KYUNGPOOK Math. J. 54(2014), 263-269
http://dx.doi.org/10.5666/KMJ.2014.54.2.263

Reflexive Index of a Family of Sets

Zhao Dongsheng
Mathematics and Mathematics Education, National Institute of Education, Nanyang Technological University, Singapore
e-mail : dongsheng.zhao@nie.edu.sg

Abstract. As a further study on reflexive families of subsets, we introduce the reflexive index for a family of subsets of a given set and show that the index of a finite family of subsets of a finite or countably infinite set is always finite. The reflexive indices of some special families are also considered.

Given a set X, let $\mathrm{Sub}(X)$ denote the set of all subsets of X and $\mathrm{End}(X)$ denote the set of all endomappings $f : X \longrightarrow X$. For any $\mathcal{A} \subseteq \mathrm{Sub}(X)$ and $\mathcal{F} \subseteq \mathrm{End}(X)$ define

$$\mathrm{Alg}(\mathcal{A}) = \{f \in \mathrm{End}(X) : f(A) \subseteq A \text{ for all } A \in \mathcal{A}\},$$
$$\mathrm{Lat}(\mathcal{F}) = \{A \in \mathrm{Sub}(X) : f(A) \subseteq A \text{ for all } f \in \mathcal{F}\}.$$

A family $\mathcal{A} \subseteq \mathrm{Sub}(X)$ is called *reflexive* if $\mathcal{A} = \mathrm{Lat}(\mathrm{Alg}(\mathcal{A}))$, or equivalently, $\mathcal{A} = \mathrm{Lat}(\mathcal{F})$ for some $\mathcal{F} \subseteq \mathrm{End}(\mathrm{X})$.

As was shown in [9], $\mathcal{A} \subseteq \mathrm{Sub}(X)$ is reflexive iff it is closed under arbitrary unions and intersections and contains the empty set and X. The reflexive families $\mathcal{F} \subseteq \mathrm{End}(X)$ were also introduced and characterized as those subsemigroups $\mathcal{L}$ of $(\mathrm{End}(X), \circ)$ such that $\mathcal{L}$ is a lower set and contains all existing suprema of subsets of $\mathcal{L}$ with respect to a naturally defined partial order on $\mathrm{End}(X)$. The similar work in functional analysis is on the reflexive invariant subspace lattices and reflexive operator algebras [1-6].

For any $\mathcal{A} \subseteq \mathrm{Sub}(X)$, let $\hat{\mathcal{A}} = \mathrm{Lat}(\mathrm{Alg}(\mathcal{A}))$. Then $\hat{\mathcal{A}}$ is the smallest family of subsets containing $\mathcal{A}$ which is closed under arbitrary unions and intersections containing empty set $\emptyset$ and X, and $\hat{\mathcal{A}}$ is finite if $\mathcal{A}$ is finite. Furthermore, $\hat{\mathcal{A}} = \mathrm{Lat}(\mathcal{F})$, where $\mathcal{F} = \mathrm{Alg}(\mathcal{A})$.

It is, however, still not known whether for any finite family $\mathcal{A}$ there is a finite $\mathcal{F} \subseteq \mathrm{End}(X)$ such that $\hat{\mathcal{A}} = \mathrm{Lat}(\mathcal{F})$.

In this short paper, we shall answer the above problem. It will be shown that the answer is positive if and only if X is a finite or countably infinite set. For

Received July 19, 2012; accepted April 24, 2013.
2010 Mathematics Subject Classification: 54C05, 54C60, 54B20, 06B99.
Key words and phrases: reflexive families, reflexive index, endomapping.

any family $\mathcal{A}$ of subsets of a set, we define a cardinal $\kappa(\mathcal{A})$, which, in a certain sense, reflects how the sets in $\mathcal{A}$ are interrelated. This cardinal will be called the reflexive index of $\mathcal{A}$. The reflexive indices of some special families are computed. For instance, we show that if $\mathcal{A}$ is a finite chain of subsets of $\mathbb{N}$ (the set of all natural numbers) with more than one member, then $\kappa(\mathcal{A}) = 2$.

