
Title	Exploring the introduction of fractions in Germany, Singapore, and South Korea mathematics textbooks
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Exploring the Introduction of Fractions in Germany, Singapore, and South Korea Mathematics Textbooks*

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This exploratory study focuses on analyzing three mathematics textbooks in Germany, Singapore and South Korea to reveal similarities and differences in their introductions of fraction concepts. Findings reveal that all three countries' textbooks introduce fraction concepts predominantly by using pictorial representations such as area models, but the introductions of multiple fraction constructs vary. The Singaporean and South Korean textbooks predominantly used a part-whole construct to introduce fractional concepts while the German textbook introduced various constructs sequentially in the first pages using several scenarios from different real-life situations. The findings were represented using visual representations, which we called textbook signatures. The textbook signatures provided configurations of the textbook features across the three countries. At the end of paper, we share insights and limitations about the use of textbook signatures in the research on textbook analysis.

Keywords: Fractions, Textbook Analysis

MESC Classification: U20

MSC2010 Classification: 97U20

I. INTRODUCTION

Teaching fractions is one of the most challenging tasks for teachers (Clarke et al., 2008; Brown & Quinn, 2006; Lamon, 2007). Key issues include the multiplicity of fraction conceptualizations (Smith, 2002), sequencing of fraction concepts (Lee & Lee, 2021), introducing different fraction constructs (Lamon, 2007), and representing fractions in

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instruction and learning materials (Lee & Lee, 2020; Vig et al., 2014). As highlighted by Smith (2002), the different meanings expressed by fractions “can wreak havoc with our communication” as we teach fractions (p. 4). Thus, to teach fractions competently, teachers should have a strong grasp of these different concepts and representations (Schoenfeld & Kilpatrick, 2008), alternate among them smoothly, and clearly communicate them to students. Textbooks play a key role in supporting this work by demarcating what mathematical ideas are to be taught, how they might be taught, and how they might be sequenced (Nicol & Crespo, 2006). It is therefore not surprising that teachers rely on textbooks for their lesson preparation and teaching (Huntley, 2008; Usiskin, 2013). Given that textbooks play an important role in shaping what mathematics is taught in schools and how (Charalambous et al., 2010; Choy et al., 2015, 2020), an analysis of how fractions are presented in textbooks can offer insights into the learning opportunities they afford.

Although textbook analyses are potentially important and have attracted the attention of researchers, they are often under-theorized and more complicated than they seem (Fan et al., 2013). Despite these challenges, mathematics educators see textbook analyses as an important means to investigate the relationships between textbooks and the educational contexts in which they are used. In particular, Pepin et al. (2001) advised that cross-national comparisons of textbooks may “point to possible [new] directions that could be followed” and can potentially “sharpen the focus of analysis of the subject under study by suggesting new perspectives” (p. 158).

In the case of fractions, this cross-national analysis is even more important as the introduction of fractions at earlier grades have implications for developing a more robust understanding of fractions at the later grades (Simon et al., 2018). For example, as Simon et al. (2018) have argued, introducing fractions as “part of a whole” is limiting and may hinder students in developing more “powerful conceptions of fractions” (p. 123). Following the same argument, we see cross-national comparisons of how fractions are introduced in textbooks as opportunities to provide new perspectives on how topics such as fractions can be taught. The aim of this study, therefore, is to compare how fractions are initially introduced and developed in widely used textbooks in three countries, Germany, Singapore and South Korea, by focusing on the fraction constructs and representations used in the textbooks to reveal similarities and differences in how they guide the teaching of fractions. In this analysis, we address the following questions: (1) Which fraction constructs are introduced and in which order? (2) Which representations are used for the constructs of fractions? Based on comparison of how fractions are initially introduced to students in the different countries, we discuss possible implications for teaching fractions and suggest ways to improve the teaching and learning of this difficult topic.

II. LITERATURE REVIEW

1. TEXTBOOK ANALYSIS

Textbooks are written to provide the mathematical ideas and embody the teaching practices recommended by curriculum documents (Huntley, 2008; Usiskin, 2013). Teachers consult textbooks for their lesson planning and students study them to learn new mathematical concepts or consolidate their mathematical understanding. In this regard, textbooks are crucial resources for teaching and learning mathematics (O’Keeffe & O’Donoghue, 2015). In addition, textbooks are often nationally authorized or approved for use in the classrooms, implying that they express a country’s values, visions, and perspectives with regard to mathematics teaching and learning (Pepin et al., 2001).

While textbook analysis has been important in mathematics education, its scope and complexity are widely varied according to its purpose (Fan et al., 2013). Textbook analysis can be as simple as comparing the uses of a single or a series of textbooks within the same country, or as complex as examining how a focal topic is developed in the textbooks used in different countries. To compare particular variables such as mathematical concepts in a focal topic (Charalambous et al., 2010; Wang et al., 2015), various researchers have looked into the types of problems (Li, 2000; Stigler, Fuson, Ham, & Kim, 1986), or types of representations presented in textbooks (Chang et al., 2016), while others have focused on tables, figures, and textual excerpts in targeted texts (Valverde et al., 2002). For example, Valverde et al. (2002) analyzed 630 Mathematics and Science textbooks from 40 countries to compare them with regard to their physical features, structures, and presentation of content and then represented the results in the form of schematic diagrams. Nevertheless, in all variations and despite the complexities, textbook analysis provides important insights into the teaching and learning of mathematics across settings in different countries (Son & Diletti, 2017).

