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When exactly is Scott sober?

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Abstract

A topological space is sober if every nonempty irreducible closed set is the closure of a unique singleton set. Sobriety is precisely the topological property that allows one to recover completely a topological space from its frame of opens. Because every Hausdorff space is sober, sobriety is an overt, and hence unnamed, notion. Even in non-Hausdorff settings, sober spaces abound. A well-known instance of a sober space appears in domain theory: the Scott topology of a continuous dcpo is sober. The converse is false as witnessed by two counterexamples constructed in the early 1980's: the first by P.T. Johnstone and the second (a complete lattice) by J. Isbell. Since then, there has been limited progress in the quest for an order-theoretic characterization of those dcpo's for which their Scott topology is sober. This paper provides one answer to this open problem.

1 Introduction

A subset of a topological space is irreducible if it is not the union of two proper closed subsets. A topological space is sober if the singleton closures are the only nonempty irreducible closed sets. In the theory of Hausdorff topological spaces, the notion of sobriety is overtly invisible since every Hausdorff (also known as T_2) space is sober. In turn, every sober space is T_0. With regards to

the separation axioms, one has the following chain of implications: $T_2 \implies$ sober $\implies T_0$. Clearly, not every T_1 space is sober since every infinite set endowed with the cofinite topology is such an example. Even for non-Hausdorff settings, sober spaces abound. We begin with two examples below:

Algebraic geometry. For any commutative ring R, the prime spectrum $\mathrm{Spec}(R)$ with the Zariski topology is a compact sober space. Moreover, every compact sober space arises in this way [12].

Pointless topology. A complete lattice is a *frame* if binary meets distribute over arbitrary joins. One trivial example of a frame is the two-element chain $\mathbf{2}$. For a topological space X, the lattice of its open sets, $\mathcal{O}X$, ordered by inclusion is a frame. Given a frame L, one can endow on its spectrum $\mathrm{Spec}\, L$ (i.e., the set of frame homomorphisms $p : X \longrightarrow \mathbf{2}$) the *hull-kernel topology*. The opens are defined to be sets of the form $\Box a := \{p : X \longrightarrow \mathbf{2} \mid p(a) = 1\}$. It turns out that such a topology is always sober. Moreover, a topological space X is sober if and only if it is homeomorphic to the hull-kernel topology defined on $\mathrm{Spec}(\mathcal{O}X)$. In view of this, sobriety is precisely the condition by which any topological space X may be recovered from its frame of opens $\mathcal{O}X$. Sobriety is also an essential ingredient in Thron's characterization of those topological spaces X such that for any topological space Y, $X \cong Y$ as spaces if and only if $\mathcal{O}X \cong \mathcal{O}Y$ as frames [27, 4].

Another rich resource of sober spaces is domain theory. Domain theory may be construed as the topology of ordered structures. Given a poset, several intrinsic topologies can be defined on it. One of the most important topologies, purportedly, is the Scott topology. The Scott opens of a poset P are just the upper sets which are inaccessible by directed suprema. This topology was first introduced by Dana Scott while manufacturing denotational models for programming languages [25]. In the Scott model, data types are interpreted as domains and programs their elements. Roughly speaking, domains are directed complete partial orders (dcpo, for short) which supports approximation – every data is the limit of its 'finite' approximants. The technical way of saying this is that domains are continuous dcpo's.

Continuous dcpo's have many beautiful properties in connection to the Scott topology, and amongst these is the well-known result that the Scott topology on a continuous dcpo is sober. In a somewhat opposite direction, it is equally well-known that the specialization order of a sober space is a dcpo. This leads to a natural question of whether every dcpo has a sober Scott

topology. The answer to this is negative. Johnstone [16] constructed the first dcpo (and later Isbell [15] the first complete lattice) with a non-sober Scott topology. Recently, H. Kou constructed yet another counterexample of a U_k-admitting dcpo which is not sober with respect to its Scott topology [20]. Since Johnstone's construction is not U_k-admitting, all these three partial orders are pairwise non-isomorphic. To date, as far as the authors of this paper know, these are the only three distinct dcpo's whose Scott topologies are not sober. The rarity of such counterexamples is anticipation of the extreme difficulty involved in identifying precisely those dcpo's which admit sober Scott topologies. The degree of difficulty has been articulated in the form of an open problem posed on p.155 of [9], which we quote below:

> **Problem.** Find an order-theoretic characterization for those **dcpo**'s which are sober with respect to their Scott topology.

Since then, limited progress has been made in settling the problem [30, 10].

In this paper, we introduce a new transitive relation, on each poset, analogous to the way-below relation $\ll$ heavily used in domain theory. With this new relation, we are able to formulate and prove the main result in this paper, i.e., a necessary and sufficient condition for a dcpo to have sober Scott topology. Applying our main result reported herein, we (1) give pathological explanations of why three well-known dcpo's cannot have sober Scott topologies, and (2) establish that two dominated dcpo's are isomorphic if and only if their Scott closed set lattices are isomorphic – thereby sharpening a key result of [11].

The layout of this paper is as follows. In Section 2, we gather at one place several well-known and essential definitions and results. In Section 3, we introduce the new transitive relation $\lhd$ and the H-continuous posets. We establish the union-completeness of the H as a subset selection in Section 4. We then develop, in Section 5, the notion of an H-compact element. After this, we single out a special class of dcpo's called the dominated dcpos, which will be useful in the subsequent development. This is done in Section 6. The machinery invented thus far will result in the characterization of dcpo's of the form $H(P)$ for some dominated dcpo P, and this is done in Section 7. This then brings the reader to Section 8, the climax of this paper, where we present an answer to the aforementioned open problem. The paper culminates in Section 9 with some categorical housekeeping of the main results obtained in the preceding sections.

Readers are expected to be familiar with domain theory and category theory. For in-depth treatment of these topics, one may consult [1, 9] for domain theory and [21, 24] for category theory.

2 Preliminaries

A partially ordered set will be called a *poset*. In this paper, we shall use $\sqsubseteq$ to denote the order relation, and $\bigsqcup E$ and $\bigsqcap E$ the supremum and infimum of a subset E, respectively. A non-empty subset D of a poset is said to be *directed* if any two elements in D have an upper bound in D. A poset P is called a *dcpo* (short for directed complete poset) if every directed subset of P has a supremum in P. A subset M of a dcpo P is called a *sub-dcpo* of P if for all directed subsets D of M, it holds that $\bigsqcup_M D \in M$ exists, and thus $\bigsqcup_M D = \bigsqcup_P D$. Since the empty set is not directed by definition, a dcpo may fail to have a least element (called a bottom). A dcpo with a bottom $\bot$ is said to be *pointed*. A poset in which the supremum of every pair of elements exists is called a *join-semilattice*. Dually, one defines the *meet-semilattice*. A poset which is both a join-semilattice and a meet semi-lattice is called a *lattice*. A lattice which has all suprema and infima is called a *complete* lattice.