Definition 1. Let $\mathcal{A}$ be a family of subsets of a set X. The *reflexive index* of $\mathcal{A}$ is defined as

$$\kappa_X(\mathcal{A}) = inf\{|\mathcal{F}| : \mathcal{F} \subseteq \mathrm{End}(X),\ \hat{\mathcal{A}} = \mathrm{Lat}(\mathcal{F})\},$$

where $|\mathcal{F}|$ is the cardinal of $\mathcal{F}$.

We shall write $\kappa(\mathcal{A})$ for $\kappa_X(\mathcal{A})$ if the set X is clearly assumed.

If $C \subseteq X$ and $f \in \mathrm{End}(X)$ such that $f(C) \subseteq C$, then we say that C is invariant under f. Thus $\mathrm{Lat}(\{f\})$ is the set of all subsets which are invariant under f. Also note that $\mathrm{Lat}(\mathcal{F}) = \bigcap\{\mathrm{Lat}(\{f\}) : f \in \mathcal{F}\}$ for any $\mathcal{F} \subseteq \mathrm{End}(X)$.

In the following we shall use $\mathbb{N}$ to denote the set of all natural numbers.

Remark 2.

(1) For any $\mathcal{A} \subseteq \mathrm{Sub}(X)$, $\hat{\hat{\mathcal{A}}} = \hat{\mathcal{A}}$, thus $\kappa_X(\hat{\mathcal{A}}) = \kappa_X(\mathcal{A})$.

(2) For any $g \in \mathrm{End}(X)$, where X is an infinite set, $\mathrm{Lat}(\{g\})$ is an infinite family. To see this, consider any $a \in X$. If $\{g^k(a) : k \in \mathbb{N}\}$ is an infinite set, then $g^k(a) \neq g^i(a)$ whenever $k \neq i$. In this case, $\{\{g^i(a) : i \geq k\} : k \in \mathbb{N}\}$ is an infinite subfamily of $\mathrm{Lat}(\{g\})$ (g^k is the composition of k copies of g). Now assume that for each $a \in X$, $\{g^k(a) : k \in \mathbb{N}\}$ is a finite set, then there are infinitely many sets of the form $\{g^k(a) : k \in \mathbb{N}\}$ $(a \in X)$, each of them is a member of $\mathrm{Lat}(\{g\})$. Therefore $\mathrm{Lat}(\{g\})$ is infinite.

Lemma 3. *Let X be a nonempty set.*

(1) *If X is a countably infinite set, then there are two mappings*

$$\mu_X^0,\ \mu_X^1 : X \longrightarrow X$$

such that for any nonempty $B \subseteq X$, if $\mu_X^0(B) \subseteq B$ and $\mu_X^1(B) \subseteq B$ then $B = X$.

(2) *If X is a finite set, there is one mapping $\mu_X^0 : X \longrightarrow X$ such that for any nonempty $B \subseteq X$, $\mu_X^0(B) \subseteq B$ implies $B = X$.*

Proof. (1) If $X = \{a_1, a_2, \cdots\}$ is a countably infinite set, define $\mu_X^0(a_i) = a_{i+1}$ and $\mu_X^1(a_i) = a_1$ for each i. Then μ_X^0 and μ_X^1 satisfy the requirement.

(2) If $X = \{a_1, a_2, \cdots, a_n\}$ is a finite set, define $\mu_X^0(a_i) = a_{i+1}(1 \leq i < n)$ and $\mu_X^0(a_n) = a_1$. Then $B \subseteq X$ and $\mu_X^0(B) \subseteq B$ will imply $B = \emptyset$ or $B = X$. □

Proposition 4. *Let X be a nonempty set.*

(1) *If X is countably infinite, $\kappa_X(\{\emptyset, X\}) = 2$.*

(2) *If X is a finite set, $\kappa_X(\{\emptyset, X\}) = 1$.*

Proof. First, note that the family $\{\emptyset, X\}$ is closed under arbitrary intersections and unions, so it is reflexive, i.e. $\mathrm{Lat}(\mathrm{Alg}(\{\emptyset, X\})) = \{\emptyset, X\}$ (Theorem 1 of [9]).

