# On a certain vector crank modulo 7

Michael D. Hirschhorn

School of Mathematics and Statistics
University of New South Wales
Sydney, NSW, 2052, Australia

m.hirschhorn@unsw.edu.au

Pee Choon Toh*

Mathematics & Mathematics Education,
National Institute of Education, Nanyang Technological University
1 Nanyang Walk, Singapore 637616

peechoon.toh@nie.edu.sg

Submitted: Apr 25, 2014; Accepted: Jan 10, 2015; Published: Feb 9, 2015
Mathematics Subject Classifications: 11P83, 05A17

**Abstract**

We define a vector crank to provide a combinatorial interpretation for a certain Ramanujan type congruence modulo 7.

**Keywords:** partitions; congruences; crank

## 1 Introduction

In [7], one of the authors established several new Ramanujan type identities and congruences modulo 3, 5 and 7 for certain types of partition functions. For example, define $Q_{po,\overline{p}}(n)$ as the number of partitions of $n$ into two colors, where the red colored parts form a partition into odd parts and the blue colored parts form an overpartition. Using the standard notation

$$
\begin{aligned}
(a;q)_n &= \prod_{j=0}^{n-1}(1-aq^j),\\
(a;q)_\infty &= \lim_{n\to\infty}(a;q)_n,\\
(a_1,\ldots,a_m;q)_\infty &= (a_1;q)_\infty\cdots(a_m;q)_\infty,
\end{aligned}
$$

*Supported by the NIE Academic Research Fund RI 3/12 TPC.

for $|q| < 1$ and $a, a_1, \ldots, a_m \neq 0$, we can write the generating function of $Q_{po,\overline{p}}(n)$ as

$$\sum_{n=0}^{\infty} Q_{po,\overline{p}}(n) q^n = \frac{1}{(q;q^2)_\infty} \times \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{(-q,-q;q)_\infty}{(q;q)_\infty}.$$

Toh [7] proved that

$$\sum_{n=0}^{\infty} Q_{po,\overline{p}}(7n+2) q^n \equiv 0 \pmod 7. \tag{1}$$

Zhou [9] subsequently provided alternative proofs of all of the congruences in [7] with the exception of (1). She re-interpreted these partition functions as partitions into multi-colors, introduced what she termed as *multiranks* – which are essentially vector cranks as defined by Garvan [4] – and proved that these vector cranks divided the partitions into equinumerous parts. The aim of this article is to define a vector crank that will explain (1) combinatorially.

## 2 A vector crank

If $\lambda$ is a partition, we define $\sigma(\lambda)$ and $n(\lambda)$ as the sum of the parts and the number of parts of $\lambda$ respectively. Let $\mathcal{D}, \mathcal{O}, \mathcal{P}$ denote the sets of partitions into distinct parts, partitions into odd parts, and unrestricted partitions respectively. Define the cartesian product

$$\mathcal{V} = \mathcal{D} \times \mathcal{D} \times \mathcal{O} \times \mathcal{O} \times \mathcal{P} \times \mathcal{P}.$$

For a vector partition $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \in \mathcal{V}$ define a sum of parts $s$, a weight $w$ and a crank $r$ by

$$s(\vec{\lambda}) = 2\sigma(\lambda_1) + \sigma(\lambda_2) + \sigma(\lambda_3) + \sigma(\lambda_4) + 2\sigma(\lambda_5) + 2\sigma(\lambda_6), \tag{2a}$$

$$w(\vec{\lambda}) = (-1)^{n(\lambda_1)}, \tag{2b}$$

$$r(\vec{\lambda}) = 2n(\lambda_3) - 2n(\lambda_4) + n(\lambda_5) - n(\lambda_6). \tag{2c}$$

The weighted count of vector partitions of $n$ with crank $m$, denoted by $N_{\mathcal{V}}(m,n)$, is given by

$$N_{\mathcal{V}}(m,n) = \sum_{\substack{\vec{\lambda} \in \mathcal{V} \\ s(\vec{\lambda}) = n \\ r(\vec{\lambda}) = m}} w(\vec{\lambda}). \tag{3}$$