For any subset A of a poset P, the subset $\uparrow A$ is defined by

$$\uparrow A = \{x \in P \mid \exists a \in A.\ a \sqsubseteq x\}.$$

A subset A of a poset P is *upper* if $A = \uparrow A$. The lower subsets are defined dually.

A subset U of a poset is called *Scott-open* if (i) U is upper, and (ii) for any directed subset D of P, $\bigsqcup D \in U$ implies $U \cap D \neq \emptyset$ whenever $\bigsqcup D$ exists. The set of all Scott-open sets of P forms a topology on P, called the *Scott topology*, denoted by $\sigma(P)$. We use the notation ΣP to denote the Scott topological space $(P, \sigma(P))$ of a poset P.

The complements of Scott-open sets are the *Scott-closed* sets. We use $\Gamma(P)$ to denote the set of all Scott-closed sets of P. Thus a subset $F \subseteq P$ is Scott-closed if and only if (i′) F is lower, and (ii′) for any directed subset $D \subseteq F$, if $\bigsqcup D$ exists then $\bigsqcup D \in F$. Both $\sigma(P)$ and $\Gamma(P)$ are complete, distributive lattices with respect to the inclusion relation.

An element x of a join-semilattice P is said to be *coprime* if for any $a, b \in P$, $x \sqsubseteq a \vee b$ implies that $x \sqsubseteq a$ or $x \sqsubseteq b$. The set of all coprime elements of a join-semilattice P is denoted by $\mathrm{COPRIME}(P)$. For any complete lattice L, $\mathrm{COPRIME}(L)$ is a sub-dcpo of L.

In a given topological space X, a subset A of X is said to be *irreducible* if whenever $A \subseteq B \cup C$ for any closed sets B and C then $A \subseteq B$ or $A \subseteq C$. A singleton and its closure are always irreducible. Also, a subset A is irreducible if and only if its topological closure $\mathrm{cl}(A)$ is irreducible. Because the lattice of closed sets of a topological space is distributive, a closed subset of a topological space X is irreducible if and only if whenever $A = B \cup C$

for closed sets B and C, one has $A = B$ or $A = C$. It follows that a closed subset of a topological space X is irreducible if and only if it is coprime in the closed set lattice of X. A topological space is said to be *sober* if every irreducible closed subset is the closure of a unique singleton.

The set of all irreducible Scott-closed sets of a poset P, ordered by set inclusion, is denoted by $H(P)$. In other words, $H(P) := \text{COPRIME}(\Gamma(L))$. Because $\Gamma(P)$ is a complete lattice, the poset $(H(P), \subseteq)$ is a dcpo.

Let P be a poset. The *way-below* relation $\ll$ on P is defined by $a \ll b$ for $a, b \in P$ if for any directed subset D of P for which $\bigsqcup D$ exists, $b \sqsubseteq \bigsqcup D$ implies $a \sqsubseteq d$ for some $d \in D$.

A poset P is *continuous* if for any $a \in P$, the set $\mathop{\Downarrow} x := \{y \in P \mid y \ll x\}$ is directed and its supremum is a.

3 H-continuous posets

Given a poset P, we define an auxiliary relation on P which is instrumental for capturing certain order-theoretic properties of the dcpo $H(P)$.

Definition 3.1. Let P be a poset and $x, y \in P$. We say that x is *under* y, denoted by $x \lhd y$, if for every non-empty irreducible Scott-closed subset $C \subseteq P$ for which $\bigsqcup C$ exists, the relation $y \sqsubseteq \bigsqcup C$ always implies $x \in C$.

Warning.   The reader needs to be wary that the symbol $\lhd$ used in this article is *not* the long-way-below relation of G. Raney, as used in [11] for instance.

Here are some expected properties of the relation $\lhd$.

Proposition 3.2. *Let P be a poset and $u, v, x, y \in P$. Then the following statements hold:*

   (i) $x \lhd y$ implies $x \sqsubseteq y$.

   (ii) $u \sqsubseteq x \lhd y \sqsubseteq v$ implies $u \lhd v$, and

   (iii) if P is pointed, then $\bot \lhd x$ always holds.

Remark 3.3. Some early remarks set our perspective right.

   1. For a dcpo $(P, \sqsubseteq)$ which is sober with respect to its Scott topology, the relation $\lhd$ coincides with the underlying order. This indicates that for most dcpo's (as to which one, we shall characterize them later), the relation $\lhd$ is trivially $\sqsubseteq$.

2. Specializing the above remark on any continuous dcpo, one can easily distinguish between the relations $\ll$ and $\lhd$ as $(P), \sigma(P))$ is sober for continuous P.

3. For the purpose of characterizing Scott-closed set lattices, a special relation $\prec$ was employed in [11] and is defined as follows. Let P be a poset and $x, y \in P$. Then $x \prec y$ if for every non-empty Scott-closed subset C of P, whenever $\bigsqcup C$ exists, $\bigsqcup C \sqsupseteq y$ implies $x \in C$. Trivially, $\prec \subseteq \lhd$. It is interesting to note that this relation $\prec$ has also been considered independently by [7] for the characterization of injective locales over perfect sublocale embeddings.

Proposition 3.4. *For any element a of a given poset P, we have*

$$H(a) := \{x \in P \mid x \lhd a\} \in \Gamma(P).$$

Proof. By virtue of Proposition 3.2, it is clear that $H(a)$ is a lower subset of P. Let D be any directed subset of $H(a)$ where $\bigsqcup D$ exists. We aim to show that $\bigsqcup D \in H(a)$. Take any non-empty irreducible Scott-closed subset C of P with $\bigsqcup C \sqsupseteq a$. By definition, for each $d \in D$, it holds that $d \in C$ since $d \lhd a$. Thus, $D \subseteq C$. Because C is, in particular, a Scott-closed subset of P, it follows that $\bigsqcup D \in C$. Consequently, $\bigsqcup D \in H(a)$. $\square$

Definition 3.5. A poset P is said to be *H-continuous* if it satisfies the following approximation axiom with respect to $\lhd$:
For each $a \in P$, the following conditions hold:

1. $H(a) \in H(P)$, and

2. $a = \bigsqcup H(a)$.