Interestingly, Charalambous et al. (2010) propose that textbooks within the same country may have a “textbook signature, that is, “uniform distinctive patterns” (p. 146). Taking a cue from Valverde et al. (2002), in this paper, we extend our prior study (Choy et al., 2015, 2020; Mizzi et al., 2016) to compare how fractions are presented in textbooks across three different countries using our conception of textbook signature. In our prior studies, we develop visual representations, which we termed as textbook signature, to highlight possible distinctive patterns in the type of conceptualizations presented, how these concepts are represented, and the cognitive demand of tasks used for these concepts. As an example, we refer to Figure 1, a textbook signature for the German textbook we used in this present study.

Referring to Figure 1, we observe that the horizontal axis represents the pages of the chapter used in the analysis and the vertical axis represents the different constructs of fraction. In this case, a gray cell at the intersection of page 1 and “part-whole” indicates the presence of a part-whole conceptualization of fraction on page 1 of the chapter.

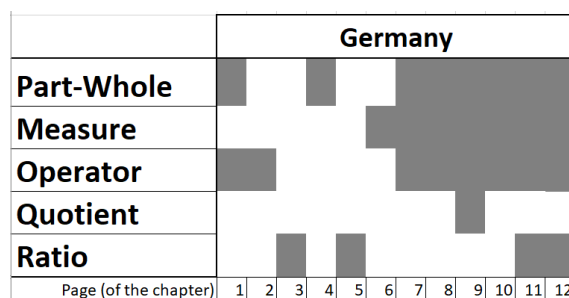


Figure 1. Textbook signature for a German Textbook.

When viewed together, this visualization provides a way for us to see the kind of constructs presented and how they are distributed across the chapter. It also gives us an idea of the multiplicity of concepts used on a single page, which may have implications on teaching and learning. By creating similar textbook signatures for different countries, we can pick up patterns of similarities and differences across the countries at the macro level. The page numbers indicated also provides a means for us to analyze the actual content at a micro level if any interesting patterns were seen. It is the dual levels of zoom afforded by our textbook signature for the different textbook variables that make for a more comprehensive textbook analysis.

2. FIVE CONSTRUCTS OF FRACTIONS

Understanding fractions is critical for learning other topics in mathematics (Lee & Hackenberg, 2014; NMAP, 2008). However, research has revealed that students generally lack proficiency in fraction concepts (Brown & Quinn, 2006; Lamon, 2007). For example, Brown and Quinn (2006), who conducted an analysis of ninth grade students' fraction errors, found an overall lack of facility in basic fraction concepts, which should have been acquired earlier through manipulating various concrete referents. Thus, investigating how fractions are covered in textbooks across different countries can give us insight into how to teach fractions effectively in order to establish a sound foundation for more advanced mathematical knowledge.

There are five important fraction constructs: Part-whole, Measure, Operator, Quotient, and Ratio (Kieren, 1980; Lamon 2007), each of which is illustrated using the fraction $\frac{3}{5}$ in Table 1. As a part-whole construct, $\frac{3}{5}$ is interpreted as three out of five equal parts of

the unit (whole), and as a measure, it is interpreted as the distance of three $\frac{1}{5}$ units from zero on the number line. As an operator construct, finding $\frac{3}{5}$ of a given quantity involves multiplying the quantity by $\frac{3}{5}$. The quotient construct views $\frac{3}{5}$ as the amount which each of five persons receives when they equally share three units of a quantity. Finally, as a ratio, $\frac{3}{5}$ represents a multiplicative comparison or relation between two quantities in a given order. That is, if for every 3 boys, there are 5 girls in a class, then the relationship between the number of boys to the number of girls can be written as 3:5 or $\frac{3}{5}$.

Among these representations, the part-whole construct is the most common conceptualization encountered in mathematics textbooks when fractions are first introduced. As highlighted by Simon et al. (2018), “fractions are typically introduced with circle or rectangle representations in which a fraction $\frac{1}{n}$ is one of n identical parts of the given figure” (p. 123). This notion of fraction is found not only in the typical textbooks used by different states but also in alternative curricula. Although the part-whole construct is most frequently used to represent fractions in early mathematics textbooks, many researchers contend that students would acquire a deeper understanding if they are exposed to other constructs of fractions (Clarke, Roche, & Mitchell, 2008; Lamon, 2007).

Furthermore, the introduction of fraction as solely a part-whole construct may limit students’ understanding of fractions as arrangements of equal parts (Simon, 2006) with no idea of a fraction as a single number. Instead, students may see fractions as two numbers—the number of shaded parts of a part-whole diagram (the numerator) and the total number of equal parts (the denominator), which may hinder their understanding of improper fractions (Stafylidou & Vosniadou, 2004). In this study, we aim to characterize the conceptualizations of fractions used in the textbooks and explore possible implications for teaching fractions.

Table 1. Conceptualizations of fractions (Lamon, 2012)

Conceptualization	Explanation
Part-whole	Fractions are considered as part-whole comparison, which assigns a number of equal parts of a unit out of the total number of equal parts into which the unit is divided. E.g., $\frac{3}{5}$ is perceived as three out of five equal parts of the unit (whole)
Measure	Fractions are seen as the distances of certain points from zero in terms of some unit distance. In other words, a number. E.g., $\frac{3}{5}$ is perceived as three $\frac{1}{5}$ units from zero on a number line, where the unit is 1.
Operator	Fractions are seen as functions or mappings which act as a set of instructions for carrying out a process. E.g., $\frac{3}{5}$ of a set of 20 marbles describes how a set of 20 marbles can be partitioned into 5 equal groups, of which we are interested in three of these groups.
Quotient	Fractions are seen as the result of a division problem. E.g., $\frac{3}{5}$ is the answer to $3 \div 5$.
Ratio	Fractions are seen as multiplicative comparisons of two quantities. E.g., $\frac{3}{5}$ is interpreted as 3:5.

