SIGNATURES OF MATHEMATICAL MODELLING: TRACING PRIMARY 6 PUPILS' MATHEMATICAL MODELLING PROCESS AND MODEL DEVELOPMENT

Chan Chun Ming Eric

National Institute of Education, Nanyang Technological University

Paper presented at the 4th Redesigning Pedagogy International Conference

May/June 2011, Singapore

Abstract

As mathematical modelling is gaining prominence in the field of mathematics education, there is little to inform what it is like in the Singapore primary mathematics classroom due to the lack of research on mathematical modelling carried out locally. Without this knowledge of what it looks like or entail, mathematics classroom pedagogies tend to border on the traditional. This paper is part of a larger study investigating Primary 6 pupils' mathematical modelling process where mathematical modelling takes on a problem-solving perspective. This paper focuses on a group of four pupils’ mathematical modelling endeavour towards capturing their conceptual representations and the related mathematical translations which are seen as models. A protocol analysis method was used to code pupils' problem-solving behaviours for interpreting their modelling actions. A macro-level analysis of the pupils' modelling endeavour was carried out to construct and trace the pupils’ model development with respect to their conceptual representations and mathematical translations. The results suggest that the pupils underwent different modelling stages that were characteristic of certain modelling actions and developed a range of models. The pupils were found to develop alternative models, and they tested and revised their models towards better or newer models with the aim of attaining the best solution model. The mathematical translations for model development were based on recognizing the structure between the quantities and variables in relation to the context. Having pupils to engage in mathematical modelling is a promising platform towards realizing the important components of the Singapore Mathematics Curriculum Framework.

SIGNATURES OF MATHEMATICAL MODELLING: TRACING PRIMARY 6 PUPILS' MATHEMATICAL MODELLING PROCESS AND MODEL DEVELOPMENT

Introduction

Mathematical problem solving has been a topic of research for some three decades now since the 1980s when it gained immense attention. As the field of mathematical problem solving continues to evolve, some researchers claimed that not enough has been done to inform how students solve problems that befit realistic situations (English & Sriraman, 2010). According to English and Sriraman, (ibid), the limiting factors were the push to teach students to do well in tests and the usual practice of solving routine problems that apparently led to a decline in research to inform about concept development through problem solving. The limited knowledge of students' problem solving beyond the classroom also contributed to not knowing enough about how students can apply what they know in real-world situations.

Recently, researchers have been advocating that mathematical modelling be the future-oriented perspective on the teaching and learning of problem solving (Lesh & Zawojewski, 2007; Lesh & Sriraman, 2005; Niss, Blum, & Galbraith, 2007; Barbosa, 2009). This call is consistent with reformed efforts that promote pedagogies that teach for understanding where students' cognitive reasoning could be deeply engaged through problem-based learning (Erickson, 1999; Hiebert et al, 1996), open-ended problems (Chan 2007; Becker & Shimada, 1997), and teaching *through* problem solving (Lambdin, 2003). The rationale for change apparently is tied to equipping students for a knowledge-based workforce possessing competencies and skills beyond school to solve real-world problems (Chan, 2008a; English &

Sriraman, 2010). It is not surprising then that the revised Singapore Mathematics Curriculum Framework (2007) has included *applications and mathematical modelling* to be part of the problem-solving process component to signal its relevance in the changing educational landscape. However, as mathematical modelling is still in its formative stage since the revision, there is no record of mathematical modelling research done for primary school pupils locally prior to 2010 except the ones by the author (Chan, 2008a, 2008b, 2009, 2010).

This paper is part of a larger research in investigating Primary 6 pupils' mathematical modelling endeavour that traces the pupils' modelling process and their model development where modelling is seen as a problem-solving activity to support the mathematics curriculum.