The second condition we term it the *approximation axiom.*

Remark 3.6. By invoking Proposition 3.4, the task of verifying that a certain poset P is H-continuous simplifies to showing that the sets of the form $H(a)$ are irreducible subsets of P with respect to the Scott topology and that P satisfies the approximation axiom.

4   Union completeness of H

The following proposition will come in handy in establishing the union-completeness of H, in the sense of [28, 29, 5].

Proposition 4.1. *Let L be a complete lattice and E be a non-empty irreducible Scott-closed subset of* COPRIME(L). *Then it holds that $\bigsqcup_L E \in$* COPRIME(L).

Proof. Suppose that $\bigsqcup_L E \sqsubseteq a \vee b$. Then for each $e \in E$, $e \sqsubseteq \bigsqcup_L E \sqsubseteq a \vee b$. Since each $e \in E$ is an irreducible element of L, it follows that $e \in E_a$ or $e \in E_b$, where $E_a := \downarrow a \cap E$ and $E_b := \downarrow b \cap E$. Hence $E = E_a \cup E_b$. So E_a and E_b are Scott-closed subsets of COPRIME(L). By assumption on E, it follows that $E = E_a$ or $E = E_b$. Thus, $\bigsqcup_L E \sqsubseteq a$ or $\bigsqcup_L E \sqsubseteq b$. The set equality in the last part holds since $\bigsqcup_L E$ is, in particular, an irreducible element of L. $\square$

In what follows, we show that the subset system defined by $H(P)$ for each poset P is *union-complete* in the sense of [28, 5].

Lemma 4.2. *Let P be a poset and $\mathcal{C} \in \Gamma(H(P))$ be non-empty. Then*

$$\bigcup \mathcal{C} \in \Gamma(P).$$

In particular, $\bigsqcup_{\Gamma(P)} \mathcal{C}$ exists and is equal to $\bigcup \mathcal{C}$.

Proof. Let E be any directed subset of $\bigcup \mathcal{C}$ whose supremum, $\bigsqcup_P E$, exists. We aim to show that $\bigsqcup_P E \in \bigcup \mathcal{C}$, i.e., there exists $C \in \mathcal{C}$ such that $\bigsqcup_P E \in C$. Consider the set $\mathcal{E} = \{\downarrow e \mid e \in E\}$. For each $e \in E$, $e \in C$ for some $C \in \mathcal{C}$ and so $\downarrow e \subseteq C$. Since $\mathcal{C} \in \Gamma(H(P))$ and all principal ideals are irreducible Scott-closed sets, it is lower in $H(P)$ and thus $\downarrow e \in \mathcal{C}$. Thus $\mathcal{E}$ is a directed subset (w.r.t. the order $(H(P), \subseteq)$) of $\mathcal{C}$. Next, we claim that $\bigsqcup_{H(P)} \mathcal{E}$ exists and is equal to $\downarrow(\bigsqcup_P E)$. Firstly, $\downarrow(\bigsqcup_P E)$ is clearly an upper bound of $\mathcal{E}$ since $\downarrow(\bigsqcup_P E)$ is an irreducible Scott-closed subset of P for which $\downarrow e \subseteq \downarrow(\bigsqcup_P E)$ for all $e \in E$. Secondly, for any given non-empty irreducible Scott-closed set X of P where $\downarrow e \subseteq X$ for all $e \in E$, it must be that $E \subseteq X$. Since E is a directed subset of P and X is, in particular, a Scott-closed subset of P, it follows that $\bigsqcup_P E \in X$. Because X is lower, $\downarrow(\bigsqcup_P E) \subseteq X$. Thus, $\bigsqcup_{H(P)} \mathcal{E} = \downarrow(\bigsqcup_P E)$. Together with the fact that $\mathcal{C} \in \Gamma(H(P))$, it follows that $\downarrow(\bigsqcup_P E) \in \mathcal{C}$. Thus $\bigsqcup_P E \in C$ for some $C \in \mathcal{C}$. $\square$

Proposition 4.3. *Let P be a poset and $\mathcal{C} \in H(H(P))$ be non-empty. Then*

$$\bigcup \mathcal{C} = \bigsqcup_{H(P)} \mathcal{C} \in H(P).$$

Proof. Apply Lemma 4.2 and Proposition 4.1 for the complete lattice $L = \Gamma(P)$. $\square$

Definition 4.4. A poset is said to be *ISC-complete* if it has suprema of all irreducible Scott-closed subsets.

Remark 4.5. By Proposition 4.3, $H(P)$ is an ISC-complete dcpo for any poset P.

5 H-compact elements

In this section, we define an important set of distinguished elements. Such elements are analogous to the compact elements of a poset and hence are expected to play an important role of characterizing those irreducible Scott-closed subsets of P which are principal ideals.

Definition 5.1. An element of P of a poset P is called H-compact if $x \lhd x$. We use $K(P)$ to denote the set of all the H-compact elements of P.

Proposition 5.2. *If M is a dcpo and $K(M) \neq \emptyset$, then $K(M)$ is a sub-dcpo of M.*

Proof. Take any directed subset D of $K(M)$. Since M is a dcpo, $\bigsqcup_M D$ exists in M. We claim that $\bigsqcup_M D \in K(M)$. For any irreducible Scott-closed subset C of M, whenever $\bigsqcup_M C \sqsupseteq \bigsqcup_M D$, then $\bigsqcup_M C \sqsupseteq d$ for every $d \in D$. But for each $d \in D$, $d \lhd d$ so that by definition of $\lhd$ we have that $d \in C$, i.e., $D \subseteq C$. Since C is a Scott-closed subset of M, it follows that $\bigsqcup_M D \in C$, as desired. $\square$

In what follows, we aim to build a two-way passage between the lattice of lower subsets of a dcpo M with its set of compact elements $K(M)$. To do this, we start with a more general setting as follows:

Definition 5.3. Let M be a poset and S a non-empty subset of M. Define for each $A \subseteq M$, a corresponding subset $A^\sharp \subseteq S$ given by

$$A^\sharp := A \cap S$$

and for each $B \subseteq S$, a corresponding subset of M given by

$$B^\flat := \{ m \in M \mid \exists b \in B . m \sqsubseteq b \} = \downarrow_M B.$$

Lemma 5.4. *Let M be a poset and S a non-empty subset of M. Then the following statements hold.*

  1. Both $\sharp$ and $\flat$ are monotone with respect to set inclusion.

2. If $A \subseteq M$ is lower in M, then $A^\sharp$ is lower in S. For any $B \subseteq S$, $B^\flat$ is lower in M.

3. For any $x \in S$, $(\downarrow_S x)^\flat = \downarrow_M x$ and $(\downarrow_M x)^\sharp = \downarrow_S x$.

4. If A and B are lower subsets of M and S respectively, then

$$(A^\sharp)^\flat \subseteq A \text{ and } (B^\flat)^\sharp = B.$$

Proof. We only check (4). Let A be a lower subset of M. For each $x \in (A^\sharp)^\flat$, there exists $y \in S \cap A$ such that $x \sqsubseteq y$. Since $y \in A$, it follows that $x \in A$. Hence $(A^\sharp)^\flat) \subseteq A$.