3. REPRESENTATIONS OF FRACTIONS

Jerome Bruner (1996) theorized learning as a process of three stages of representations: enactive (action-based), iconic (image-based) and symbolic (language or notation-based). He argues that students understand a mathematical concept better when they begin with concrete manipulation and pictorial illustration of the concept before moving to a symbolic notation or algorithm (Bruner, 1966). Many studies also support the importance of using multiple representations in initial fraction learning. For example, Cramer et al. (2002) found that students' engagement with multiple modes of representations, such as pictorial, manipulative, verbal, real-world, and symbolic, to be significantly helpful for initial fraction learning. Similarly, Petit et al. (2010) asserted that "students need to interact with multiple models that differ in perceptual features, which causes them to rethink and ultimately generalize the mathematical concepts being investigated with the models" (p. 5).

Also, for effective understanding of fractions, van de Walle et al. (2013) recommended the use of all three types of pedagogical models (area models [e.g., circular pieces], linear models [e.g., fraction strips, number lines], and set models [e.g., tallies]) as each has pros and cons. In addition, publications from the Rational Number Project highlighted the use of three different ways of representing fractions such as real-world situations, verbal symbols, and manipulatives (Cramer & Whitney, 2010). These studies support the importance of using varied representations in initial learning of fractions, which is relevant to the second research question of this study.

Another important aspect to consider lies in the idea that a particular conceptualization of fraction may be best represented by a specific model. For example, while the pictorial area model may seem intuitive for students when they learn about the part-whole concept of fractions (see Figure 2), it may not be a suitable model when teachers want to bring out the measure concept of fractions. The choice of representations is especially crucial when students learn about the concept of fraction as ratio, which may not necessarily involve partitioning and thus is "conceptually different from the part-whole and quotient constructs" (Getenet & Callingham, 2019, p. 204). Moreover, in cases where the same conceptualization can be represented in different ways (see Figure 2), it is important for teachers to make clear how the different representations are related to the construct.

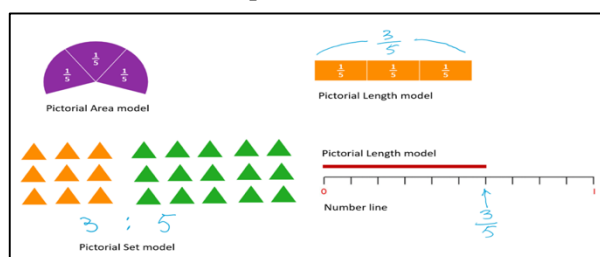


Figure 2. Different representations of $\frac{3}{5}$

In this paper, we first examined content (fraction constructs), and instructional aids (representations) in three officially sanctioned textbooks in Germany, Singapore, and South Korea and then determined their similarities and differences in terms of these two variables using our notion of “textbook signature.” For the first variable, textbook signature comprised the five basic constructs of fractions presented on each page (refer to Figure 1). For the second variable, textbook signature indicates the configurations for the frequencies of occurrence in each page of the three textbooks by its histogram (refer to Figure 10). Using textbook signature to compare each variable across the three textbooks enabled us to discover patterns characterizing each textbook’s treatment of fractions and present them visually and numerically.

III. RESEARCH METHODOLOGY

1. CONTEXTS OF THE FOCAL COUNTRIES

Germany, Singapore and South Korea offer a wide spectrum of educational contexts for comparison. Unlike Germany, both Singapore and South Korea have a centralized educational system. Grades 1 through 12 are compulsory in Germany and South Korea, but only Grades 1 through 6 in Singapore. In the upper grades, the German national mathematics curriculum focuses mainly on modelling, problem solving, argumentation and reasoning, and communication and conceptual understanding. The South Korean mathematics curriculum emphasizes conceptual understanding and aims to develop mathematical attitudes, thinking and communication skills in a creative way (Ministry of Education, 2015); whereas the Singapore curriculum focuses on problem solving and stresses the development of conceptual understanding, skills, proficiency, mathematical processes, attitudes, and metacognition (Ministry of Education-Singapore, 2012). Although South Korea and Singapore seem to very similar in terms of its educational system, which reflects, to a large extent, a strong Confucian heritage culture (Leung, 2014), Leong and Kaur (2019) have argued that Singapore is more pragmatic in its educational approach and adopts an eclectic set of teaching approaches which often stem from contrasting ideals (Wong, 2015). Therefore, taken together, the three countries lie along the spectrum of educational contexts at different points—with Germany on a predominantly western view of education and Korea on the other end of the spectrum, reflecting a more Confucian heritage view of education.