Mathematical Modelling and Model Development

Mathematical modelling has been defined variously in literature. This paper adopts a modelling perspective that sees mathematical modelling as a problem-solving activity. Mathematical modelling is in this paper is defined as a problem-solving activity that involves the process of using mathematical language and mathematical reasoning to give meaning to specific situations as pupils work on contextual modelling tasks. Models are developed during the modelling process and are the conceptual interpretation or representation of specific situations characterized by elements of ideas, concepts, constructs and relationships. Thus when students paraphrase, explain, draw diagrams, categorize, find relationships, dimensionalize, quantify, or make predictions, they are generally developing their conceptual systems or models through the mathematizing. The representations are expressed in the form of text, diagrams, abstractions, or verbal explanations through the connections and operationalization of these elements. Models differ in sophistication and quality depending on how adequately the pupils develop them through the use of their problem-solving behaviours.

An emerging model is a model that can be utilized to become part of or that evolves into another model.

The mathematical modelling endeavour is situated in a problem-based learning setting. The problem-based approach is intentionally factored to highlight the significance of the task driving the learning (Tan, 2003; Stephien & Pyke, 1997; Hmelo-Silver, 2004). The task (a modelling task in this case) is differentiated from traditional problem solving in that it is based on authentic and contextually rich information used as the stimulus for triggering deep and high extents of mathematical and metacognitive thinking. The task has to be engaged collaboratively as it appeals to different ways of managing it and can result in a variety of solutions. In a problem-based setting, the teacher facilitates rather than gives direct instructions.

The Modelling Process

Figure 1 shows a generic mathematical modelling process that involves four modelling stages, namely, *Describing*, *Manipulating*, *Predicting* and *Verifying* (Ang, 1991; Blum & Niss, 1991; Chan, 2008a; Lesh & Doerr, 2003; Mousouslides, Sriraman, Pittalis, & Christou, 2007; Swetz & Hartzler, 1991) although the terms used may be slightly different amongst some researchers. The common premise is that the starting point is a real-world problem or situation that can be formulated into a mathematical problem and where the mathematical solutions are used to interpret the real-world situation.

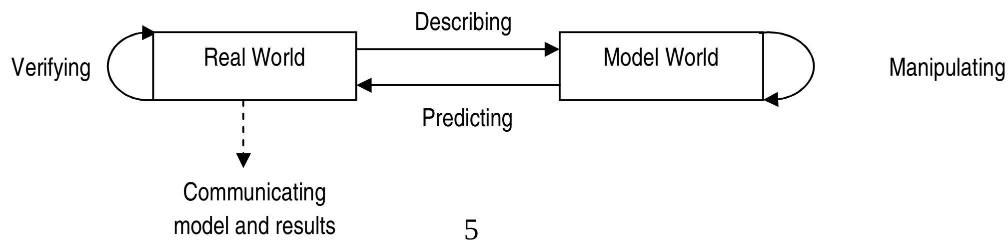


Figure 1. Modelling stages

In this paper, the four stages have been identified as *Describing*, *Manipulating*, *Predicting* and *Optimization* where *Verifying* is subsumed under *Predicting*. As part of defining some operational terms, *Description* refers to attempts at understanding the problem to simplifying it. This includes drawing inferences from text, diagrams, formulas or whatever given data to make sense of the task details. *Manipulation* refers to attempts at establishing relationships between variables and task details through constructing hypotheses, critically examining contextual information, retrieving or organizing information, mathematizing, or using strategies towards developing a mathematical model. Most of the mathematical computations and reasoning take place in this stage. *Prediction* refers to attempts at interpreting the models that they have conceived to ensure that they fit the parameters given or established through affirming, verifying or making decisions to justify their cause. *Optimization* refers to attempts at improving or extending their model solutions to achieve an ideal solution that is quantity efficient and yet maximizing value. Optimization could only be achieved when the preceding stage has been achieved, that is, a workable model has to be attained.

Method

This section describes in brief the participants involved in this study, the modelling task they were engaged in and the data collection and analysis method.