Let B be a lower subset of S. For each $b \in B$, it holds trivially that $b \in S$ and, moreover, $b \sqsubseteq b$. So, by definition, $b \in (B^\flat)^\sharp$ and thus, $B \subseteq (B^\flat)^\sharp$. Next, for each $x \in (B^\flat)^\sharp$, there exists $b \in B$ such that $x \sqsubseteq b$. Since B is lower in S, it follows that $x \in B$. This implies $(B^\flat)^\sharp \subseteq B$. Consequently, $B = (B^\flat)^\sharp$. $\square$

Remark 5.5. In summary, $\langle \flat, \sharp \rangle$ forms an e-p pair (i.e., *embedding-projection* pair) between the lattices of lower subsets of M and that of S.

Proposition 5.6. *Let M be a dcpo with a non-empty $K(M)$. Then for each $A \in \Gamma(M)$, it holds that $A^\sharp = A \cap K(M) \in \Gamma(K(M))$.*

Proof. By virtue of Lemma 5.4 (2), it suffices to show that $A^\sharp$ is closed under directed joins in $K(M)$. Let D be a directed subset of $A^\sharp$ whose sup exists in $K(M)$. We want to show that $\bigsqcup_{K(M)} D \in A^\sharp$. Clearly, D is a directed subset of A in M. Consequently, $\bigsqcup_M D \in A$. By Proposition 5.2, $\bigsqcup_M D = \bigsqcup_{K(M)} D$ and so $\bigsqcup_{K(M)} D \in A \cap K(M) = A^\sharp$. $\square$

Proposition 5.7. *Let P be a poset and X be a non-empty irreducible Scott-closed subset of P. Then for each $x \in X$, $\downarrow x \lhd X$ holds in $H(P)$.*

Proof. Let $x \in X$ be given. Suppose that $\mathcal{C} \in H(H(P))$ with $\bigsqcup_{H(P)} \mathcal{C} \supseteq X$. Then, by Proposition 4.3, $X \subseteq \bigcup \mathcal{C}$. Hence $x \in C$ for some $C \in \mathcal{C}$. Thus, $\downarrow x \subseteq C$. Since $\mathcal{C} \in H(H(P))$ is lower in $H(P)$ and that $\downarrow x \in H(P)$, it follows that $\downarrow x \in \mathcal{C}$. $\square$

Corollary 5.8. *Let P be a poset. Then for each $x \in P$, $\downarrow x \in K(H(P))$.*

To characterize dcpo's of the form $H(P)$ for some kind of posets P, it is expected to formulate some version of algebraicity relevant to our current context.

Definition 5.9. A poset P is *H-prealgebraic* if for each $a \in P$,

$$a = \bigsqcup \{x \in K(P) \mid x \sqsubseteq a\}.$$

A *H-prealgebraic* poset P is *H-algebraic* if for any $a \in P$, the following conditions hold:

1. $\downarrow_P \{x \in K(P) \mid x \sqsubseteq a\} \in H(P)$, and

2. $\{x \in K(P) \mid x \sqsubseteq a\} \in H(K(P))$.

Remark 5.10. Readers who are familiar with the existing concept of algebraicity (with respect to the way-below relation $\ll$) might have already noticed that the condition (ii) in Definition 5.9 seems to be an additional requirement. As this condition is required later (and for a purely technical reason), we have decided to include it as part of the definition of H-algebraicity as opposed to inventing another definition later.

Proposition 5.11. *For any poset P, the dcpo $H(P)$ is H-prealgebraic.*

Proof. Since $C = \bigsqcup_{H(P)} \{\downarrow x \mid x \in C\}$ for any $C \in H(P)$, the desired result then follows from Corollary 5.8. $\qquad\square$

Proposition 5.12. *Let P be an H-algebraic poset. Then for any $a \in P$, it holds that*

$$(H(a)^\sharp)^\flat = H(a).$$

Proof. Since P is an H-algebraic poset, the set $\downarrow_P H(a)^\sharp = \downarrow_P \{x \in K(P) \mid x \lhd a\} = \downarrow_P \{x \in K(P) \mid x \sqsubseteq a\} \in H(P)$ with $\bigsqcup_P \downarrow_P H(a)^\sharp = a$. For each $x \in H(a)$, $x \lhd a$ then implies that there exists $c \in \downarrow_P H(a)^\sharp$ such that $x \sqsubseteq c$. There exists $c' \in H(a)^\sharp$ such that $c \sqsubseteq c'$. Since $x \sqsubseteq c \sqsubseteq c' \lhd c' \sqsubseteq a$, Proposition 3.2 forces $x \in (H(a)^\sharp)^\flat$. This proves that $H(a) \subseteq (H(a)^\sharp)^\flat$, which is sufficient in light of Lemma 5.4(ii). $\qquad\square$

Proposition 5.13. *Let P be an H-algebraic poset. Then P is H-continuous.*

Proof. Since P is an H-algebraic poset. It follows that for any $a \in P$, the set $B_a := \downarrow_P \{x \in K(P) \mid x \sqsubseteq a\} \in H(P)$. Now $\bigsqcup B_a = \bigsqcup \{x \in K(P) \mid x \sqsubseteq a\} = a$. So, for any $x \in H(a)$, it holds that $x \in B_a$. On the other hand, if $y \in B_a$, then $y \sqsubseteq d$ for some $d \lhd d \sqsubseteq a$. This implies that $y \in H(a)$. thus, $H(a) = B_a \in H(P)$. It follows that P is H-continuous. $\qquad\square$

6 Dominated dcpo's

In this section, we introduce a new class of dcpo's called the dominated dcpo's. Such a class of dcpo's is instrumental in the subsequent development of our theory presented in this paper.