In terms of curricular sequence, Singapore’s and South Korea’s approach may be similar to a large extent. In Singapore, for instance, fractions are introduced in Grade 2 (See Figure 3) and South Korea in Grade 3. Also, in both Singapore and South Korea, students should have learned about whole numbers up to four digits and the four operations by the time

they encounter fractions. In addition, they would be familiar with some of the basic measurement topics such as length and time. However, area is not explicitly covered in Grade 2 or Grade 3. Singapore's and South Korea's approach to developing fraction concepts differs distinctively from Germany's. First, in Singapore and South Korea, the notion of a fraction is introduced in the early grades, whereas in Germany fractions are not formally treated until the sixth grade (age 11-12). Moreover, the textbooks from the two Asian countries revisit the operations taught previously, and the operations get progressively more complex as students progress through the grades (See Figure 3). In contrast, the selected German textbook tends to introduce multiple operations of a particular concept simultaneously. As a basis for comparison, we only focus on the grade level at which fractions are first introduced in each country and select a representative textbook for the corresponding grade for analysis.

Grade 2	Grade 3	Grade 4	Grade 5	Grade 6
1. Fraction of a whole 1.1 fraction as part of a whole 1.2 notation and representations of fractions 1.3 comparing and ordering fractions with denominators of given fractions not exceeding 12 • unit fractions • like fractions 2. Addition and subtraction 2.1 adding and subtracting like fractions within one whole with denominators of given fractions not exceeding 12	1. Equivalent fractions 1.1 equivalent fractions 1.2 expressing a fraction in its simplest form 1.3 comparing and ordering unlike fractions with denominators of given fractions not exceeding 12 1.4 writing the equivalent fraction of a fraction given the denominator or the numerator 2. Addition and subtraction 2.1 adding and subtracting two related fractions within one whole with denominators of given fractions not exceeding 12	1. Mixed numbers and improper fractions 1.1 mixed numbers, improper fractions and their relationships 2. Fraction of a set of objects 2.1 fraction as part of a set of objects 3. Addition and subtraction 3.1 adding and subtracting fractions with denominators of given fractions not exceeding 12 and not more than two different denominators 3.2 solving up to 2-step word problems involving addition and subtraction	1. Fraction and division 1.1 dividing a whole number by a whole number with quotient as a fraction 1.2 converting fractions to decimals 2. Four operations 2.1 adding and subtracting mixed numbers 2.2 multiplying a proper/improper fraction and a whole number without calculator 2.3 multiplying a proper fraction and a proper/improper fractions without calculator 2.4 multiplying two improper fractions 2.5 multiplying a mixed number and a whole number 2.6 solving word problems involving addition, subtraction and multiplication	1. Four operations 1.1 dividing a proper fraction by a whole number without calculator 1.2 dividing a whole number/proper fraction by a proper fraction without calculator 1.3 solving word problems involving the 4 operations

Figure 3. Development of fraction across the grades in Singapore.

2. TEXTBOOK SELECTION

The textbooks selected for analysis are Lambacher Schweizer 6 for Germany, Shaping Maths Coursebook 2B (3rd edition) for Singapore, and Soohak 3 for South Korea. Most German textbooks for Grades 1-3 do not address the concept of fractions, although fractions may be introduced at Grade 4, and then formally treated at Grade 6. Therefore, a sixth-grade mathematics textbook, Lambacher Schweizer 6 (published by Klett), which is widely endorsed and used by most German federal states (especially those highly populated) was chosen for this study. In Singapore, all three primary level textbook series have to be approved by the Ministry of Education-Singapore for adherence to the national curriculum before being published. Consequently, every textbook in Singapore introduces fraction concepts in Grade 2 (age 8) and follows a very similar sequence of presentation. The textbook analyzed in this paper is Shaping Math Coursebook 2B (3rd edition), which has

been widely adopted by many schools. Korea currently has only one textbook published by government (Ministry of Education, 2015) although different textbooks published by private company will be available within a few years. As Korean schools operate on a semester basis, there are two volumes of the mathematics textbook per grade level, one for each semester. The fraction concept is first introduced in Grade 3 (age 9-10) and there is one fraction chapter in each volume, so we selected two fraction chapters in Grade 3 for this study. In this paper, we focus only on pages dealing with initial fraction learning, which excludes comparing the sizes of fractions and decimal concepts, and types of fractions (mixed fractions) and addition and subtraction of fractions with the same denominators.

3. DATA ANALYSIS

Following Herbel-Eisenmann (2007), in our analysis we viewed textbooks as having “objectively given structure” (p. 346) and focused on their content presentation and structure without referring to how they might be used by teachers and students. The relevant chapters in the selected textbooks from the three countries were analyzed for how they introduced fractions at two levels, the page level and the chapter level, and what they implied about the teaching and learning of fractions. For each page, we conducted an item-by-item analysis. That is, every example, question, task, and explanatory text or representation was counted as an item. We then analyzed the items based on the following variables: constructs of fractions including part-whole, measure, operator, quotient, and ratios (Kieren, 1980; Lamon, 2007) and representations of fractions, which were subcategorized as concrete, pictorial (area, length and set models), verbal, and symbolic.

Specifically, for constructs of fractions, we coded the items based on clear descriptors given in the text without assumptions about the students’ or the textbook authors’ perspectives. For example, in Figure 7, we coded the task on page 206 as the measure construct because it involves the descriptor of “figure out how many unit fractions are iterated in $\frac{2}{3}$.” Also, for representations of fractions, we coded the items based on clear definitions of each representation. Concrete representation was defined as manipulatives or objects from real-life as seen in Figure 5. Pictorial representation was defined as pictures or diagrams directly related to the concept of fractions as seen in Figure 7. Verbal representation was defined as any speech bubbles or balloons directly related to the concept of fractions as seen in Figure 8. Symbolic representation was defined as any mathematical notations directly related to the concept of fraction.