Participants

The participants were Primary 6 pupils from two classes of the same neighbourhood schools. Each class comprised small groups of four or five pupils. Two groups from each class were selected by their respective mathematics teacher to be the target group for video-recording.

Modelling Task

The pupils were engaged in solving a different modelling problem each week for five weeks. In this paper, only the Floor-Covering Problem is discussed. The task in this paper (see Figure 1) is an adaptation of the modelling task from Gravemeijer, Pligge and Clarke (1998).

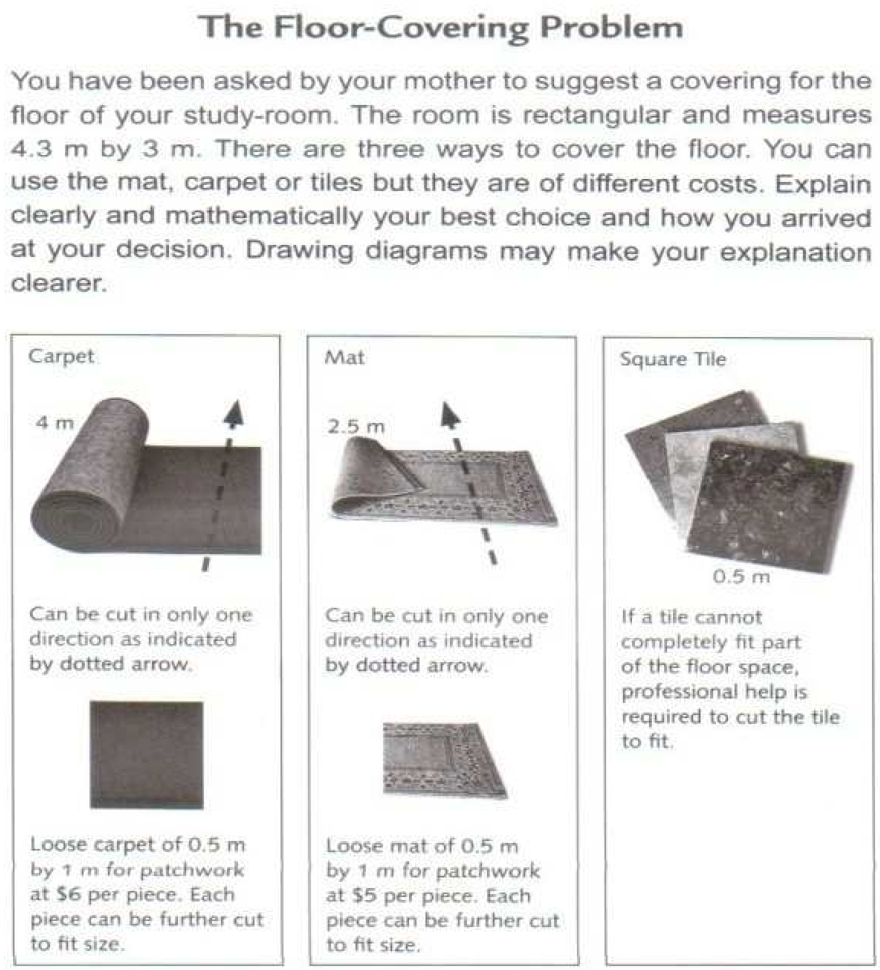


Figure 1. The Floor-Covering Problem

The task is situated in the context of determining the most economic way of covering the floor of a study room given the different floor-covering materials that were priced differently. The pupils had to identify the key mathematics components from the task, conceive layout designs, establish key variable relationships and analyze their designs to determine which design was most economical.

Data Collection and Analysis

The target groups were video-recorded during their engagement of the modelling task. Other sources of data included my field notes, the pupils' written work and journals. The video data were transcribed and reviewed several times for evidence of conceptual interpretations that suggested the progression of the pupils' thinking for emerging models through modelling stages. Although a problem-solving coding scheme was used to code the pupils' protocols for specific problem-solving behaviours, this paper is limited to highlighting the modelling stages and the models that the group of pupils developed in the modelling process.