The statement (2) clearly follows from Lemma 3(2).

To prove (1), by Lemma 3(1), we have $\kappa_X(\{\emptyset, X\}) \leq 2$. Also, by Remark 2 (2), for any $f \in \mathrm{End}(X)$ the set $\mathrm{Lat}(\{f\})$ is infinite, implying $\{\emptyset, X\} \neq \mathrm{Lat}(\{f\})$. Hence $\kappa_X(\{\emptyset, X\}) = 2$. $\square$

Proposition 5. *If X is a noncountable infinite set, then $\kappa_X(\{\emptyset, X\}) = |X|$, where $|X|$ is the cardinal of X.*

Proof. Let $\emptyset \neq \mathcal{F} \subseteq \mathrm{End}(X)$ and $|\mathcal{F}| < |X|$. Take $\mathcal{F}^*$ to be the subsemigroup of $(\mathrm{End}(X), \circ)$ generated by $\mathcal{F}$, where $\circ$ is the composition operation. If $\mathcal{F}$ is finite, then $\mathcal{F}^*$ is finite or countably infinite. Since X is uncountable it follows that $|\mathcal{F}^*| < |X|$. If $\mathcal{F}$ is infinite, then $|\mathcal{F}^*| = |\mathcal{F}| < |X|$. Chose one element $a \in X$ and let $\mathcal{F}^*a = \{f(a) : f \in \mathcal{F}^*\}$, called the orbit of a under $\mathcal{F}$. Clearly $\mathcal{F}^*a$ is a member of $\mathrm{Lat}(\mathcal{F})$. However, $|\mathcal{F}^*a| \leq |\mathcal{F}^*| < |X|$, implying $\mathcal{F}^*a \neq X$. Also as $\mathcal{F}^*a \neq \emptyset$, so $\mathrm{Lat}(\mathcal{F}) \neq \{\emptyset, X\}$. It thus follows that $\kappa_X(\{\emptyset, X\}) \geq |X|$. Now consider $\mathcal{K} = \{f_a : a \in X\} \subseteq \mathrm{End}(X)$, where $f_a : X \longrightarrow X$ is the constant mapping that sends every $x \in X$ to a. Then for any nonempty set $B \subseteq X$, if $f_a(B) \subseteq B$ for all $a \in X$, then $X = B$. Therefore $\mathrm{Lat}(\mathcal{K}) = \{\emptyset, X\}$ and so $\kappa_X(\{\emptyset, X\}) \leq |\mathcal{K}| = |X|$. All these show that $\kappa_X(\{\emptyset, X\}) = |X|$. $\square$

Now we prove the main result of this paper.

Theorem 6. *Let X be a finite or countably infinite set. Then for any finite family $\mathcal{A} \subseteq Sub(X)$, $\kappa(\mathcal{A})$ is finite.*

Proof. Since the conclusion is clearly true if X is a finite set, we only give the proof for countably infinite sets X. To simplify the argument we take $X = I\!N$ (the set of all natural numbers) and denote $\kappa_N(\mathcal{F})$ simply by $\kappa(\mathcal{F})$. Without lose of generality, we assume that $I\!N \in \mathcal{A}$. By rearranging, if necessary, we can let $\mathcal{A} = \{A_1, A_2, \cdots, A_m\}$ such that $A_1 = I\!N$ and $j > i$ if $A_j \subset A_i$ (however, $j > i$ need not imply $A_j \subset A_i$). Let $\Theta = \{(i_1, i_2, ..., i_k) : 1 \leq i_1 < i_2 < \cdots < i_k \leq m, 1 \leq k \leq m\}$. For each $\sigma = (i_1, i_2, ..., i_k) \in \Theta$, define $X_\sigma = \bigcap_{t=1}^{k} A_{i_t} - \bigcup\{A_s : s \neq i_t, t = 1, 2, \cdots, k\}$. If $\sigma = (i_1, i_2, ..., i_k) \in \Theta$, we call each $i_t, 1 \leq t \leq k$ a component of σ.