Definition 6.1. Let P be a poset and $X \subseteq P$ be non-empty. A subset F of P is X-*dominated* if there exists $x \in X$ such that $F \subseteq\, \downarrow x$. A family $\mathcal{F}$ of subsets of P is said to be X-*dominated* if every $F \in \mathcal{F}$ is X-dominated. A poset P is called *dominated* if for every irreducible Scott-closed subset X of P, the set $\bigcup \mathcal{D}$ is itself X-dominated for all directed X-dominated family $\mathcal{D}$ of irreducible Scott-closed subsets of P.

Recall that for any directed subset D of a poset, the Scott closure $\mathrm{cl}\,(D)$ is irreducible.

Proposition 6.2. *Every dominated poset is a dcpo.*

Proof. Let P be a dominated poset and $D = \{x_i \mid i \in I\}$ be a directed subset of P. So, $\mathcal{D} := \{\downarrow x_i \mid i \in I\}$. Then $\mathcal{D}$ is $\mathrm{cl}(D)$-dominated. Here, $\mathrm{cl}(D)$ refers to the Scott-closure of D. Since P is dominated, $\bigcup \mathcal{D}$ is dominated by $\mathrm{cl}(D)$, i.e., there exists $x \in \mathrm{cl}(D)$ such that $\downarrow x \supseteq\, \downarrow x_i$ for all $i \in I$. We now claim that $x = \bigsqcup D$. Firstly, it is clear that x is an upper bound of D. Next, if $x_i \sqsubseteq y$ for all $i \in I$, then $\downarrow x_i \subseteq\, \downarrow y$. This implies that $D \subseteq\, \downarrow y$. Thus, $\downarrow x \subseteq \mathrm{cl}(D) \subseteq\, \downarrow y$, which then implies that $x \sqsubseteq y$. $\qquad\square$

Hereafter, we only use the term 'dominated dcpo' instead of 'dominated poset'.

For a poset P and $X \in \Gamma(P)$, we use the notation:

$$\mathcal{C}_X := \{F \in H(P) \mid \exists x \in X.F \subseteq\, \downarrow x\}.$$

Proposition 6.3. *For any irreducible Scott-closed set X of P, it holds that*

$$\bigsqcup_{H(P)} \mathcal{C}_X = X.$$

Proof. This is because $\downarrow x \in \mathcal{C}_X$ for each $x \in X$. $\qquad\square$

Theorem 6.4. *Let P be a poset. Then the following statements are equivalent.*

(i) P is dominated.

(ii) *For every irreducible Scott-closed subset X of P, $\bigsqcup_{H(P)} \mathcal{D}$ is X-dominated for all directed X-dominated family $\mathcal{D}$ of irreducible Scott-closed subsets of P.*

(iii) *For every irreducible Scott-closed subset X of P,*

$$\mathcal{C}_X \in H(H(P)).$$

(iv) *For every irreducible Scott-closed subset X of P,*

$$\mathcal{C}_X \in \Gamma(H(P)).$$

Proof. (i) $\implies$ (ii): Let $X \in H(P)$ and $\mathcal{D}$ a directed X-dominated family of irreducible Scott-closed subsets of P. Note that $\bigcup \mathcal{D} \subseteq \bigsqcup_{H(P)} \mathcal{D}$. Now, by assumption, there exists $a \in X$ such that $\bigcup \mathcal{D} \subseteq\downarrow a$. Since $\downarrow a \in H(P)$ is an upper bound of $\mathcal{D}$, it follows that $\bigsqcup_{H(P)} \mathcal{D} \subseteq\downarrow a$.

(ii) $\implies$ (i): Suppose X and $\mathcal{D}$ were as above, and there exists, by assumption, $a \in X$ such that $\bigsqcup_{H(P)} \mathcal{D} \subseteq\downarrow a$. Because $\bigcup \mathcal{D} \subseteq \bigsqcup_{H(P)} \mathcal{D}$, it follows that $\bigcup \mathcal{D} \subseteq\downarrow a$.

(i) $\implies$ (iii): Since $\mathcal{C}_X$ is clearly a lower subset of $H(P)$, it remains to show that (1) $\mathcal{C}_X$ is closed under directed joins, and that (2) $\mathcal{C}_X$ is an irreducible subset of $H(P)$.

To achieve (1), we pick an arbitrary directed subset $\mathcal{E}$ of $\mathcal{C}_X$. Since $H(P)$ is a sub-dcpo of $\Gamma(P)$, it follows that $\bigsqcup_{H(P)} \mathcal{E} = \bigsqcup_{\Gamma(P)} \mathcal{E}$. We must show that $\bigsqcup_{\Gamma(P)} \mathcal{E} \in \mathcal{C}_X$. Notice that $\mathcal{E}$ is a directed X-dominated family of irreducible Scott-closed subsets of P. Since P is a dominated dcpo, it follows that $\bigsqcup_{H(P)} \mathcal{E} \subseteq\downarrow x$ for some $x \in X$. Thus, $\bigsqcup_{\Gamma(P)} \mathcal{E} \subseteq\downarrow x$ and so, $\bigsqcup_{\Gamma(P)} \mathcal{E}$ is X-dominated. Hence $\bigsqcup_{H(P)} \mathcal{E} = \bigsqcup_{\Gamma(P)} \mathcal{E} \in \mathcal{C}_X$.

We now verify that (2), i.e., $\mathcal{C}_X$ is an irreducible subset of $H(P)$. Suppose that

$$\mathcal{C}_X \subseteq \mathcal{A} \cup \mathcal{B}$$

for two Scott-closed subsets $\mathcal{A}$ and $\mathcal{B}$ of $H(P)$. Taking set union on both sides, we have that

$$\bigcup \mathcal{C}_X \subseteq \bigcup \mathcal{A} \cup \bigcup \mathcal{B}.$$

Now, by Proposition 4.2, we know that the three sets $\bigcup \mathcal{C}_X$, $\bigcup \mathcal{A}$ and $\bigcup \mathcal{B}$ are Scott-closed subsets of P. We denote the latter two sets by A and B respectively. By Proposition 4.2, we know that $\bigcup \mathcal{C}_X = \bigsqcup_{\Gamma(P)} \mathcal{C}_X$. Crucially, at this juncture, note that $\bigsqcup_{\Gamma(P)} \mathcal{C}_X = X$. Thus $X \subseteq A \cup B$. Since X is an irreducible Scott-closed subset of P, it follows that $X \subseteq A$ or $X \subseteq B$. Without loss of generality, assume that $X \subseteq A$. For each $C \in \mathcal{C}_X$, there is

an $x \in X$ such that $C \subseteq\, \downarrow x$. Now $\downarrow x \subseteq X \subseteq A$ implies that $x \in A$. This means that $x \in Y$ for some $Y \in \mathcal{A}$, and consequently, $C \in \mathcal{A}$ since it is a lower subset of $H(P)$. Thus, we have shown that $\mathcal{C}_X \subseteq A$. This completes the proof of (2) that $\mathcal{C}_X$ is an irreducible subset of $H(P)$.