Findings

The findings of one group of four pupils engaged in the deliberation of the Floor-Covering Problem are presented in two subsections: the modelling process and the model development of the pupils.

The Mathematical Modelling Process

Figure 2 shows a timeline diagram that captures the modelling stages the pupils went through during the modelling process. The bracketed numbers printed in certain stages imply the number of models the students had developed within those stages and the "T" implies the presence of teacher in providing scaffolding.

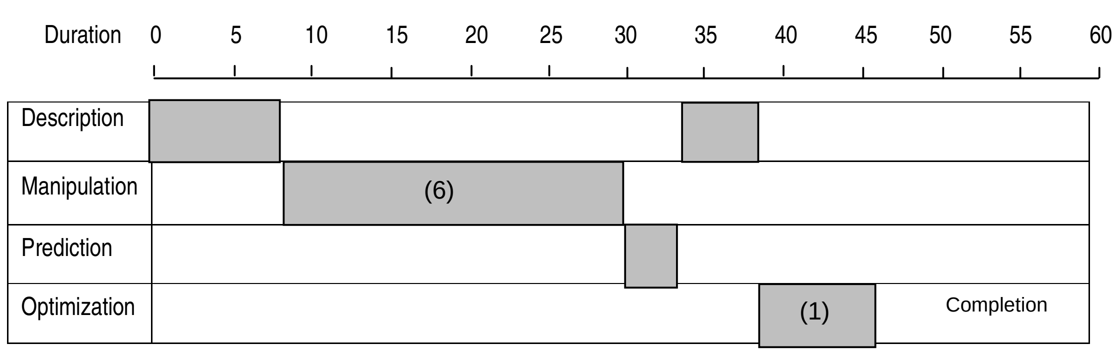


Figure 2. Pupils' mathematical modelling process

The pupils took 46 minutes to complete the modelling task. The modelling endeavour was characterized by the pupils being involved in the four modelling stages but it was not entirely linear from start to finish. The modelling pathway was *Description* → *Manipulation* → *Prediction* → *Description* → *Optimization*.

The pupils had spent quite a substantial amount of time (approximately 13 minutes) during the *Description* stages. This suggested the need to understand the problem task more deeply before they could proceed to work on and manipulate the variables. By returning to the *Description* stage a second time, it shows the need to further make sense of the task details before proceeding again. The following excerpt shows how the pupils engaged one another to clarify details towards sense making during the first *Description* stage.

55	S2	Why don't we calculate the best way and the cheapest way for each one first?
56	S4	Can I say something? They never state to find the cheapest way, so we can choose not to.
57	S3	Yes, "explain clearly mathematically your best choice" *(referring S4 to the task sheet).*
58	S4	They say the best choice, not necessarily the cheapest.
59	S3	What do you mean by the best choice?
60	S2	The cheapest choice lah. The one that you spend the least money.

The contemplation saw them defining what they wanted to achieve through giving meaning to the words "best choice" which kept them focused on their goal. Clarifying the terms enabled them to move on in one accord.

The *Manipulation* stage registered the longest duration, approximately 22 minutes. This was expected as the pupils were involved in establishing variable relationships and constructing models. During this stage, the pupils developed six models. Much of the mathematizing (dimensionalizing, analyzing, explaining, hypothesizing, conjecturing, comparing, etc.) took place here as pupils related dimensions of the floor-covering materials to the dimensions of the floor and as well compared areas and costs. Models developed were compared and were either improved upon or for some decisions to be made to select a particular floor-covering design. The models that the pupils have constructed are discussed in the section Model Development.

During the *Prediction* stage, the pupils interpreted by verifying that they had obtained a workable model.