For each $\sigma \in \Theta$, we assume that $X_\sigma = \{a_1^\sigma, a_2^\sigma, \cdots\}$ such that $a_1^\sigma < a_2^\sigma < \cdots$ when $X_\sigma \neq \emptyset$.

For $\sigma_1 = (s_1, s_2, \cdots, s_k)$, $\sigma_2 = (t_1, t_2, \cdots, t_l) \in \Theta$, define $\sigma_1 < \sigma_2$ if $\{t_1, t_2, \cdots, t_l\} \subset \{s_1, s_2, \cdots, s_k\}$.

It's easy to see that the following statements are true:

(a) X_σ and X_β are disjoint if $\sigma \neq \beta$;

(b) for each $A_i \in \mathcal{A}$, $A_i = \bigcup\{X_\sigma : i$ is a component of $\sigma\}$;

(c) for each $\sigma = (i_1, i_2, \cdots, i_k) \in \Theta$,

$$\bigcap\{A_{i_t} : t = 1, 2, \cdots, k\} = \bigcup\{X_\beta : \beta \in \Theta, \beta \leq \sigma\}.$$

Now let $f^0 : I\!N \longrightarrow I\!N$ be a mapping such that $f^0|_{X_\sigma} = \mu^0_{X_\sigma}$ as constructed in the proof of Lemma 3 for each set X_σ (note that $X'_\sigma s$ are disjoint sets and their union is $I\!N$). Let $f^1 : I\!N \longrightarrow I\!N$ be the mapping such that $f^1(x) = a_1^\sigma$ for each $x \in X_\sigma$.

For any $\sigma = (s_1, s_2, \cdots, s_k) > \beta = (t_1, t_2, \cdots, t_m)$ define $f_{\sigma,\beta} : I\!N \longrightarrow I\!N$, if X_σ and X_β are nonempty, as follows:

$$f_{\sigma,\beta}(x) = \begin{cases} x, & \text{if } x \notin X_\sigma, \\ a_1^\sigma, & \text{if } x \in X_\sigma - \{a_1^\sigma\}, \\ a_1^\beta, & \text{if } x = a_1^\sigma. \end{cases}$$

Now consider the finite family $\mathcal{F} = \{f^0, f^1\} \bigcup \{f_{\sigma,\beta} : \sigma, \beta \in \Theta, \sigma > \beta, X_\sigma \neq \emptyset, X_\beta \neq \emptyset\}$.

(1) Let $A_i \in \mathcal{A}$. By above (a) and (b), A_i is a disjoint union of some $X_\sigma' s$. Since $f^0(X_\sigma) \subseteq X_\sigma, f^1(X_\sigma) \subseteq X_\sigma$, thus $f^0(A_i) \subseteq A_i$ and $f^1(A_i) \subseteq A_i$.

Now let $\sigma, \beta \in \Theta$ such that $\sigma > \beta$. If $x \in A_i$ and $x \notin X_\sigma$ then $f_{\sigma,\beta}(x) = x \in A_i$. If $x \in X_\sigma$, then i is a component of σ, so i is also a component of β. Now $f_{\sigma,\beta}(x) \in X_\sigma$ or $f_{\sigma,\beta}(x) \in X_\beta$. But X_σ, $X_\beta \subseteq A_i$, so $f_{\sigma,\beta}(x) \in A_i$, therefore $f_{\sigma,\beta}(A_i) \subseteq A_i$. It follows that $\mathcal{A} \subseteq \mathrm{Lat}(\mathcal{F})$. Then $\mathrm{Alg}(\mathcal{A}) \supseteq \mathrm{Alg}(\mathrm{Lat}(\mathcal{F}))$ and so $\hat{\mathcal{A}} = \mathrm{Lat}(\mathrm{Alg}(\mathcal{A})) \subseteq \mathrm{Lat}(\mathrm{Alg}(\mathrm{Lat}(\mathcal{F}))) = \mathrm{Lat}(\mathcal{F})$, the last equation holds for any $\mathcal{F}$ (see Lemma 1(3) of [9]).