(iii) $\implies$ (iv): Trivial.

(iv) $\implies$ (i): Let X be any irreducible Scott-closed subset of P and $\mathcal{D}$ be a directed X-dominated family of irreducible Scott-closed subset of P. Then $\mathcal{D}$ is a directed subset of $\mathcal{C}_X$ by definition. Since $\mathcal{C}_X$ is a Scott-closed subset of P, it follows that $\bigsqcup_{H(P)} \mathcal{D} \in \mathcal{C}_X$. Thus, there exists $x \in X$ such that $\bigsqcup_{H(P)} \mathcal{D} \subseteq\, \downarrow x$. But $\bigcup \mathcal{D} \subseteq \bigsqcup_{H(P)} \mathcal{D}$ implying that $\mathcal{D}$ is X-dominated. $\qquad \square$

The definition of a dominated dcpo seems so unnatural and complicated that one may be led to believe in their rarity. Hopefully, the following examples should convince the reader of their abundance.

Example 6.5. 1. Every complete semilattice is dominated. Recall that L is a complete semilattice if every subset which is bounded above has supremum. Equivalently, a poset is a complete semilattice if every non-empty subset has an infimum. All complete lattices and complete semilattice are dominated. In particular, all complete lattices are dominated.

2. If P is a poset where every irreducible Scott-closed subset bounded above has a supremum, then P is dominated. In particular, every ISC-complete poset is dominated. Because of this, if a poset P is such that ΣP is sober, then P is dominated.

Proposition 6.6. *For any poset P, the dcpo $H(P)$ is dominated.*

Proof. This is because $H(P)$ is ISC-complete by Lemma 4.3. $\qquad \square$

Remark 6.7. The above proposition is a concrete instance of Example 6.5(2). We shall need this important example later in the development of our theory.

7 Characterization of $H(P)$ for dominated dcpo's P

The dcpo $H(P)$ has been used (and called the *directed completion* of P) by X. Mao and L. Xu [23] to study B-posets and FS-posets, which are generalizations of FS-domains created earlier by A. Jung in [17]. In their study, P has always been restricted to a continuous dcpo. However, little is known about the order-theoretic properties of $H(P)$ for non-continuous posets P. In this

section, we make use of the H-compact elements of $H(P)$ to recover P for a dominated dcpo, and subsequently arrive at our desired characterization of $H(P)$ over the class of dominated dcpo's P.

Theorem 7.1. *Let P be a dominated dcpo. Then $X \in K(H(P))$ if and only if X is a principal ideal, i.e., $X =\downarrow x$ for some $x \in P$.*

Proof. By Corollary 5.8, it suffices to show the "only if " part. Suppose that $X \in K(H(P))$. By Theorem 6.4(iii), $\mathcal{C}_X$ is an irreducible Scott-closed subset of $H(P)$. As said earlier, it is clear that $\bigsqcup_{H(P)} \mathcal{C}_X = X$. Since $X \lhd X$, it follows that $X \in \mathcal{C}_X$. Thus, $X \subseteq\downarrow x$ for some $x \in X$. Hence $X =\downarrow x$, as desired. $\qquad\square$

Corollary 7.2. *Let P be a dominated dcpo. Then the principal ideal mapping $\eta_P : P \to K(H(P))$, $x \mapsto\downarrow x$ is an order-isomorphism.*

Corollary 7.3. *For dominated dcpos P and Q, the following statements are equivalent:*

(i) $P \cong Q$.

(ii) $\Gamma(P) \cong \Gamma(Q)$.

(iii) $H(P) \cong H(Q)$.

Proof. (i) $\iff$ (iii): Immediate from Corollary 7.2.
(i) $\implies$ (ii): Trivial.
(ii) $\implies$ (iii): Obvious since $H(P) = \text{COPRIME}(\Gamma(P))$. $\qquad\square$

Remark 7.4. The equivalence of (i) and (ii) strengthens a similar result known to be true for complete-semilattices P and Q (c.f. Corollary 5.7 of [11]).

Corollary 7.5. *For any dominated dcpo P, the dcpo $H(P)$ is H-algebraic.*

Proof. We first show that for any $X \in H(P)$, the set

$$\downarrow_{H(P)}\{C \in K(H(P)) \mid C \subseteq X\} \in H(H(P)).$$

By Lemma 7.1, every H-compact element of $H(P)$ is a principal ideal. Thus, this set is just $\mathcal{C}_X$ as defined earlier, which by Lemma 6.4 is an irreducible Scott-closed subset of $H(P)$ because P is dominated.

Next, we show that the set

$$\{C \in K(H(P)) \mid C \subseteq X\} \in H(K(H(P))).$$

Now the above set is just $\{\downarrow x \mid x \in X\}$. By Corollary 7.2, η_P is an iso-morphism between P and $K(H(P))$. Thus $\{\downarrow x \mid x \in X\} = \eta_P(X)$. Since $X \in H(P)$, it follows that $\eta_P(X) \in H(K(H(P)))$. $\qquad\square$

Corollary 7.6. *For a dominated dcpo P, the dcpo $H(P)$ is H-continuous.*

Proof. A direct consequence of Corollary 7.5 and Proposition 5.13. □

Corollary 7.7. *Let P be a dominated dcpo. Then every irreducible Scott-closed subset of $K(H(P))$ has a supremum in $H(P)$.*

Proof. Let $\mathcal{C} \in H(K(H(P)))$ be given. Since P is dominated, the map $\eta_P^{-1} : K(H(P)) \longrightarrow P$ is an isomorphism. This induces an order isomorphism $\theta : H(K(H(P))) \cong H(P)$. We claim that $\theta(\mathcal{C}) = \bigsqcup_{H(P)} \mathcal{C}$. To establish this, we write $\mathcal{C} = \{\downarrow x_i \mid i \in I\}$. This is legitimate since every member of $K(H(P))$ for a dominated dcpo P is a principal ideal. So $\theta(\mathcal{C}) = \{x_i \mid i \in I\}$ which is an irreducible Scott-closed subset of P since θ is an isomorphism between $H(K(H(P))$ and $H(P)$. Clearly, for any $i \in I$, $\downarrow x_i \subseteq \theta(\mathcal{C})$ because the latter is lower. Furthermore, for any irreducible Scott-closed subset X of P which contains $\mathcal{C}$ as a subset, it must that for all $i \in I$, it holds that $x_i \in \downarrow x_i \subseteq X$, implying $x_i \in X$ (since X is lower). Thus $\{x_i \mid i \in I\} \subseteq X$ which implies that $\theta(\mathcal{C})$ is the least upper bound of $\{\downarrow x_i \mid i \in I\}$ taken in $H(P)$. □

We add to the list of equivalent conditions for a dcpo P to be dominated the following item.

