MATHEMATICS THROUGH THE LENS OF GEOMETRY

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SYNOPSIS

How do we create an enriching learning environment that promotes learning? One way to move our students from learning mathematical concepts and skills in isolation to seeing how they are inter-linked is a paradigm shift that will cause our students to learn mathematics holistically and even to appreciate its intrinsic beauty. This paper discusses how mathematics teachers can stimulate the visual thinking or geometrical intuition of our students by looking at algebraic problems through the lens of geometry.

BACKGROUND

Visual thinking in Mathematics is a powerful way to aid analytical understanding of mathematics. As teachers we should make full use of it in our teaching and as well as to help our students cultivate it. There used to be more geometry in our mathematics syllabus, though not now. What we are concerned with is not just the content of geometry but more importantly the geometrical approach to problem solving which is not very frequently employed by mathematics teachers. For example, the geometrical approach is able to help pne explore the use of graphs beyond representation of algebraic equations to providing convincing visual proofs without words. It goes beyond getting the solutions to an algebraic problem to giving the student a visual understanding of the nature of the solution, e.g. real or imaginary roots. In othe words, mathematics should be looked at through the lens of geometry more often than what we are doing now. We begin by exploring the historical development of geometry to help us appreciate the current reality in the schools.

Historical Development

Geometry was taught for hundreds of years as an axiomatic system. That is, we begin with some undefined terms such as points and lines, we use these underfined terms to formulate some statements without proofs such as the parallel postulate, and we call them axioms. Then we proceed to prove the properties in geometry as theorems. The approach was not very different from what Euclid did in his famous book Elements, written over 2 000 years ago. Classical geometry and algebra deal with finite operations, whereas calculus involves infinite operations, namely, limits. In order to provide a sound foundation for operations involving infinity, in the nineteenth century, pure mathematics was born. Pure mathematics emphasizes on rigor. In other words, every statement must be proved. Henceforth mathematics as it evolved was taught in the sequence of definitions, theorems, and proofs. That is, mathematics was supposed to be a rigorous science. The approach by Euclid was not rigorous. The results in Euclidean geometry cannot be proved logically with the axioms given. As it happened, Hilbert and others provided a rigorous approach to Euclidean geometry. Using the axiomatic system provided by Hilbert we can deduce logically all the results in Euclidean geometry. Hilbert's axioms fill up 4 pages classified into 5 groups. In a way, Hilbert saved Euclidean geometry and as a consequence also killed it. What was being done in schools was no longer mathematically correct. As it was being taught, it is not an axiomatic system as required. To make it mathematically correct, the approach was too difficult for the schools. The maturity required went beyond the level of the students. Hence a substantial amount of geometry was taken out of the Syllabus C after the Mathematics Reform in the 60's. The present Syllabus D was inherited from Syllabus C and therefore, like Syllabus C, has much less geometry than before.

Current Situation

What is lost is not so much the content of geometry but the rich learning environment that geometry used to provide when it was studied in the traditional way. Several factors create a rich learning environment. It is one that causes students to see different perspectives of one topic. In the study of mechanics, the scenarios or context in which the principles of mechanics that can be applied is very wide. Hence there will be many ways a teacher can test a student the concepts of mechanics, with varying or differing levels of difficulty. The second factor that makes a rich learning environment is challenge. As the possibilities of seeing the same subject matter from different perspective increases the challenge to understand and solve new problems increases as well. In this kind of environment, there is no one set answer to any problem, and students have to engage in higher order thinking of analysis and synthesis. In short the third factor that makes a rich learning environment is one that promotes thinking. In the same way, the study of mechanics provides a rich learning environment whereas numerical analysis does not, at least at the school level. Teaching of proof is no longer restricted to geometry. There is a model, known as Van Hiele levels, describing five levels of learning difficulty in school geometry described by Grinstein & Lipsey (2001). The last two levels refer to geometrical proofs. Since the proof no longer plays the same role in school geometry as before, it renders Van Hiele levels obsolete. We have to re-think what Mathematical proof is and how it can be taught. Furthermore, geometry should not be synthetic geometry alone, which is geometry without reference to analytical methods. Despite the many attempts to reform school geometry, there has been no consensus on what school geometry should be and how it should be taught. Recently, there is a revival of geometry at the undergraduate level. There is a growing interest in geometry as people begin to see more of it in other Mathematics fields as well as its practical application to technology. For example, projective geometry discussed by Silvester (2001) deals with the theoretical framework of representing three dimensional images in a two dimensional plane.