(2) Given any $C \subseteq I\!N$ such that $C \in \mathrm{Lat}(\mathcal{F})$, we show that $C \in \hat{\mathcal{A}} = \mathrm{Lat}(\mathrm{Alg}(\mathcal{A}))$. First, if $C \cap X_\sigma \neq \emptyset$, then there is a point $x \in C \cap X_\sigma$, so as $f^1 \in \mathcal{F}$, $f^1(x) = a_1^\sigma \in C$. Hence $a_2^\sigma = f^0(a_1^\sigma) \in C$, $a_3 = f^0(a_1^\sigma) \in C$, etc. It then follows that $X_\sigma \subseteq C$. Now if $\beta < \sigma$, then $a_1^\beta = f_{\sigma,\beta}(a_1^\sigma) \in C$, so $X_\beta \cap C \neq \emptyset$, therefore we also can deduce that $X_\beta \subseteq C$. For any element $x \in C$, there exists $\gamma = (i_1, i_2, \cdots, i_k)$ such that $x \in A_{i_t}$ for each $t = 1, 2, \cdots, k$ and $x \notin A_j$ for all $j \notin \{i_1, i_2, \cdots, i_k\}$. Then $x \in X_\gamma$. In addition, $X_\gamma \subseteq C$ because $x \in X_\gamma \cap C$ which implies $X_\gamma \cap C \neq \emptyset$. By property (c), $\bigcap\{A_{i_t} : t = 1, 2, \cdots, k\} = \bigcup\{X_\beta : \beta \leq \gamma\} \subseteq C$. In addition, $x \in \bigcap\{A_{i_t} : t = 1, 2, \cdots, k\} \in \hat{\mathcal{A}}$ ($\hat{\mathcal{A}}$ is closed under arbitrary intersections and each $A_i \in \mathcal{A}$). All these show that C is a union of members of $\hat{\mathcal{A}}$, thus $C \in \hat{\mathcal{A}}$ because $\hat{\mathcal{A}}$ is closed under arbitrary unions. Hence $\mathrm{Lat}(\mathcal{F}) \subseteq \hat{\mathcal{A}}$.

The combination of (1) and (2) implies that $\mathrm{Lat}(\mathcal{F}) = \hat{\mathcal{A}}$. Since $|\mathcal{F}|$ is finite, the proof is completed. $\square$

Now we consider $\kappa(\mathcal{A})$ for some special families $\mathcal{A}$ of subsets of $I\!N$.

Example 7. Let $\mathcal{A} = \{2I\!N,\ 3I\!N,\ 5I\!N\}$. We show that $\kappa(\mathcal{A}) \leq 4$.

Let $I\!N - (2I\!N \cup 3I\!N \cup 5I\!N) = \{a_k : k = 1, 2, \cdots\}, 2I\!N - (3I\!N \cup 5I\!N) = \{b_k^1 : k = 1, 2, \cdots\}, 3I\!N - (2I\!N \cup 5I\!N) = \{b_k^2 : k = 1, 2, \cdots\}, 5I\!N - (3I\!N \cup 2I\!N) = \{b_k^3 : k = 1, 2, \cdots\}, 10I\!N - 3I\!N = \{c_k^1 : k = 1, 2, \cdots\}, 6I\!N - 5I\!N = \{c_k^2 : k = 1, 2, \cdots\}, 15I\!N - 2I\!N = \{c_k^3 : k = 1, 2, \cdots\}, 30I\!N = \{d_k : k = 1, 2, \cdots\}$.