We also define the weighted count of vector partitions of $n$ with crank congruent to $k$ modulo $t$ by

$$N_{\mathcal{V}}(k,t,n) = \sum_{m=-\infty}^{\infty} N_{\mathcal{V}}(mt+k, n) = \sum_{\substack{\vec{\lambda} \in \mathcal{V} \\ s(\vec{\lambda}) = n \\ r(\vec{\lambda}) \equiv k \pmod t}} w(\vec{\lambda}). \tag{4}$$

Finally, we have the following generating function for $N_{\mathcal{V}}(m,n)$,

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} N_{\mathcal{V}}(m,n)z^m q^n = \frac{(q^2;q^2)_\infty(-q;q)_\infty}{(z^2q;q^2)_\infty(z^{-2}q;q^2)_\infty(zq^2;q^2)_\infty(z^{-1}q^2;q^2)_\infty}. \tag{5}$$

**Theorem 1.** *The following equation holds for all nonnegative integers $n$.*

$$N_{\mathcal{V}}(0,7,7n+2) = N_{\mathcal{V}}(1,7,7n+2) = \cdots = N_{\mathcal{V}}(6,7,7n+2) = \frac{Q_{po,\overline{p}}(7n+2)}{7}.$$

The main ingredient in the proof of the theorem is Winquist's identity [8], which is a variant of the $B_2$ case of the Macdonald identities [5]. We state the identity in the following symmetric form [6, Eq. (3.1)]. If we define

$$F_1(x) = \sum_{j=-\infty}^{\infty}(-1)^j q^{3j^2}(x^{3j}+x^{-3j}), \tag{6a}$$

$$F_2(x) = \sum_{k=-\infty}^{\infty}(-1)^k q^{3k^2+2k}(x^{3k+1}+x^{-3k-1}), \tag{6b}$$

we have

$$F_1(x)F_2(y) - F_1(y)F_2(x) = -\frac{2}{x}\left(xq, \frac{q}{x}, yq, \frac{q}{y}, xy, \frac{q^2}{xy}, \frac{x}{y}, \frac{yq^2}{x}, q^2, q^2; q^2\right)_\infty. \tag{6c}$$

*Proof of Theorem 1.* If we set $\zeta = \exp(2\pi i/7)$ in (5), we obtain

$$\begin{aligned}
&\sum_{t=0}^{6}\zeta^t\sum_{n=0}^{\infty}N_{\mathcal{V}}(t,7,n)q^n \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}N_{\mathcal{V}}(m,n)\zeta^m q^n \\
&= \frac{(q^2;q^2)_\infty}{(q,\zeta^2q,q/\zeta^2,\zeta q^2,q^2/\zeta;q^2)_\infty} \\
&= \frac{(\zeta q,q/\zeta,\zeta^3q,q/\zeta^3;q^2)_\infty}{(q^7;q^{14})_\infty}\times\frac{(q^2,q^2,\zeta^2q^2,q^2/\zeta^2,\zeta^3q^2,q^2/\zeta^3;q^2)_\infty}{(q^{14};q^{14})_\infty} \\
&= \frac{F_1(\zeta^3)F_2(\zeta)-F_1(\zeta)F_2(\zeta^3)}{2\zeta(1-\zeta^2)(1-\zeta^3)(q^7;q^7)_\infty},
\end{aligned}$$

where we used (6c) with $x=\zeta^3$ and $y=\zeta$.

Since $3j^2 \equiv 0,3,5,6 \pmod 7$ and $3k^2+2k \equiv 0,1,2,5 \pmod 7$, the power of $q$ in $q^{3j^2+3k^2+2k}$ is congruent to 2 modulo 7 exactly when $j \equiv 0 \pmod 7$ and $k \equiv 2 \pmod 7$. This means that the coefficient of $q^{7n+2}$ in

$$F_1(\zeta^3)F_2(\zeta) - F_1(\zeta)F_2(\zeta^3)$$

is zero since

$$(-1)^{j+k}(\zeta^{9j}+\zeta^{-9j})(\zeta^{3k+1}+\zeta^{-3k-1})-(-1)^{j+k}(\zeta^{3j}+\zeta^{-3j})(\zeta^{9k+3}+\zeta^{-9k-3})=0$$

when $j \equiv 0 \pmod 7$ and $k \equiv 2 \pmod 7$. Thus

$$\sum_{t=0}^{6} N_{\mathcal{V}}(t,7,7n+2)\zeta^t = 0. \tag{7}$$

Since the minimal polynomial for $\zeta$ over the rational numbers is

$$p(x) = 1 + x + x^2 + \cdots + x^6,$$

we conclude that

$$N_{\mathcal{V}}(0,7,7n+2) = N_{\mathcal{V}}(1,7,7n+2) = \cdots = N_{\mathcal{V}}(6,7,7n+2). \qquad \square$$

