs and explain which type of graph should be used or both can be used" (MOE, 2012, p. 52). The following illustrates an example to exemplify opportunities to develop this connection: Anna drew a line graph to display her recent test scores for English, Mathematics and Science. Do you think a line graph is appropriate to present Anna's test scores? Why? Why not? When can we use line graph to present information? When designing instructional materials, opportunities for making connections across different types of graphic representations can be made explicit. For example, opportunities to make connections between pie charts and bar graphs to represent the favourite food (different ethnic food) of 240 adults (Lee, Koay, Collars, Ong, & Tan, 2018, p. 44). Such learning experience enable students to develop graph sense, the relationship between different types of graphs e.g. “data represented in circle graph can be displayed in bar graph but the reverse is not always the case” (Friel, Curcio & Bright, 2001, p. 128).

Conceptual-procedural connections

Conceptual-procedural connections promotes relational understanding (Skemp, 1976) with an emphasis on ‘the understanding of the underlying mathematical principles (Lee et al., 2019, p. 39). One challenging algorithm that some students face is the multiplication algorithm for 2-digit by 1-digit with regrouping. Different representations and materials, such as, base ten sets and number disc can be found in various instructional materials to illustrate the algorithm. Perhaps, if the various representations are meaningfully connected for relational understanding, the extension of the algorithm to larger numbers such as multiply 2-digit by 2-digit might make more sense for students (Figure 6).