Two main questions guided our analysis: First, what construct of fractions was referred to in this item? Second, what representation of fractions was used? The coded data were used to create a textbook signature for each country, which represented the occurrence of the constructs and representations of fractions. While only the presence of each construct

of fractions per page was documented, the frequency of occurrence of representations across all pages was noted to create a profile of the kinds used. These data were visualized in two histograms for both variables of each country (see Figures 4 and 11). These visualizations provide the basis for our Level 2 analysis in which we draw inferences about the teaching and learning of fractions in each country. When appropriate, we compare our findings about the types of fraction constructs and representations to curricular documents, context of each country, and extant literature on teaching and learning fractions.

IV. RESULTS AND DISCUSSION

1. CONSTRUCT OF FRACTIONS

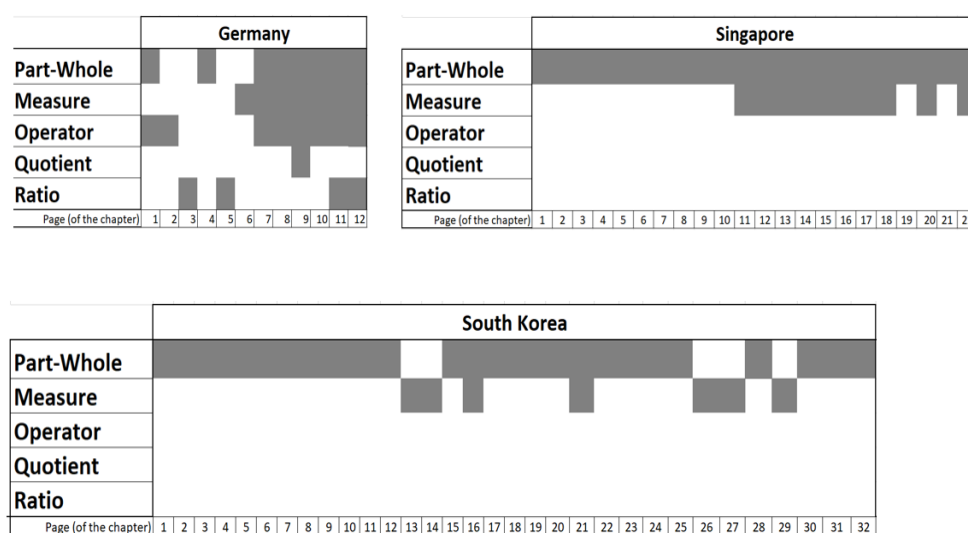


Figure 4. Fraction constructs used across the three textbooks of the different countries.

As explained earlier, our textbook signature in Figure 4 shows the different constructs of fractions (part-whole, measure, operator, quotient and ratio), which are dealt with in the introductions to fractions in the three textbooks (Grade 2 in Singapore, Grade 3 in Korea, and Grade 6 in Germany). It can be observed that the part-whole construct of fractions predominates in the Singaporean and South Korean introductions to fractions at second and third grade levels respectively, whereas all constructs are introduced in the German introduction of fractions at the sixth-grade level. In the German textbook, most of the constructs are introduced sequentially in the first pages using several concrete tasks such as folding paper sheets so that portions represent different fractions (part-whole construct)

(see Figure 5), drawing the human body and observing its proportions (ratio construct), and determining amounts of ingredients of a recipe (measure construct). The second half of the German introduction to fractions features all the different constructs, mainly part-whole, measure and operator (see task 10 in Figure 6), with some attention to quotient and more to ratio (see task 6 in Figure 6).

The focus on multiple constructs in the German textbook reflects the importance of conceptual understanding, which is focused at in the German mathematics curriculum. These introductory examples also support the second pillar in German mathematics curriculum: reasoning and argumentation. The simultaneous introduction of several fraction constructs fosters comparison of different approaches to the topic of fractions as a foundation for in-depth conceptual understanding required for reasoning about fractions. However, the numerous constructs introduced in the German textbook might be a catching-up strategy because fractions are formally introduced at a later stage in German schools in comparison to other countries, such as Singapore and South Korea. Nevertheless, by requiring students to learn different facets of fractions within a short time, the German textbook prepares them to navigate among different constructs more fluently.

Faltet ein Blatt Papier so, dass ihr ein Zwölftel des Papiers erhaltet. Findet möglichst viele unterschiedliche Weisen, das Papier zu falten. Vergleicht eure Ergebnisse untereinander. Nun einmal systematisch: Nehmt mehrere Blätter und faltet noch andere Anteile. Beginnt zunächst mit $\frac{1}{2}$. Aber Achtung: Jetzt ist diagonal falten nicht mehr erlaubt. Faltet dann $\frac{1}{3}$, $\frac{1}{4}$ usw. bis $\frac{1}{8}$. - Was fällt euch auf? Sortiert die Anteile nach der Anzahl der Faltmöglichkeiten. - Warum kann man bei manchen Anteilen häufiger falten als bei anderen?	diagonal falten 
<i>Translation:</i> Fold a paper so that you obtain one twelfth of the paper. Find as many different ways as possible to fold the paper. Compare your results with each other. And now systematically: Take more sheets of paper and fold other portions. First, begin with $\frac{1}{2}$. But be careful: Diagonal folding is no longer allowed. Afterward fold $\frac{1}{3}$, $\frac{1}{4}$ etc. till $\frac{1}{8}$. - What do you recognize? Sort the portions according to the number of possibilities of folding. - Why are there more ways to fold some portions rather than the other ones?	