Define the mappings f, g_1, g_2, g_3 in $\mathrm{End}(I\!N)$ as follows:

$$f(x)=\begin{cases} a_{k+1}, & \text{if } x=a_k\,(k\geq 1),\\ b^i_{k+1}, & \text{if } x=b^i_k\,(i=1,2,3, \text{ and } k\geq 1),\\ c^i_{k+1}, & \text{if } x=c^i_k\,(i=1,2,3, \text{ and } k\geq 1),\\ d_{k+1}, & \text{if } x=d_k\,(k\geq 1).\end{cases}$$

$$g_1(x)=\begin{cases} a_1, & \text{if } x=a_{k+1}\,(k\geq 1),\\ b^1_1, & \text{if } x=a_1,\\ c^1_1, & \text{if } x=b^i_1\,(i=1,3),\\ c^3_1, & \text{if } x=b^2_1,\\ b^i_1, & \text{if } x=b^i_{k+1}\,(i=1,2,3, \text{ and } k\geq 1),\\ c^i_1, & \text{if } x=c^i_{k+1}\,(i=1,2,3, \text{ and } k\geq 1),\\ d_1, & \text{if } x=d_k\,(k\geq 1) \text{ or } c^i_1\,(i=1,2,3).\end{cases}$$

$$g_2(x)=\begin{cases} a_1, & \text{if } x=a_{k+1}\,(k\geq 1),\\ b^2_1, & \text{if } x=a_1,\\ c^2_1, & \text{if } x=b^i_1\,(i=1,2),\\ c^3_1, & \text{if } x=b^3_1\\ b^i_1, & \text{if } x=b^i_{k+1}\,(i=1,2,3, \text{ and } k\geq 1),\\ c^i_1, & \text{if } x=c^i_{k+1}\,(i=1,2,3, \text{ and } k\geq 1),\\ d_1, & \text{if } x=d_k\,(k\geq 1) \text{ or } x=c^i_1\,(i=1,2,3).\end{cases}$$

$$g_3(x)=\begin{cases} b^3_1, & \text{if } x=a_1,\\ x, & \text{otherwise }.\end{cases}$$

Let $\mathcal{F}=\{f,g_1,g_2,g_3\}$ and $A\in \mathrm{Lat}(\mathcal{F})$.

(i) Each of $2\mathbb{N},3\mathbb{N}$ and $5\mathbb{N}$ is invariant under every mapping in $\mathcal{F}$. Thus $\hat{\mathcal{A}}\subseteq \mathrm{Lat}(\mathcal{F})$.

(ii) If $A\cap(\mathbb{N}-(2\mathbb{N}\cup 3\mathbb{N}\cup 5\mathbb{N}))\neq\emptyset$, then, as $g_1(A)\subseteq A$, it follows that $a_1\in A$. Then, each $a_{k+1}, k\geq 1$ is in A because $f(A)\subseteq A$. Since $g_i(A)\subseteq A$ it follows that $b^i_1\in A\,(i=1,2,3)$. Again, as $f(A)\subseteq A$, we deduce that A contains each of $2\mathbb{N}-(3\mathbb{N}\cup 5\mathbb{N}), 3\mathbb{N}-(2\mathbb{N}\cup 5\mathbb{N})$ and $5\mathbb{N}-(2\mathbb{N}\cup 3\mathbb{N})$. Now A contains each of $c^i_1\,(i=1,2,3)$. With a similar argument we deduce that A contains each of $6\mathbb{N}-5\mathbb{N}, 10\mathbb{N}-3\mathbb{N}, 15\mathbb{N}-2\mathbb{N}$ and $30\mathbb{N}$. Hence $A=\mathbb{N}\in \mathrm{Lat}(\mathcal{F})$.