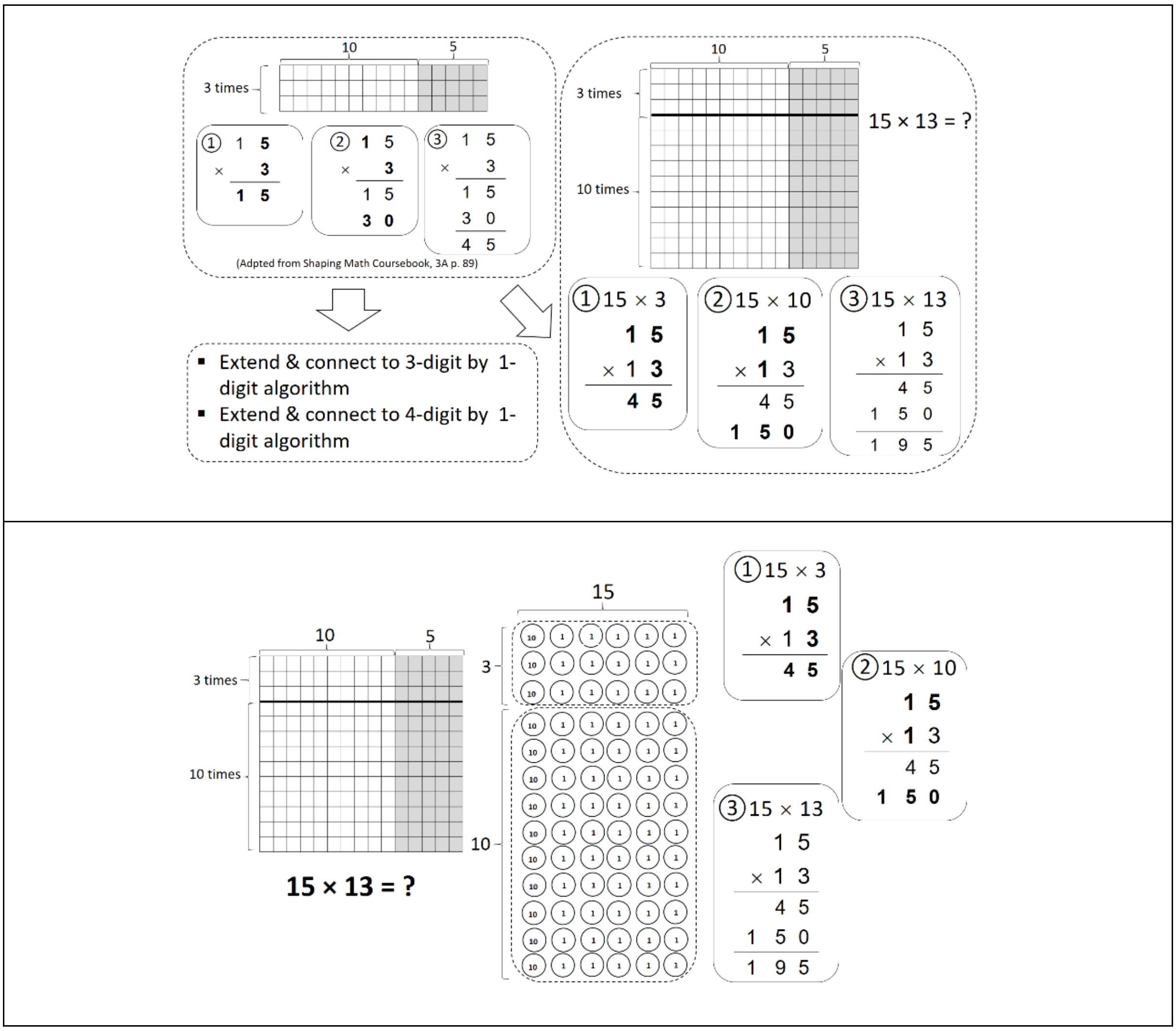


Figure 6. Representations for relational understanding.

The ability to translate from one representation of the concept to another promotes success in problem solving (Gagatsis & Elia, 2004). Thoughtful use of mathematical symbols and notations are also important to help students connect mathematical ideas. For example, the consistent use of various forms of crutches for whole number addition and addition algorithm can enhance students' procedural fluency and relational understanding. Figure 7a shows that the crutch being read as 15 ones in subtraction. However, in addition algorithm, it is treated as 1 ten plus 2 tens. Cheng and Ko (2011) suggested the consistent use of the crutch as its positional value for both the addition and subtraction algorithm (Figure 7b) to facilitate connections in the algorithms.

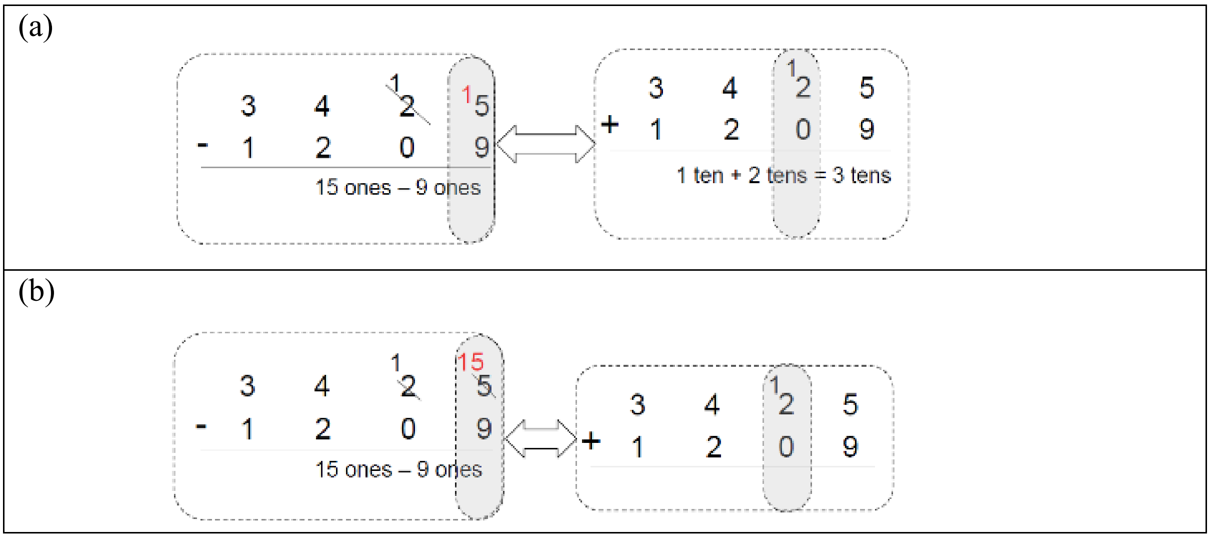


Figure 7. Consistent use of crutch to facilitate connections.

Transitional learning connections

The model method provides a strong connection between primary and secondary mathematics learning and addresses transitional connections (Lee et al., 2019). Figure 8 provides an example of transitional connection from primary to secondary mathematics using area model of multiplication (Figure 6) for the factorization of $x^2 + 5x + 6$.