Figure 5. Example of tasks which highlight folding to introduce the concept of fractions in the German textbook.

6 a) In der Klasse 6c ist das Verhältnis von Mädchen zu Jungen 3:2. Wie groß ist der Anteil an Mädchen in dieser Klasse? Wie viele Mädchen könnten in der Klasse sein? b) Wie groß ist das Verhältnis von Mädchen und Jungen in eurer Klasse? Bestimmt auch die Anteile. 10 Wie viel sind ... a) $\frac{1}{4}$ von 8 kg, b) $\frac{2}{3}$ von 60 m, $\frac{3}{10}$ von 20 Äpfeln, $\frac{1}{2}$ von 40 Eiern, $\frac{1}{100}$ von 2 €, $\frac{4}{30}$ von 2 min,
<i>Translation:</i> 6 a) The ratio of girls to boys in class 6c is 3:2. How much is the portion of girls in the class? How many girls could there be in class? b) What is the ratio of girls to boys in your class? Determine the portions too. 10 How much is ... a) $\frac{1}{4}$ of 8 kg b) $\frac{2}{3}$ of 60 m $\frac{3}{10}$ of 20 apples $\frac{1}{2}$ of 40 eggs $\frac{1}{100}$ of 2 Euros $\frac{4}{30}$ of 2 mins

Figure 6. A task addressing the fraction construct of ratio (Task 6) and another one for the fraction construct of operator (Task 10) in the German textbook.

In South Korea, similar to Singapore, the concept of fractions is first introduced based on part-whole construct and heavily depends on the construct in initial fraction learning in Grade 3. However, after covering part-whole construct enough, it also deals with the measure construct briefly by focusing on iterations of a unit fraction (see Figure 7, left) and introduction of a number line (see Figure 7, right). Addressing these different constructs of fractions at initial stage of fraction learning seems to be related to the focus of South Korean mathematics curriculum, which emphasizes profound conceptual understanding.

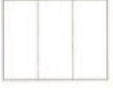
Grade 3-1 (p. 206)  $\frac{2}{3}$는 $\frac{1}{3}$이 몇 개인지 알아보시오. • $\frac{2}{3}$와 $\frac{1}{3}$만큼 각각 색칠하시오. $\frac{2}{3}$  $\frac{1}{3}$  • $\frac{2}{3}$는 $\frac{1}{3}$이 개입니다.	Grade 3-2 (p.112)  분모가 5인 분수를 수직선에 나타내어 보시오. • 수직선을 보고 안에 알맞은 수를 써넣으시오.  • $\frac{5}{5}$인 곳을 찾아 수직선에 $\uparrow$로 나타내어 보시오. • $\frac{6}{5}$인 곳을 찾아 수직선에 $\uparrow$로 나타내어 보시오.
<i>Translation:</i> How many $\frac{1}{3}$s are there in $\frac{2}{3}$? Color the amounts of $\frac{2}{3}$ and $\frac{1}{3}$ and figure out how many unit fractions ($\frac{1}{3}$) are iterated in $\frac{2}{3}$.	<i>Translation:</i> Fill in the corresponding blanks in the number line. Then mark where $\frac{5}{5}$ and $\frac{6}{5}$ are.

Figure 7. Examples of tasks related to the measure construct in the South Korean textbook.

As mentioned above, the part-whole construct is explicitly featured in the Singaporean textbook throughout the chapter. This is aligned with the curriculum for Grade 2, which specifies “fractions as part of a whole” (Ministry of Education-Singapore, 2012, p. 38). However, the use of the measure construct of fractions towards the end of the chapter is suggested. Here, we follow Simon et al.’s (2018) idea that when “a fraction is understood as a measure, $1/n$ and m/n have meaning independent of a whole unit; it can be understood as a single number” (p. 123). In other words, seeing fraction as a measure allow students to build on their understanding of whole numbers and see iteration of unit fractions in ways similar to the iteration of “1” in the building up of whole numbers. Understanding this construct is thus the basis of doing arithmetic involving fractions without the context—seeing the fraction $\frac{1}{4}$ as a number independent of the context.

A possible reason for the predominant use of the part-whole construct in Grade 2 is the spiral nature of the Singapore Mathematics curriculum—the idea that fractions will be revisited in terms of other constructs at higher levels. For instance, the measure construct of fractions is more explicit at Grade 4, in which students learn to place fractions on the number line when learning about improper fractions, similar to the task in the Korean textbook in Figure 7. However, no connections between the two constructs are made at Grade 2, even though the notion of a fraction as a part-whole comparison does not lend itself naturally to the arithmetic operations made more explicit by the measure construct. Instead, the part-whole construct, as represented by the areas of circles and rectangles, is used to make sense of ordering and operations.

2. REPRESENTATIONS OF FRACTIONS IN INTRODUCING THE FRACTION CONCEPT

When analyzing the representations of fractions in the initial tasks of the textbooks from all three countries, pictorial representations are quite commonly used to introduce the concept of fractions. The most common type of pictorial representation is the area model followed by the length model, which is introduced later in the chapter. However, the set model is used only in the South Korean textbook to introduce the fraction concepts towards the end of the chapter. Also, across the three textbooks, concrete representations are used at the beginning of the chapter, and symbolic representations are also used throughout the rest of the chapter. However, in the Korean textbook, concrete representations are used fairly consistently across the chapter while they are phased out at the end of the chapters in the German and Singaporean texts (see Figure 10).