224	S3	OK, so for the carpet we have $162, for the mat we have $145, and for the tiles, we have $162 again. So it is mat, correct? Mat, Method 2.
		(S1 records)
225	S2	Although for the loose pieces, it has lots of leftover, it is still the cheapest method.
226	S1	I agree
227	S3	Same here.
228	S2	Shall we double check?
229	S3	I don't think so. We should do the survey now.
		(S2 takes out their Group Journal sheet and prepares to write)
230	S3	Thus our conclusion is that (…) if we use mat, method 2, (…) eh, XXX, mat method 2 how to make?

As seen, in protocol line (PL) 224, they were verifying their choice of materials by comparing what they had found out about their costs. This was affirmed in PL 225, 226 and 227. They made their conclusion in PL 230. Although PL 230 sounded vague, they were basically contemplating how to go about writing their conclusion and not contemplating about their arrived decision.

The pupils were aided by the teacher in extending their thinking. This enabled the pupils to improve on their model which they had not thought about before. The revising and improvement took place in the *Optimization* stage since they were then involved in maximizing value for cost and material savings. The excerpt is shown below:

263	T	OK, so here is 0.2, right? So you'll have these pieces. Can you use these remaining pieces?
264	S2	You mean you can cut along the loose pieces?
265	T	It’s a carpet. What did they say? You can "further cut". (*T points to the task sheet)* Think about it.
266	S3	We can cut it as small as possible.
267	T	Further cut. Think about it.
268	S2	OK, lets go back to this one. Carpet. 4 times 3. So how do we do it? (S2 begins to unroll the serviette again model the situation)
269	S3	If she is saying that we can use the remaining loose carpet to fix instead of buying another loose carpet to fix, that means instead of buying another loose carpet, we can use the remaining to fix.
270	S2	0.3 here. 0.3 times 3. Here is 0.2, 0.2. Then here is 0.4.
271	S2	Then we only need 2.
272	S3	Yeah, we only need 2.
273	S2	OK, then 2 lor. Then change lor (S2 uses the correction fluid to make changes)
274	S3	So instead of 18 and instead of 3 pieces, now we have 2 pieces of loose carpet.
275	S1	Then how about this?
276	S2	So this is 4 x 4, so 0.3 times 3. Here also 12 (S2 uses the correction fluid to make changes)
277	S4	OK, we have only last 5 minutes.
278	S2	So now we have changed the total cost.
279	S1	I feel this is the cheapest.
280	S2	2.5 by 3 times 2.
281	S3	You can buy one, and the loose carpet still got remaining, we can fix also. Maybe we can buy one instead of 2.

As background information, the pupils had thought that they could patch a certain area of floor gap using 3 loose pieces of mat material. As seen in PL 268 to 270, the pupils revisited their model and determined that they could optimize the potentially wasted material through re-using it. It resulted in the realization that they needed only to purchase 2 loose pieces instead of 3 to patch the floor gap (see PL 271, 272, and 281).

Model Development

The models that the pupils had conceptualized are shown in Figures 3 (*Manipulation* stage) and 4 (*Optimization* stage).

In Figure 3, the pupils conceptualized six models or designs in the ways the floor could be covered by the various floor-covering materials during the *Manipulation* stage. The designs for the models (a) to (e) were based on the different ways the material, carpet and mat, could be unrolled against the floor and the different orientations the loose materials could be used to patch the floor gap. Each model resulted in a different cost because different amounts of materials were used. For model (f), there was only one way to cover the floor through tiling. The interpretations of the models were derived from analyzing the group's verbal protocols and they are presented in the last column of Figure 3. It suggests that pupils were dominantly involved in mathematical reasoning and aspects of mathematizing in translating their conceptual models.