DESCRIPTION OF STRATEGY

The analytical approach to geometry is powerful for stimulating the visual intuition of our students as it provides them a rich environment to learn Mathematics. We should regard it as part of geometry. Geometry should never be taught in isolation of other topics in mathematics. Geometry should be linked with algebra and algebra with geometry. Indeed, there is geometry in algebra and algebra in geometry. This is a paradigm that mathematics teachers have to shift to in order to teach mathematics holistically. Learning in the 21st century should be holistic in nature as there are many inter and intra connections among various disciplines. When learning takes place this way, our students will be able to create new knowledge by making meaningful connections among seemingly diverse fields.

Descartes invented the coordinates system in the seventeen century. It served as a bridge between the land of geometry and the land of algebra. However the migration through the bridge has been mostly one way, namely, from geometry to algebra. For example, vectors are now regarded as algebra, though it was geometry before. We should make it a two-way approach, i.e. to see algebra through geometrical pictures. The geometrical approach is a useful tool in teaching. For example, the use of visual verification can be very convincing for our students. We should make better use of the geometrical approach.

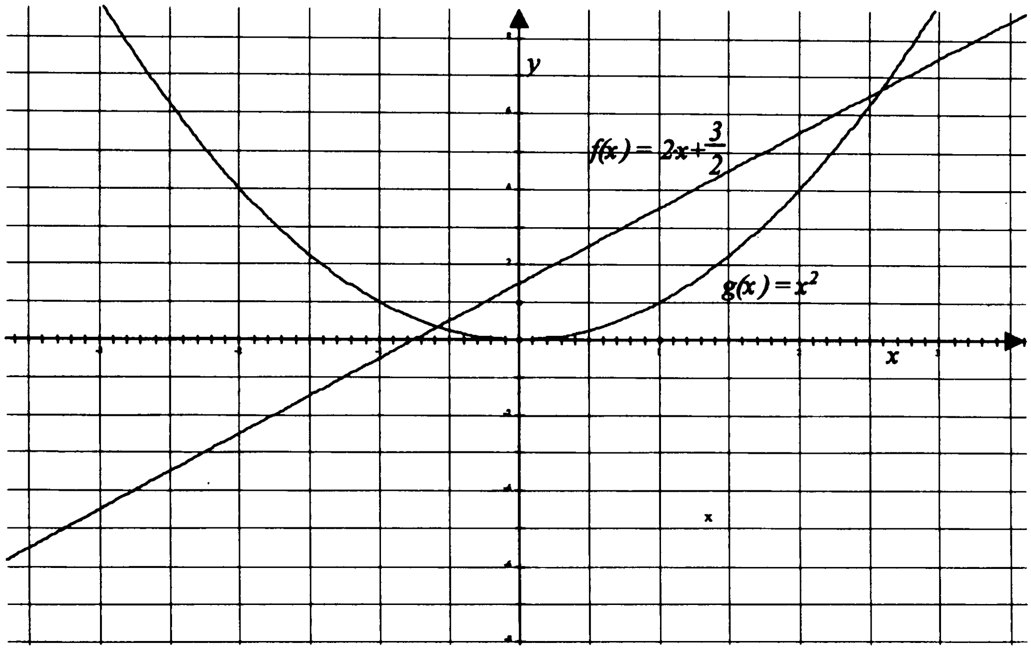
The following examples are given to show how one can view algebra through the lens of geometry. Though not every algebraic problem can be turned into a geometrical picture, some do as shown in the following examples. Hence when we have an algebraic problem we ask whether we can draw a picture of it. Then ask whether we can solve it geometrically. If we try it often enough, we should be able to see more clearly the geometry within an algebraic problem. Hence it begins by creating a awareness in ourselves and our students that it is indeed possible to solve algebraic problems geometrically. This needs to be followed by constant and conscientious practice to develop the geometrical thinking in our students.