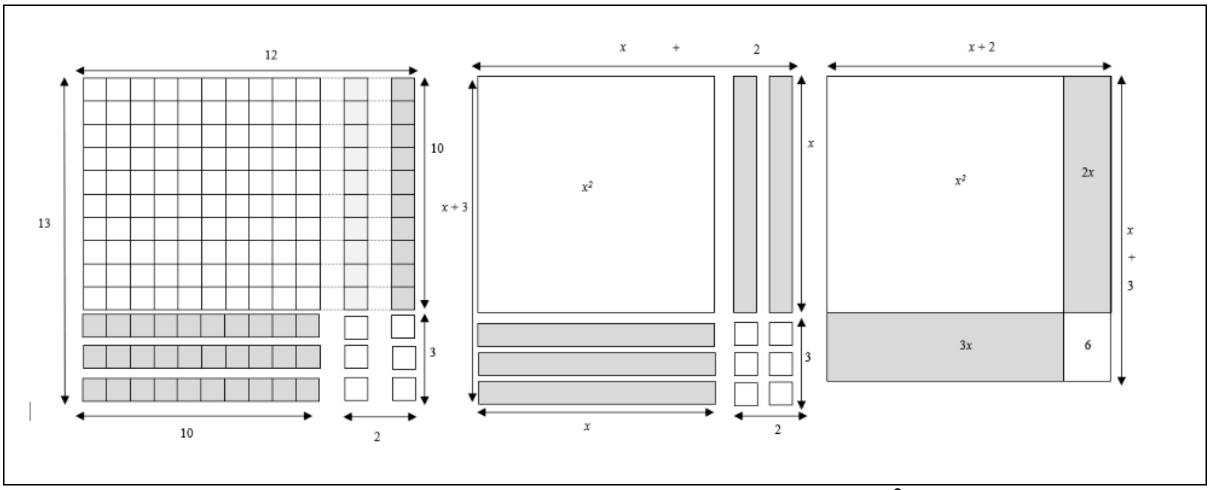


Figure 8. Area model to represent the factorization of $x^2 + 5x + 6$.

Executive control connections

Executive control connections involve metacognition or 'thinking about thinking', 'awareness of', 'monitoring of' and 'regulation of' one's thinking and learning (Lee et al., 2019). The use of problem solving frameworks such as The Problem Wheel (Lee, Chang & Lee 2001) and Pólya's (1957) four phases of mathematical problem-solving "guide students in exercising metacognitive knowledge, metacognitive monitoring and metacognitive regulation throughout the entire problem solving process" (Loh and Lee, 2017, p. 178). Several examples of Pólya's (1957) four phases of mathematical problem solving can be found in the instructional materials for primary mathematics. In teaching problem solving, teachers are encouraged to "demonstrate the use of Pólya's four-step problem solving strategy and models thinking aloud to make visible the thinking process" (MOE, 2012, p. 24).

Real-life connections

According to the introduction in Pappas (1989), "to experience the joy of mathematics is to realize mathematics is not some isolated subject that has little relationship to the things around us". In fact, Pappas pointed out that there is inseparable relationship between mathematics and the world. Real-life connections can be emphasised through the process of addressing real-world problems using mathematics, solving problems in real-world contexts and mathematical modelling (Lee et al., 2019). An example will be 'Take a look at some graphical representations from the newspapers. What are the various graphical representations that you can find? Discuss.' Although learners might make the connections spontaneously, Hodgson (1995) and Weinberg (2001), argued that teachers cannot assume that the connection will be made without some intervention. Some students may have difficulties in making connections (de Jong et al., 1998; Van Someren et al., 1998) and need to be supported in doing so (Ainsworth et al., 2002; Rau et al., 2014). Hodgson (1995) and Weinberg (2001), also highlighted that teachers play a critical role in making the different possible mathematical connections explicit to their students through instruction. One of the interventions in the cognitive psychology literature that supports students in making connections is sense-making problems (Rau et al., 2014). Understanding and sense-making processes involves "explicit, verbally mediated learning in which students attempt to understand or reason" (Koedinger et al., 2012, p. 775). They added that "sense making can be conceived as linking non-verbal with verbal forms of knowledge" (Koedinger et al., 2012, p. 775). Research on sense-making processes in connection making argue that "domain experts connect concepts across multiple visual representations" (Rau, 2018, p. 813). The multiple representations literature (e.g. Ainsworth, 2006) recognises the central role of sense-making processes in connection making and suggests several design principles for instructional interventions that support sense-making processes or problems (Rau, 2018). However, learners may not be able to "relate the multiple representations systematically" or even "detect relevant structures within representations" resulting in "disjointed mental representations" (Bodemer & Faust, 2006, p. 28).