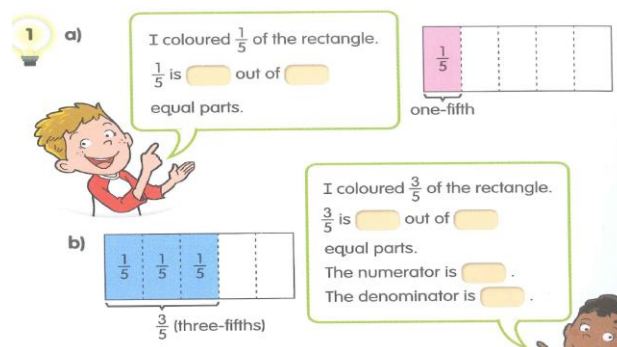


Figure 8. Combination of different representations in the Singaporean textbook.

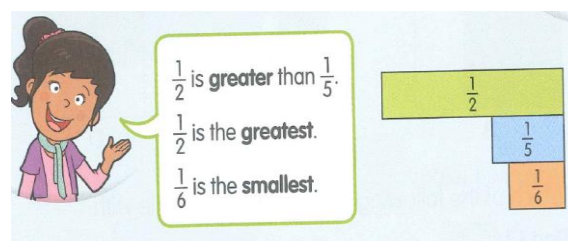


Figure 9. Use of pictorial length models to compare unit fractions in the Singaporean textbook.

Also, in the Singaporean textbook, regardless of the representations used, the tasks are often accompanied by both verbal and symbolic representations to emphasize the correspondence between different representations. For example, in Figure 8, a pictorial area model based on the part-whole notion is used in conjunction with the symbolic representation $\frac{1}{5}$ and the verbal description of 1 out of 5 equal parts. In addition, even though only the part-whole notion of fraction is presented in the Singapore textbook, a pictorial length model is used to compare fractions (See Figure 9). However, it should be noted that the bar model is often interpreted in terms of its length or its area (Lee & Lee, accepted).

Further examples of concrete representations and symbolic representation can be found in the tasks in Figure 5 and Figure 6 from the German textbook respectively. It is important to highlight that multiple representations were often used for one construct. The use of the symbolic together with another representation was quite common. Consider, for example, the task in Figure 8 which introduces the part-whole construct using the pictorial and symbolic representation in the Singaporean textbook. Such observations are crucial in order to understand the textbook signature regarding the representations in Figure 10.

In contrast to the signature of the constructs, the frequency of the different representations on each page of the focal chapters in the three textbooks was noticed (see

Figure 10). Each task on each page of the introductory chapter was assigned one dominant representation in order to visualize the signature of the three textbooks in Figure 10 for a better comparison among them. The concrete representation is distributed throughout all the chapters but more evenly in the Korean chapter. The length model of pictorial representation is more common in the Singaporean book than in the German and Korean books, which address the length model on just two or three pages respectively. In the Singaporean book, the use of the area model to represent fractions is equally distributed throughout the whole chapter, whereas the South Korean and German textbooks focus on the length model in the specific part of the chapter. Whereas only the Korean textbook provides a set model as a pictorial representation for fractions, the Singaporean textbook emphasizes the verbal representation of fractions more than the German and Korean textbooks. Moreover, while all three textbooks use the symbolic representation of fractions across the whole chapter, in the German and Singaporean textbooks its frequency increases progressively whereas in the Korean book it is used more sporadically.

Taken as a whole, the representations used in the textbooks are directly related to the fraction constructs highlighted in the chapters. The multiple use of fraction constructs in the German textbook (Figure 4) supports the increasing abstraction represented in the results in Figure 10 (at the beginning of the chapter more concrete and at the end more symbolic). The initial focus on hands-on tasks and real-life situations (concrete representation) in the German textbook reflects the importance placed on modelling in the German curriculum, which is deemed to be highly effective in facilitating students' understanding of the practical use of knowledge of fractions. The increasing abstraction requires more use of symbolic representation which explains the pattern in Figure 10 for the German textbook.

To cover the part-whole construct of fractions, both Singaporean and Korean textbooks tended to frequently use concrete or area model of pictorial representations at the beginning of the chapter. However, even though both Singaporean and Korean books introduce only two fraction constructs such as part-whole and measure, the Singaporean textbook tends to focus on both concepts simultaneously from the middle of the chapter. This might explain the increased use of the symbolic representation in the Singaporean chapter, which is not staggered in the Korean representation signature in Figure 10. Herein lies an opportunity for teachers to bring in different fraction constructs and their accompanying representations. The amount and order of fraction constructions and representations for introducing fractions lead to many interesting questions to follow up beyond this study.