In a similar way we can show the following statements are true:

(iii) If $A\cap(2\mathbb{N}-(3\mathbb{N}\cup 5\mathbb{N}))\neq\emptyset$ then A contains $2\mathbb{N}$. If $A\cap(3\mathbb{N}-(2\mathbb{N}\cup 5\mathbb{N}))\neq\emptyset$, then A contains $3\mathbb{N}$. If $A\cap(5\mathbb{N}-(2\mathbb{N}\cup 3\mathbb{N}))\neq\emptyset$, then A contains $5\mathbb{N}$.

(iv) If $A\cap(6\mathbb{N}-5\mathbb{N})\neq\emptyset$, respectively, $A\cap(10\mathbb{N}-3\mathbb{N})\neq\emptyset$, $A\cap(15\mathbb{N}-2\mathbb{N})\neq\emptyset$, then $A\supseteq 6\mathbb{N}$, respectively, $A\supseteq 10\mathbb{N}$, $A\supseteq 15\mathbb{N}$.

(v) If $A\cap 30\mathbb{N}\neq\emptyset$, then $A\supseteq 30\mathbb{N}$.

From (i)-(v), it follows that A either equals $\mathbb{N}$ or is a union of intersections of $2\mathbb{N},3\mathbb{N},5\mathbb{N}$, that is $A\in\hat{\mathcal{A}}$ and so $\hat{\mathcal{A}}=\mathrm{Lat}(\mathcal{F})$.

Thus $\mathrm{Lat}(\mathcal{F})=\hat{\mathcal{A}}$, so $\kappa(\mathcal{A})\leq 4$.

Remark 8. From the proof in the above example, we can see that a more general conclusion is true: if $p_1, p_2, \cdots, p_m$ are distinct primes, then $\kappa(\{\{p_i I\!N : i = 1, 2, \cdots, m\}) \leq m + 1$.

Proposition 9. *If $\mathcal{A} = \{A_1, A_2, \cdots, A_m\}$ is a finite chain of distinct subsets of $I\!N$ with $m \geq 2$, then $\kappa(\mathcal{A}) = 2$.*

Proof. Without lose of generality, we assume that $A_1 \subset A_2 \subset \cdots \subset A_m$ and $A_1 \neq \emptyset$ and $A_m = I\!N$. Let $A_1 = \{a_1^1, a_2^1, \cdots\}, A_2 - A_1 = \{a_1^2, a_2^2, \cdots\}, ..., A_m - A_{m-1} = \{a_1^m, a_2^m, \cdots\}$. Define $f, g \in \mathrm{End}(I\!N)$ as follows: for $i = 1, 2, \cdots, m$, and $k \in I\!N$,

$$f(a_k^i) = \begin{cases} a_{k+1}^i, & \text{if } a_k^i \text{ is not the last element in } A_i - A_{i-1}, \\ a_k^i, & \text{if } a_k^i \text{ is the last element in } A_i - A_{i-1}. \end{cases}$$

$$g(x) = \begin{cases} a_1^i, & \text{if } x = a_{k+1}^i, \\ a_1^i, & \text{if } x = a_1^{i+1}. \end{cases}$$

Since $\hat{\mathcal{A}}$ is the smallest family containing $\mathcal{A}$ which is closed under arbitrary unions and intersections, $\hat{\mathcal{A}} = \mathcal{A} \cup \{\emptyset\}$. Furthermore, $\mathcal{A} \cup \{\emptyset\} = \mathrm{Lat}(\{f, g\})$. Thus $\kappa(\mathcal{A}) \leq 2$. By Remark 2(2), for any $h \in \mathrm{End}(I\!N)$, $\mathrm{Lat}(\{h\})$ is an infinite family, so $\kappa(\mathcal{A}) \neq 1$, therefore $\kappa(\mathcal{A}) = 2$. □

Remark 10.