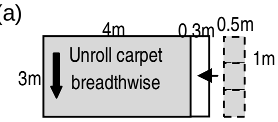
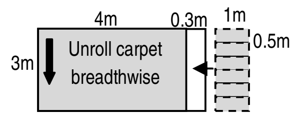
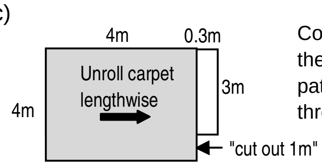
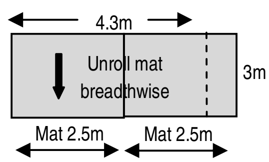
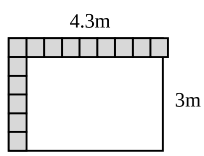
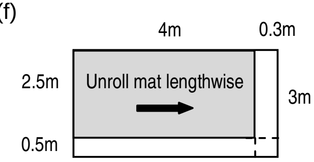
Modelling Stages	Model development (Developing conceptual representations)	Interpretation of models through students' reasoning and computations
Manipulation Stage	(a)  Conceptualized laying out the carpet breadthwise and patching the floor gap with three loose pieces as shown.	(a) and (b). Reasoned about number of loose pieces needed to patch the gap; (a) is more cost efficient than (b) because only 3 loose pieces were needed instead of 6. Worked out total cost of carpeting as: (4 x 3 x $12) + (3 x $6) = $162
	(b)  Conceptualized laying out the carpet breadthwise and patching the floor gap with six loose pieces as shown.	
	(c)  Conceptualized laying out the carpet lengthwise and patching the floor gap with three loose pieces.	(c). "*Now you turn it and paste it this way. Here 4m. So you have to cut out 1m…*"
	(d)  Conceptualized laying out two rolls of mat breadthwise.	(d). It will be 2.5 times 3 plus another one (another roll), that will be 5 times 3. 2.5 x 3 x 2
	(e)  Conceptualized laying out of tiles.	(e). Dimensions of gap worked out as (4m x 0.5m) and (0.3m x 2.5m). By aligning 1m edge of loose pieces to the gaps, 7 loose pieces were identified. Total cost of matting worked out as (4 x 2.5 x $11) + (7 x $5) = $145. However, did not unroll until 4.3m.
	(f)  Conceptualized laying out of mat lengthwise.	(f). Reasoned that 9 tiles in a row for 6 rows would be sufficient to cover the floor. Worked out the cost of tiling as 9 x 6 x $3 = $162. Cost of cutting tile not considered.

Figure 3. Model development during the Manipulation Stage

Their ideal model is presented in Figure 4 where the pupils optimized it by reusing part of the potentially wasted loose pieces to patch an area in the floor gap thus resulting in cost and material savings.

(g)

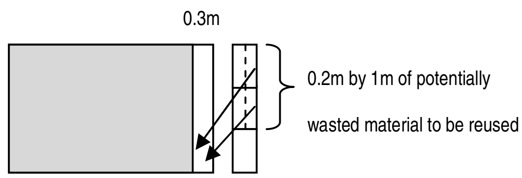


Figure 4. Pupil's attempt in optimizing floor-covering layout design

Discussion and Implications

Unlike the solving of structured problems, mathematical modelling is seen as a problem-solving activity where pupils who engage in it take a much longer time to complete the modelling task. The modelling is characterized by the pupils undergoing different stages during the modelling process where each stage depicts certain modeling behaviours manifested by the pupils. The process usually is not linear, suggesting its iterative nature. The iterative process is consistent with other studies that showed the need to revisit problem information, test and revise approaches towards improving the models (Blomhøj, 2004; Doerr & English, 2003). The development of the models as evident in this paper suggests that pupils were expressing, testing and revising their models and in so doing, they were making visible their mathematical thinking and mathematizing capabilities. It enforces the meaningfulness of problem-solving in that it went beyond using the givens and mapping them towards getting a single correct solution but that the students were thinking more deeply about making their solutions workable and better (Doerr & English, 2003). Moreover, the testing and revising shows why mathematical modelling is problem-driven; the instructional approach engages intense pupil-task interaction and as well pupil-pupil interaction thus keeping the cognitive level high. The value of the testing and revising is also seen as having their conceptualizations being put to the test repeatedly which in a sense is part of a developmental process in nurturing thinking.