PROCESS

A. Graphical solution of quadratic equations

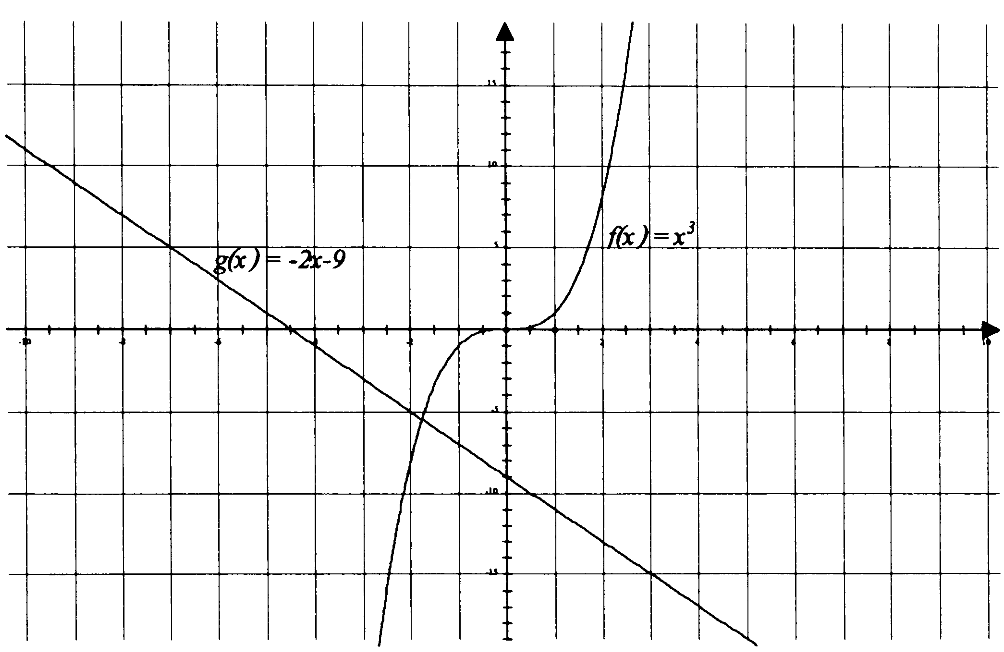
Quadratic equations need not be solved using algebra as is the normal practice. It can also be done through the graphical representation of the quadratic equations. Construct an accurate graph of $y = x^2$. Use the graph of $y = x^2$ to solve $2x^2 - 4x - 3 = 0$. This is a standard exercise. Draw the line $y = 2x + 3/2$ and find its intersection points with $y = x^2$ if any. We may use the graph to solve other quadratic equations.

However, we may also use the same graph to disscuss whether a quadratic equation has two real solutions or otherwise. The idea here is not to obtain the best result but to obtain a quick result with minimum effort. For example, by looking at the line $y = 2x + 3/2$ we see that the line has a positive y-intercept. From the graph we know that there are two real solutions. In other words, we can deduce from graphs some results easily, and at times easier than using algebra. We may consider other cases. We may even give a graphical interpretation to the discriminant $b^2 - 4ac$. In other words, we may use graphs to investigate the nature of the roots of a quadratic equation. We may extend it to cubic equations.



B. Graphical solution of cubic equations

The same process is applied in the solving of cubic equations. A graph of $y = x^3$ is carefully constructed. This graph is next used to solve the cubic equation: $x^3 + 2x + 9 = 0$. Draw the line $y = -2x - 9$ and find the intersection points with $y = x^3$. From the graph we see that there is always one solution. We may solve other cubic equations using the the graph. Also we may use the graph of $y = x^3$ to discuss the nature of roots of a cubic equation. The idea is to extend the use of graphs beyond the solution of an equation. Using graphs can make the solution of a cubic equation easier, though not always. For example, we prefer not to solve $x^3 - 3x^2 + 5x + 6 = 0$ using the above graph. A suggested way of solving such cubic equation is given in item C below.



C. Solving Cubic Equation by completing the cube

The following shows how a given cubic equation can be first manipulated to another cubic equation by completing the cube. Having done that, the solutions to the original cubic equation can be solved graphically.