Proposition 7.8. *Let P be a dcpo. The following statements are equivalent.*

  (i) *P is dominated.*

  (ii) *For every irreducible Scott-closed subset $\mathcal{C}$ of $H(P)$, it holds that*

$$(\mathcal{C}^{\sharp})^{\flat} \in H(H(P)).$$

Proof. Before we commence on the proof, let us understand what the set $(\mathcal{C}^{\sharp})^{\flat}$ for a given $\mathcal{C} \in H(H(P))$ really looks like. Using the definition of $\sharp$ and $\flat$, we may unwind the definition of $(\mathcal{C}^{\sharp})^{\flat}$ as follows:

$$(\mathcal{C}^{\sharp})^{\flat}$$
$$= (\mathcal{C} \cap K(H(P)))^{\flat}$$
$$= \{F \in H(P) \mid \exists C \in \mathcal{C} \cap K(H(P)).F \subseteq C\}.$$

(i) $\implies$ (ii): Since P is dominated, $C \in \mathcal{C} \cap K(H(P))$ if and only if $C = \downarrow x$ for some $x \in \bigcup \mathcal{C}$, relying on Theorem 7.1. Thus, we may continue to rewrite $(\mathcal{C}^{\sharp})^{\flat}$ in the following way:

$$\{F \in H(P) \mid \exists x \in \bigcup \mathcal{C}.F \subseteq \downarrow x\}.$$

Denote $\bigcup \mathcal{C}$ by X. By Proposition 4.3, $X \in H(P)$ is the least upper bound of $\mathcal{C}$ in $H(P)$. Then $(\mathcal{C}^\sharp)^\flat$ is just nothing but $\mathcal{C}_X$, recycling our earlier notation. Because P is dominated, it follows from Theorem 6.4 that $\mathcal{C}_X$, i.e., $(\mathcal{C}^\sharp)^\flat$, is a irreducible Scott-closed subset of $H(P)$.

(ii) $\implies$ (i): Given any $X \in H(P)$, just set $\mathcal{C}$ to be $\{F \in H(P) \mid F \subseteq X\}$. Apply (ii) directly to obtain the desired result. $\qquad\square$

We are able to present the first characteristic of dcpo's P such that ΣP is sober.

Proposition 7.9. *Let P be a dcpo. Then the following statements are equivalent.*

(i) ΣP is sober.

(ii) For every irreducible Scott-closed subset $\mathcal{C}$ of $H(P)$, it holds that

$$\mathcal{C} = (\mathcal{C}^\sharp)^\flat.$$

Proof. (i) $\implies$ (ii): If ΣP is sober, then P is dominated. Then we gather from the proof of Proposition 7.8 that for any $\mathcal{C} \in H(H(P))$,

$$(\mathcal{C}^\sharp)^\flat = \{F \in H(P) \mid \exists x \in \textstyle\bigcup \mathcal{C}.F \subseteq \downarrow x\}.$$

Because ΣP is sober, the irreducible Scott-closed subset $X := \bigcup \mathcal{C}$ of P is just the closure of a singleton, i.e., there exists $x \in X$ such that $X = \downarrow x$. Then for any $F \in \mathcal{C}$, we must have $F \subseteq X = \downarrow x$. Thus $F \in (\mathcal{C}^\sharp)^\flat$. By Lemma 5.4, the desired equality of sets holds.

(ii) $\implies$ (i): Let X be any irreducible Scott-closed subset of P. We aim to show that there exists $x \in X$ such that $X = \downarrow x$. Define the irreducible Scott-closed subset $\mathcal{C}$ of $H(P)$ as follows: $\mathcal{C} := \{F \in H(P) \mid F \subseteq X\}$. By the assumption in (ii), $\mathcal{C} = (\mathcal{C}^\sharp)^\flat$. In particular, $X \in \mathcal{C}$ so that the set equality forces the existence of $x \in \bigcup \mathcal{C} = X$ such that $X \subseteq \downarrow x$. This then implies that $X = \downarrow x$ since $\downarrow x \subseteq X$ always holds. $\qquad\square$

Remark 7.10. The above result characterizes the class of those dcpo's whose Scott topology are sober as those such that the natural transformation $\flat \circ \sharp :$ $H^2 \longrightarrow H^2$ gives the identity map.

Definition 7.11. Let M be a dcpo. We say that M is $\flat$-*stable* if for every $B \in H(K(M))$, it holds that $B^\flat = \downarrow_M B \in H(M)$.

Proposition 7.12. *For an H-algebraic dominated dcpo M, the following statements are equivalent.*

(i) M *is* $\flat$*-stable.*

(ii) $K(M)$ *is dominated.*

Proof. (i) $\implies$ (ii): Assume that M is $\flat$-stable. We prove that $K(M)$ is dominated. Let $X \in H(K(M))$ be given. Suppose $\mathcal{D}$ is an X-dominated directed family of irreducible Scott-closed subsets of $K(M)$. We aim to show that $\bigsqcup_{H(K(M))} \mathcal{D}$ is itself X-dominated. For each $C \in \mathcal{D}$, there exists $x \in X$ such that $C \subseteq \downarrow_{K(M)} x$. Since M is $\flat$-stable, each $C^\flat \in H(M)$. Also, we have $C^\flat \subseteq (\downarrow_{K(M)} x)^\flat = \downarrow_M x$ by Lemma 5.4(1) and (3). Since $x \in X^\flat$ for each $x \in X$, it follows that the family $\mathcal{D}^\flat := \{C^\flat \mid C \in \mathcal{D}\}$ of irreducible Scott-closed sets of M is $X^\flat$-dominated, where $X^\flat \in H(M)$ by assumption. Because M is dominated, $\bigsqcup_{H(M)} \mathcal{D}^\flat \subseteq \downarrow_M y$ for some $y \in X^\flat$. Since $y \in X^\flat$, there exists $x^* \in X$ such that $\bigsqcup_{H(M)} \mathcal{D}^\flat \subseteq \downarrow_M x^*$. Relying on the monotonicity of $\sharp$ (Lemma 5.4), $\bigsqcup_{H(K(M))} \mathcal{D} = \bigsqcup_{H(K(M))} (\mathcal{D}^\flat)^\sharp \subseteq (\bigsqcup_{H(M)} \mathcal{D}^\flat)^\sharp \subseteq \downarrow_{K(M)} x$. This shows that $\mathcal{D}$ is X-dominated. Thus, $K(M)$ is a dominated dcpo.