A fundamental aspect of the mathematical modelling process according to Llinares and Roig (2006) was to be able to recognize the underlying structure of the situation. It implies the recognition of the quantities and variables that were involved in the situation and how the students should manipulate these quantities and variables between them in achieving their goals. The more pronounced emerging models reported in this study show how the students had used a consistent structure: Area of floor x cost per unit area of material. This structure is viewed as a summation of two parts: (Area of floor covering material for certain amount of floor area x cost per unit area of material) + (Area of floor gap x cost of amount of loose material). The conceptual representations coupled with the mathematical relations bears similarities to Gravemeijer's (1997) notion of mathematical modelling as a form of organizing and translating where models emerge through the organizing and the related mathematical procedures as translation. In this sense, a basic model that was developed became the tool to use regardless of the orientation of the material as laid out on the floor or the type of material used.

The findings in this paper also suggest that pupils tend not to be able to extend their thinking towards optimizing their models. They stop when they have obtained a workable model. To them, they have solved the problem. Thus, this is where the teacher-pupil interaction is crucial to help scaffold their thinking further. As seen in this paper, a little scaffolding by the teacher (PL 263 to 267) enabled them to transform their quantities into ways that helped them maximize value.

Conclusion

The findings and discussion all point to a pedagogy that is vastly different from traditional problem solving. Pupils embarking on mathematical modelling go through different modelling stages. The iterative aspect of the modelling process enables them to evaluate and revise their models towards goal resolution. Going into a particular modelling stage provides a glimpse of the different modelling process students are involved in, from understanding and unpacking the task details, conceptualizing the models through manipulating the data, to verifying and making improvements to the models, suggesting the simulation of actual problem solving in the real world. The models the students developed reveal the ways they organize the quantities and variables as relationships through their discourse in interpreting, analyzing, explaining, hypothesizing, conjecturing, comparing, and justifying. Emerging models become the tools for generating more and better models when students recognize the structure involved.

While the findings reported in this paper holds promise in support of mathematical modelling, more research is needed in this area to link theory and practice, to find an appropriate balance between contemporary and traditional approaches, and as well to address the beliefs of teachers, pupils and parents since this domain is very new in the curriculum. The writing of this paper however has made a small but significant step in promoting mathematical modelling in the mathematics classroom.

References

Ang, K. C. (2001). Teaching mathematical modelling in Singapore schools. The *Mathematics Educator, 6*(1), 62-75.

Barbosa, J. C. (2009). Mathematical modelling, the socio-critical perspective and the reflexive discussions. In Blomhøj, M., & Carreira, S. (Eds.), *Proceeding from Topic Study Group 21 at ICME-11* (pp. 133-143). Monterrey: IMFUFA tekst n. 461, Department of science, systems and models, Roskilde University.

Becker, J., & Shimada, Y. (1997). *The open-ended approach: A new proposal for teaching mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Blomhøj, M. (2004). Mathematical modelling: A theory for practice. In B. Clarke, D. M. Clarke, G. Emanuelsson, B. Johansson, D. V. Lambdin, F. K. Lester, A. Wallby, and K. Wallby (eds.). *International perspectives on learning and teaching mathematics*. Sweden: National Center for Mathematics Education.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects – state, trends, and issues in mathematics instruction. *Educational Studies in Mathematics*, 22(1), 37-68.

Chan, C. M. E. (2007). Using open-ended mathematics problems: A classroom experience (Primary). In C. Shagar & R. B. A. Rahim (Eds.), *Redesigning pedagogy: Voices of practitioners* (pp. 129-146). Singapore: Pearson Education South Asia.

Chan, C. M. E. (2008a). Using model-eliciting activities for primary mathematics classroom. *The Mathematics Educator,11* (1/2), 47-66.