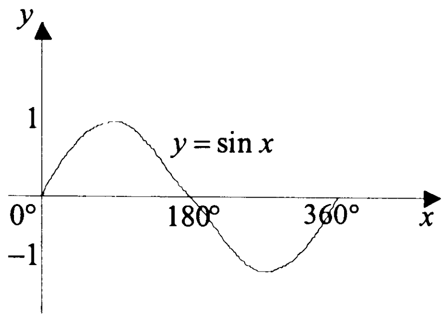
$$\begin{aligned} x^3 - 3x^2 + 5x + 6 &= (x^3 - 3x^2 + 3x - 1) + 2x + 7 \\ &= (x - 1)^3 + 2(x - 1) + 9 \\ &= y^3 + 2y + 9 \text{ (replace } x - 1 \text{ with } y) \end{aligned}$$

Now we may use the graph in item B above to solve $y^3 + 2y + 9 = 0$. Hence in turn we have solved $x^3 - 3x^2 + 5x + 6 = 0$. Solutions of cubic equations were in the syllabus in the 50's. However, it was removed later. As it can be seen, solving cubic equations can be approached easily using graphs which is a geometrical approach. It is also possible to solve a cubic equation by iteration.

D. The sine graph

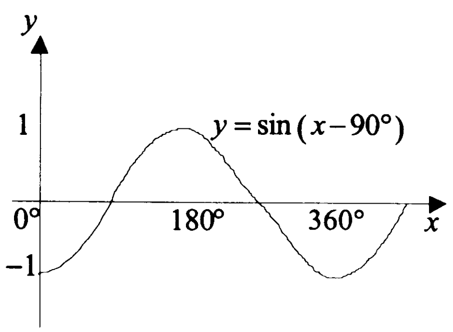
When trigonometric curves are represented geometrically i.e., graphically, it can give us a clear understanding of the relationships between the trigonometric values of various angles as well as help us make generalizations about their relationships.

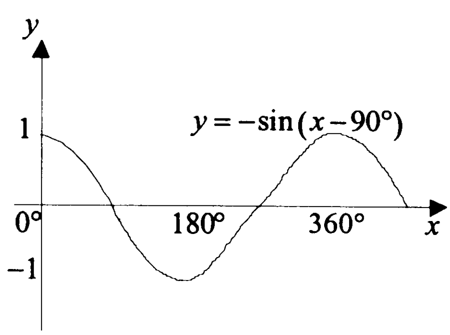
A graph of $y = \sin x$ is sketched. We can read information out of the graph. For example, from the graph we can tell whether $\sin 120°$ is positive, negative or 0? We can also read from the graph that $\sin(180° - A) = \sin A$ and $\sin(180° + A) = -\sin A$. There are many more.

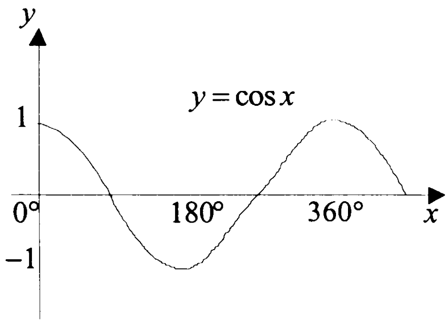


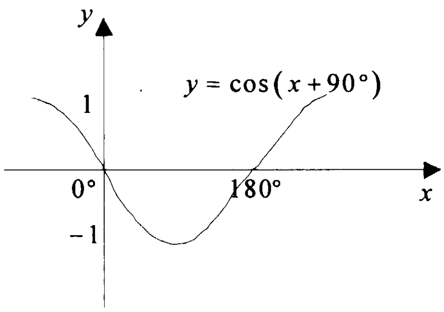
E. The cosine graph

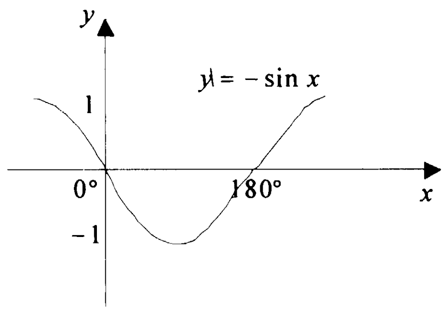
The graph of $y = \sin x$ and the graph of $y = \cos x$ are sketched. It can be seen from the graphs that $\sin(90° - x) = \cos A$ and $\cos(90° + x) = -\sin x$. Further trigonometric relationships can be surfaced by looking at the two graphs.