Mathematics Journal Prompts Using the Connected School Mathematics Curriculum Perspective

Journal writing (Cheng & Koh, 2019) can be a powerful platform for students to engage in reflective review and make connections between mathematical ideas and between mathematics and other subjects (MOE, 2012, p. 22). Affordances of journal writing activities for teaching-learning-assessment links can be optimised when students are asked to explain their ideas clearly through multiple representations such as pictures, numbers and words. The design principles for mathematics journal prompts (content, process and affective prompts) discussed in Cheng and Koh (2019) provide very useful suggestions for teachers to design meaningful journal prompts for their mathematics classrooms. The Connected School Mathematics Curriculum Perspective presented in this chapter provides an innovative perspective into the design of journal prompts beyond content, processes, affective aspect of learning. The framework also suggests connections beyond the mathematics classrooms, support across grade levels and metacognition for even greater connections (See Figure 9).

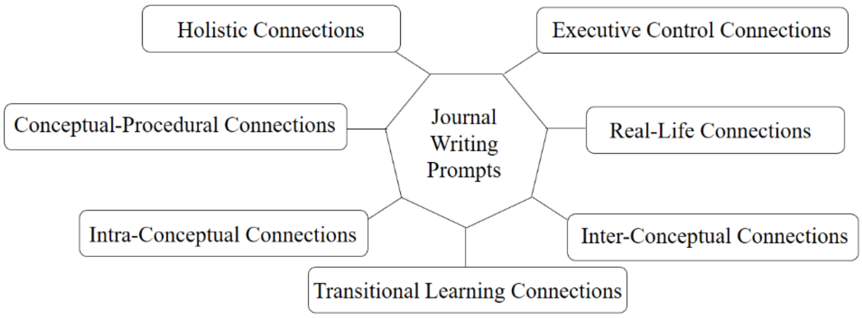


Figure 9. Designing journal writing prompts from the Connected School Mathematics Curriculum Perspective

Below, two journal prompts using the framework in Figure 9 are examined,
Journal Prompt 1: What do I know about the number 1965?
Journal Prompt 2: Mr Triangle is very sad because he does not look like Mr Rectangle. What could Mr Rectangle say to Mr Triangle to brighten up Mr Triangle?

In Journal Prompt 1, connection to the history of Singapore is made when students connect 1965 to the year when Singapore separated from Malaysia and became independent on 9 August 1965 - real-life connections. When students decompose 1965 into various part-whole relationships (e.g., decomposing into thousands, hundreds, tens and ones, expressing it in its equivalent form $1(1000) + 9(100) + 6(10) + 5(1)$, representing the decomposition using Diene's block, model diagrams etc., they are engaged in developing number sense. 1965 is divisible by 5 and this can be expressed as $1965 = 5 \times 393 = 5(300 + 90 + 3)$ leading to the distributive property in secondary mathematics (transitional connections). 1965 is an odd number and students can prove (e.g., through multiple representations) this using divisibility two rules and divisibility into two equal groups rules and connecting to odd and even alternating rules, unit digit rule, numbers in the two times table and counting on in twos rules (Frobisher, 1999) - conceptual-procedural connections, intra-conceptual connections, inter-conceptual connections. Understanding, planning, proving and checking the proof for 1965 as an odd number and divisibility by 5 require executive control connections. Through this activity, students can develop an appreciation of numbers and mathematics - holistic connections.

In Journal Prompt 2, students are required to reflect and make greater sense of triangles and rectangles, recognise and explain their similarities and differences using their understanding of the concepts (intra and inter conceptual connections). When students think beyond themselves (they had to cheer someone else, Mr Triangles) they are not just consumer of positive learning experiences, they also contribute to the positive learning environment – holistic connections. Understanding why Mr Triangles is getting upset, planning for what to say to Mr Triangles and checking one's explanation engages students in executive-control connections. Students can also relate triangles and rectangles to real-life situations (real-life connections) while answering this journal prompt.

Conclusion

This chapter attempts to discuss the various definitions of connections and presents the seven types of connections (Lee, Ng & Lim, 2019) with illustrations. Through the examples presented in this chapter, teachers can apply the potential of this framework in the design and enactment of various mathematical tasks and instructional materials for meaningful connections in their mathematics classrooms.

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