Chan, C. M. E. (2008b). Mathematical modeling experiences for mathematical development in children. *Thinking Classroom, 9*(2), 37-45.

Chan, C. M. E. (2009). Mathematical modelling as problem solving for children in the Singapore mathematics classroom. *Journal of Science and Mathematics Education in Southeast Asia, 32*(1), 36-61.

Chan, C. M. E. (2010). Mathematical modelling in a PBL setting for pupils: Features and task design. In B. Kaur, & J. Dindyal (Eds.), *Mathematical Applications and Modelling Year Book 2010* (pp. 112-128). Singapore: World Scientific.

Doerr, H. M., & English, L. D. (2003). A modelling perspective on students' mathematical reasoning about data. *Journal for Research in Mathematics Education, 34*, 110-136.

English, L. D., & Sriraman, B. (2010). Problem solving for the 21st century. In English, L. D. & Sriraman, B. (Eds.). *Theories of mathematics education: Seeking new frontiers.* Advances in Mathematics Education, Series: Springer.

Erickson, D. K. (1999). A problem-based approach to mathematics instruction. *The Mathematics Teacher, 92*(6), 516-521.

Gravemeijer, K. (1997). Commentary: solving word problems: a case of modelling? *Learning and Instruction, 7*(4), 389Y397.

Gravemeijer, K., Pligge, M. A., & Clarke, B. (1998). Reallotment. In National Center for Research in Mathematical Science Education & Freudenthal Institute (Eds.). *Mathematics in context.* Chicago: Encyclopaedia Britannica.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A., & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher, 25* (4), 12-21.

Hmelo-Silver, C. E. (2004). Problem-based learning: What and how do students learn? *Educational Psychology Review, 16*(3), 235-266.

Lambdin, D. V. (2003). Benefits of teaching through problem solving. In Lester, F. K. (Ed). *Teaching mathematics through problem solving: Prekindergarten–Grade 6* (pp. 3-13). Reston, VA: National Council of Teachers of Mathematics.

Lesh, R., & Doerr, H. (2003). Foundations of a models and modeling perspective in mathematics teaching and learning. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning and teaching* (pp. 3-34). Mahwah, NJ: Lawrence Erlbaum Associates.

Lesh, R., & Sriraman, B. (2005). John Dewey revisited—pragmatisim and the models modeling perspective on mathematical learning. In A. Beckmann, C. Michelsen, & B. Sriraman (Eds.). *Proceedings of the 1st international symposium of mathematics and its connections to the arts and sciences* (pp. 7-31). The University of Education, Schwöbisch Gmund, Germany.

Lesh, R., & Zawojewski, J. (2007). Problem solving and modeling. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 763-803). Charlotte, NC: Image Age Publishing.

Llinares, S., & Roig, A. I. (2006). Secondary school students' construction and use of mathematical models in solving word problems. *International Journal of Science and Mathematics Education, 6*(3), 505-532.

Ministry of Education. (2007). *Mathematics Syllabus* – Primary, Singapore: Curriculum Planning and Developmental Division.

Mousoulides, N., Sriraman, B., & Christou, C. (2007). From problem solving to modelling: The emergence of models and modelling perspectives. *Nordic Studies in Mathematics Education, 12*(1), 23-47.

Niss, M., Blum, W., & Galbraith, P. (2007). How to replace the word problems. In W. Blum, P. Galbraith, H-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education: The 14th ICMI study*. (pp.3-32). New York: Springer.

Stepien, W. J., & Pyke, S. L. (1997). Designing problem-based learning units. *Journal for the Education of the Gifted, 20*(4), 380-400.

Swetz, F. and Hartzler, J. S. (1991). *Mathematical modeling in the secondary school curriculum*. The National Council of Teachers of Mathematics: Reston, Virginia.

Tan, O. S. (2003). *Problem-based learning innovation: Using problems to power learning in the 21st century*. Singapore: Thomson Learning.