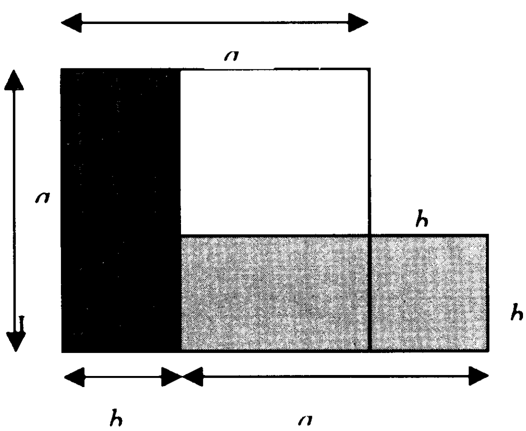






F. An inequality

It is standard to represent $(a + b)^2 = a^2 + b^2 + 2ab$ by diagrams. We may do the same for the inequality $a^2 + b^2 \geq 2ab$, as shown below.

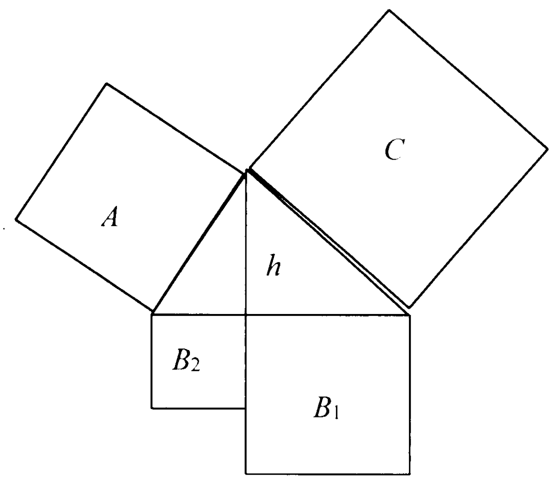


It is easy to see from the diagram that we have the equality when a = b.

Examples on Areas

A. Pythagoras theorem.

Let a, b, and c be the sides of a triangle. Write $A = a^2$, $B = b^2$, and $C = c^2$. Let b be the base, h the height and $H = h^2$. The areas B_1 and B_2 are the squares as shown in the following diagram.



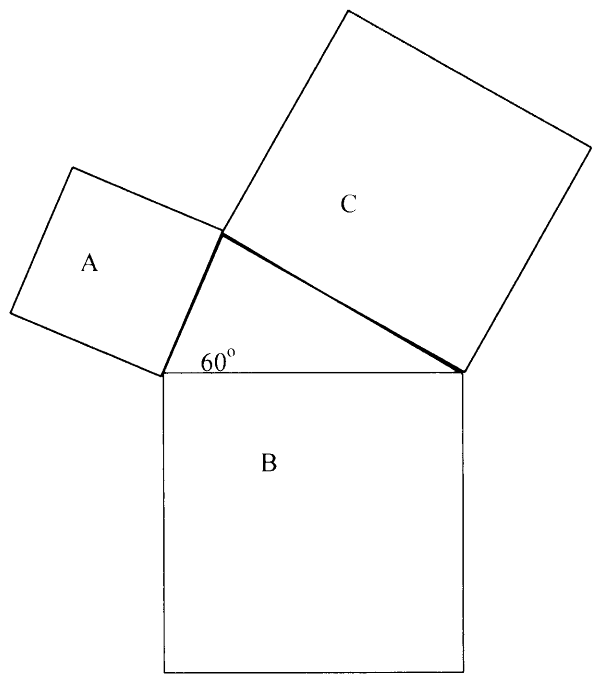
Then by the Pythagoras theorem $C = H + B_1$, and $A = H + B_2$. Thus $C - A = B_1 - B_2$. The result is required in the next item.

B. A little Pythagoras theorem.

Suppose the angle between side a and side b is 60°. Then from the diagram in item A we have $B_2 = (a/2)^2$ and $B_1 = (b - a/2)^2$. Thus $C - A = B_1 - B_2$ becomes $c^2 = a^2 + b^2 - ab$. We call it a little Pythagoras theorem for an angle of 60°. Similarly, we may obtain another little Pythagoras theorem for an angle of 30°. That is, $c^2 = a^2 + b^2 - \sqrt{3}ab$.

C. The cosine rule.

Let a triangle be given as in item A above and the shaped region as shown below be D.