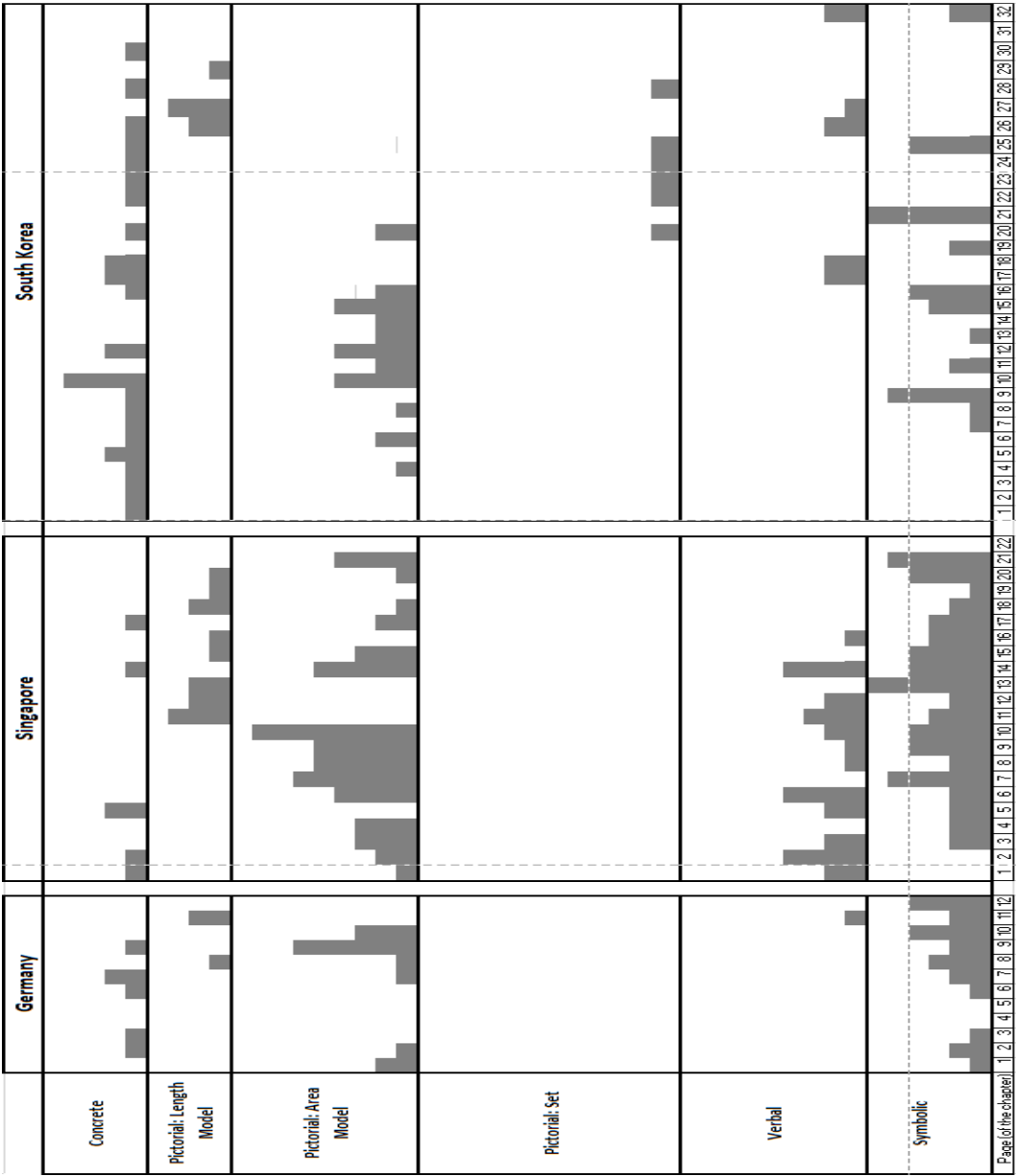


Figure 10. Representations of fractions across the three countries.

V. LIMITATIONS AND CONCLUDING REMARKS

This study supports our findings from the prior research (Choy et al., 2015, 2020) that the textbook signature of each country is unique regardless of the topics investigated. More

importantly, the textbook signatures provide visual configurations of the textbook features across different countries. For example, whereas Singaporean and Korean textbooks use one or two constructs to introduce fraction concepts in an early grade, the German book introduces various fraction constructs simultaneously in a later grade. In Germany, fractions are heavily covered at a higher level when it seems to be assumed that students are prepared to learn multiple constructs of fractions simultaneously and become able to change from one to another fluently. In contrast, in Singapore and Korea, where fractions are introduced earlier, the emphasis is on the expansion of the fraction concept starting at the cognitive readiness of younger learners. Thus, the different constructs of fractions are introduced sequentially in the Korean and Singaporean textbooks even though the exact sequence may differ. With respect to the representations of fractions, the textbooks in all three countries use multiple representations through the stages of concrete, iconic and symbolic, which is in alignment with Bruner's (1966) theory.

In our study, textbook signatures were analyzed to represent the distinctive patterns in textbooks, enabling us to visually compare how the fraction constructs are addressed in textbooks across three countries. However, there are limitations in this exploratory study. First, we examined purposefully selected samples of textbooks from each country, precluding a broader scope of investigation. Second, we constructed textbook signatures only for the introduction of fractions without examining the development of the concept across grade levels, so questions remain. For example, which approach, introducing single constructs of fractions sequentially starting at an earlier age, or multiple constructs of fractions simultaneously at a later age, is more effective for subsequent learning about fractions? Are these approaches associated with the relative performances of the three countries in international benchmark assessments?

Although our findings are preliminary and more work is required to develop our notion of textbook signatures for meaningful use in analyzing and comparing different textbooks, our exploratory study suggests some potential for their use as textbook visual representations of findings from textbook analysis. The notion of a textbook signature presents a useful way to investigate the role of textbooks in the learning and teaching of fractions in mathematics education.

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