(1) The reader may wonder whether there is a set family whose reflex index is 1. Consider $\mathcal{A} = \{\emptyset, I\!N\} \cup \{C_n : n = 1, 2, \cdots\}$, where $C_n = \{n, n+1, ...\}$. Then $\mathcal{A} = \hat{\mathcal{A}} = \mathrm{Lat}(\{f\})$, where f is defined by

$$f(m) = \begin{cases} 1, & \text{if } m = 1, \\ m - 1, & \text{if } m > 1. \end{cases}$$

(2) The following is a chain of subsets of $I\!N$ whose reflexive index is not finite. Put $\mathcal{B} = \{\emptyset, I\!N, \{k : k \in I\!N, k \geq 2\}\} \bigcup \{D_n : n \in I\!N, n > 1\}$, where for each $n > 1, D_n = \{2, 3, \cdots, n\}$. Clearly $\hat{\mathcal{B}} = \mathcal{B}$. Let $\mathcal{F} \subseteq \mathrm{End}(I\!N)$ be any finite family of endomappings on $I\!N$ satisfying $\mathcal{B} \subseteq \mathrm{Lat}(\mathcal{F})$. If $f(1) = 1$ for all $f \in \mathcal{F}$, then $\{1\} \in \mathrm{Lat}(\mathcal{F}) - \hat{\mathcal{B}}$. If there is $f \in \mathcal{F}$ with $f(1) \neq 1$, let $l = max\{f(1) : f \in \mathcal{F}\}$, then $l \geq 2$ and the subset $\{1\} \cup D_l$ is in $\mathrm{Lat}(\mathcal{F}) - \mathcal{B}$. Thus for any finite $\mathcal{F} \subseteq \mathrm{End}(I\!N)$, $\mathrm{Lat}(\mathcal{F}) \neq \mathcal{B} = \hat{\mathcal{B}}$, therefore $\kappa(\mathcal{B})$ is not finite.

Remark 11.

(1) It is possible and necessary to identify the *exact* values of the reflexive indices of more concrete families (such as $\mathcal{A} = \{p_i I\!N : i = 1, 2, \cdots, n\}$ for any distinct prime numbers $p_1, p_2, \cdots, p_n$). We leave this to interested readers to try.

(2) In [7][8], the reflexive families of closed subsets of a topological space are studied. We can also define the reflexive index for a family of closed sets. One of the natural problems would be: for which spaces, does every finite family of closed

sets have a finite reflexive index? Furthermore, one can introduce and consider the reflexive index of a family of closed subspaces of a Hilbert space.

Acknowledgements. I would like to thank Professor Carsten Thomassen for a very helpful conversation with him while he was visiting our department in 2009, which led me to prove the main result in this paper. I also must thank the referees for identifying some errors in the earlier draft and giving me valuable comments and suggestions for improvement.

References

[1] P. R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc., **76**(1970), 887-933.

[2] P. R. Halmos, *Reflexive lattices of subspaces,* J. London Math. Soc., **4**(1971), 257-263.

[3] K. J. Harrison and W.E. Longstaff, *Automorphic images of commutative subspace lattices*, Proc. Amer. Math. Soc., **296**(1986), 217-228.

[4] W. E. Longstaff, *Strongly reflexive subspace lattices,* J. London Math. Soc.,**11**(1975), 2:491-498.

[5] W. E. Longstaff, *On lattices whose every realization on Hilbert space is reflexive,* J. London Math. Soc., **37**(1988), 2:499-508.

[6] W. E. Longstaff and O. Panaia,*On the ranks of single elements of reflexive operator algebras,* Proc. Amer. Math. Soc., **125**(1997), 10:2875-2882.

[7] Z. Yang and D. Zhao, *Reflexive families of closed sets,* Fund. Math., **192**(2006), 111-120.

[8] Z. Yang and D. Zhao, *On reflexive closed set lattices,* Comment. Math. Univ. Carolinae, **51**(2010), 1:23-32.

[9] D. Zhao, *On reflexive subobject lattices and reflexive endomorphism algebras,* Comment. Math. Univ. Carolinae, **44**(2003), 23-32.