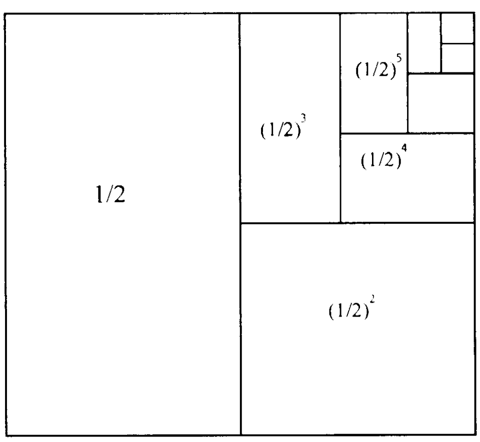


Combining the two diagrams above we see that $B = B_1 - B_2 + 2D$. Thus $C - A = B_1 - B_2$ becomes $C = A + B - 2D$. We call it the big Pythagoras theorem. In fact, this is the cosine rule and we can work out what D is.

D. Geometric series: Example 1.

Shape ½ of the unit square, then 1/4 and 1/8. If we go on, eventually we fill up the whole unit square. Hence we have the sum

$1/2 + (1/2)^2 + (1/2)^3 + (1/2)^4 \ldots = 1.$

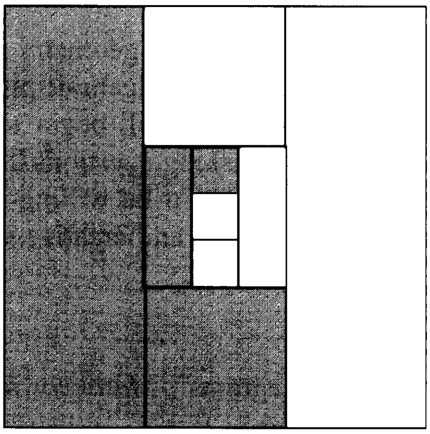


This is standard.

E. Geometric series: Example 2.

Shape 1/3 of the unit square, then 1/9. Again shape $(1/3)^3$ and $(1/3)^4$. As we can see from the diagram below, there are as many shaped regions as non-shaped regions. Hence we have

$1/3 + (1/2)^2 + (1/3)^3 + (1/3)^4 + ... = 1/2.$



F. Geometric series: Example 3.

In the same way as above, we can construct a diagram to find the sum $^1/_4 + (1/4)^2 + (1/4)^3 + (1/4)^4 + ...$ Indeed, the diagram works for $r + r^2 + r^3 + ...$ provided $0 < r < {}^1/_2$. More precisely, the ratio of the total shaped region and the total non-shaped region is r: 1-2r or kr:k(1–2r). Since $kr + k(1-2r) = 1$, then $k = 1/(1-r)$ and $r + r^2 + r^3 + ... = r/(1-r)$. For the case when $^1/_2 < r < 1$, it is harder but it can be done; see Lee (1980).

REFLECTIONS

One of the advantages of using the geometrical approach is that sometimes it makes a problem more transparent, for example, summing geometric series by diagram as in item E of the examples on areas. It makes the solution or proof so much simpler. However, one should be aware that a geometrical solution may not always simplify things, like summing $r + r^2 +r^3 + ...$ The case when $0 < r < {}^1/_2$ is easier whereas the case when $^1/_2 < r < 1$ is much harder. Thus a rule of thumb is that we should use the geometrical approach when it is simple enough for us and the students to grasp. Then we follow it with algebra. An example is solving cubic equations. When the x^2 term in a cubic equation is missing, we can solve it easily using graphs. Otherwise we should solve it by completing the cube first before using the graphical method. We should keep in mind that whenever possible we should always look at a problem algebraically and geometrically. In the same way at the primary level we solve word problems arithmetically and also pictorially. The model method is solving by pictures.

Looking at a problem from the geometrical viewpoint is a habit. The habit is that of thinking geometrically. It is always useful to look at a problem from two different perspectives, that is, approach a algebraic problem geometrically and a geometric problem algebraically. Being able to see different views helps the student to better understand the problem. This is also one way to recover what was lost due to less and less geometry in school syllabus. Many of my students, when introduced to geometrical thinking often expressed surprise at having gained this new insight, as if a new dimension in Mathematics teaching and learning has been opened to them.

The norm in many of our school teaching is to view geometry algebraically. The paradigm we want ourselves and our pupils to shift to is to think algebraic problems geometrically. The long term benefits our students can reap is that they will begin to see and appreciate the world through different perspectives. They will be asking questions like, "How else can we solve this problem? What other perspective can I see?" To be willing to see other perspectives is an important disposition that helps in collaborative and creative work in the 21st century.

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