We end by indicating how one may prove (1) directly as the details were omitted in [7]. This can be done by observing that

$$\begin{aligned}\sum_{n=0}^{\infty} Q_{po,\overline{p}}(n)q^n &= \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^3} \\ &\equiv \frac{(q^2;q^2)_\infty^9}{(q;q)_\infty^3} \times \frac{1}{(q^{14};q^{14})_\infty} \pmod 7.\end{aligned} \tag{8}$$

Thus (1) is equivalent to proving the coefficients of $q^{7n+2}$ in

$$\frac{(q^2;q^2)_\infty^9}{(q;q)_\infty^3}$$

are all divisible by 7. We offer three alternative ways of doing this. The easiest way is to appeal directly to [3, Th. 2]. Alternatively, we can use one of the Macdonald identities associated with the $C_2^\vee$ root system [5, p. 137] or [6, Eq. 3.12], to express

$$\frac{(q^2;q^2)_\infty^9}{(q;q)_\infty^3} = \sum_{\substack{\alpha\equiv 1 \pmod 8\\ \beta\equiv 3 \pmod 8}} \frac{1}{8}(\beta^2-\alpha^2)q^{\frac{\alpha^2+\beta^2-10}{16}}.$$

If the exponent of $q$ is congruent to 2 modulo 7, we have

$$\alpha^2+\beta^2 \equiv 16(2)+10 \equiv 0 \pmod 7.$$

Since $-1$ is a quadratic nonresidue modulo 7, 7 must divide both $\alpha$ and $\beta$. The third way is to apply the Hecke operator $T_7$ to $\frac{\eta(16\tau)^9}{\eta(8\tau)^3}$, a weight 3 cusp form of level 128. One can refer to [1] for examples of how this may be done.

**Acknowledgements**

We thank the anonymous referee for helpful comments that improved the article. In particular, the referee pointed out that the coefficients $N_{\mathcal{V}}(m,n)$ are nonnegative which can be proved using the following corollary [2, Eq. (4.1)] of the $q$-binomial theorem.

**Proposition 2.** *If* $|q|, |t| < 1$ *then*

$$\frac{(at;q)_\infty}{(a;q)_\infty (t;q)_\infty} = \frac{1}{(a;q)_\infty} + \sum_{n=1}^{\infty} \frac{t^n}{(aq^n;q)_\infty (q;q)_n}.$$

## References

[1] S. Ahlgren. Multiplicative relations in powers of Euler's product. *J. Number Theory*, 89:222–233, 2001.

[2] A. Berkovich, and F. G. Garvan. K. Saito's Conjecture for nonnegative eta products and analogous results for other infinite products. *J. Number Theory*, 128:1731–1748, 2008.

[3] S. Cooper, M. D. Hirschhorn, and R. Lewis. Powers of Euler's product and related identities. *Ramanujan J.*, 4:137–155, 2000.

[4] F. G. Garvan. New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7 and 11. *Trans. Amer. Math. Soc.*, 305:47–77, 1988.

[5] I. G. Macdonald. Affine root systems and Dedekind's $\eta$-function. *Invent. Math.*, 15:91–143, 1972.

[6] P. C. Toh. Generalized $m$th order Jacobi theta functions and the Macdonald identities. *Int. J. Number Theory*, 4:461–474, 2008.

[7] P. C. Toh. Ramanujan type identities and congruences for partition pairs. *Discrete Math.*, 312:1244–1250, 2012.

[8] L. Winquist. An elementary proof of $p(11m+6) \equiv 0 \pmod{11}$. *J. Combin. Theory*, 6:56–59, 1969.

[9] R. R. Zhou. Multiranks for partitions into multi-colors. *Electron. J. Combin.*, 19:#P8, 2012.