(ii) $\implies$ (i): Assume that $K(M)$ is dominated. We prove that M is $\flat$-stable. Let $B \in H(K(M))$. To show that $B^\flat \in H(M)$, we must prove that it is an irreducible Scott-closed subset of M. We start by verifying that $B^\flat$ is a Scott-closed subset of M. Firstly, by its definition, it is a lower set. Next, we prove that $B^\flat$ is closed under directed joins in M. Let D be a directed subset of $B^\flat$. For each $d \in D$, there exists $b_d \in B$ such that $d \sqsubseteq b_d$. Since M is H-algebraic, $(H(d))^\sharp \in H(K(M))$. Because $\bigsqcup_M (H(d))^\sharp = d \sqsubseteq b_d$ for each $d \in D$ and that D is directed, we have a B-dominated directed family $\mathcal{D}' := \{(H(d))^\sharp \mid d \in D\}$ of irreducible Scott-closed subsets of $K(M)$. Since $K(M)$ is dominated, there exists $b \in B$ such that $\bigsqcup_{H(K(M))} \mathcal{D}' \subseteq \downarrow_{K(M)} x$. This implies that $(\bigcup_{d \in D} (H(d))^\sharp) \subseteq \downarrow_{K(M)} b$. Now, for each $d \in D$, $\bigsqcup_M (H(d))^\sharp = d$ by the H-algebraicity of M. Thus $d \sqsubseteq b$ for each $d \in D$. Hence $\bigsqcup_M D \sqsubseteq b$. This means $\bigsqcup_M D \in B^\flat$. To complete the proof that $B^\flat \in H(M)$, we must show that it is an irreducible subset of M. Suppose that $B^\flat \subseteq X \cup Y$ for two Scott-closed subset of M. By Proposition 5.6, we notice that $X^\sharp$ and $Y^\sharp$ are Scott-closed subsets of $K(M)$. By the definition of $\sharp$, $(X \cup Y)^\sharp = X^\sharp \cup Y^\sharp$, and thus $(B^\flat)^\sharp \subseteq X^\sharp \cup Y^\sharp$. By Lemma 5.4, $B = (B^\flat)^\sharp$ and so $B \subseteq X^\sharp \cup Y^\sharp$. Since B is an irreducible subset of $K(M)$, it follows that $B \subseteq X^\sharp$ or $B \subseteq Y^\sharp$. Without loss of generality, assume that $B \subseteq X^\sharp$. Applying $\flat$ on both sides of the inclusion, we have, by Lemma 5.4, that $B^\flat \subseteq (X^\sharp)^\flat \subseteq X$, as desired. $\square$

Definition 7.13. A dcpo M which has all suprema of irreducible Scott-closed subsets of $K(M)$ is called *H-coherent*.

Remark 7.14. From Corollary 7.7, $H(P)$ is H-coherent for any dominated dcpo P.

Definition 7.15. In the following, a $\flat$-stable, H-coherent H-algebraic dcpo which has suprema of all irreducible Scott-closed subsets will be called a *stably H-algebraic* dcpo, for short.

Theorem 7.16. *A dcpo M is isomorphic to $H(P)$ for a dominated dcpo P if and only if M is stably H-algebraic.*

Proof. ($\implies$) Assume that $M \cong H(P)$ for a dominated dcpo P. By Corollary 7.5, M is H-algebraic. By Proposition 4.3, $H(P)$ has suprema of all irreducible subsets. Corollary 7.7 asserts that M is H-coherent. Since P is dominated, $K(H(P)) \cong P$ must be dominated by Corollary 7.2. Thus $K(M) \cong K(H(P)) \cong P$ must also be dominated. Thus, by Proposition 7.12, it follows that M is $\flat$-stable. Combining all these properties of M, we obtain the result that M is stably H-algebraic.

($\impliedby$) Assume that M is stably H-algebraic. Since M has suprema of all irreducible Scott-closed subsets, it follows from Example 6.5(3) that M is dominated. Thus, by Proposition 7.12, $K(M)$ is dominated. We now show that M is isomorphic to $H(P)$ with $P = K(M)$. We aim to prove that the mapping $\phi : M \to H(P)$, $x \mapsto (\downarrow x) \cap P$, is an order-isomorphism of dcpo's. By Proposition 5.2, P is a dcpo with respect to the order inherited from M. Now, the set $\phi(x) = (\downarrow x) \cap P = \{y \in P \mid y \sqsubseteq x\}$ is an irreducible Scott-closed subset of M since M is H-algebraic. This justifies that $\phi : M \to H(P)$ is well-defined. For trivial reasons, ϕ is monotone. We now proceed to show that ϕ is an order-isomorphism. To this end, we aim to prove that the mapping $\bigsqcup_M : H(P) \to M$ is the inverse of ϕ. Notice that $\bigsqcup_M$ is a well-defined mapping since M is H-coherent. Clearly, for each $x \in M$, $\bigsqcup_M \phi(x) = \bigsqcup_M (\downarrow x) \cap P = \bigsqcup_M \{y \in K(M) \mid y \sqsubseteq x\} = x$ because M is H-algebraic. Now for each $C \in H(P)$, let $\bigsqcup_M C = a$. We claim that $C = (\downarrow a) \cap P$. Since $(\downarrow a) \cap P \supseteq C$ is trivial, we only need to show that $(\downarrow a) \cap P \subseteq C$. Let $x \in (\downarrow a) \cap P$, i.e., $x \lhd x$ and $x \sqsubseteq a$. This implies that $x \lhd a = \bigsqcup_M C$. But since M is $\flat$-stable, $C^\flat$ is an irreducible Scott-closed subset of M with $\bigsqcup_M C^\flat = \bigsqcup_M C = a$. So $x \lhd \bigsqcup_M C^\flat$. Since $C^\flat \in H(P)$, it follows that $x \in C^\flat$, i.e., there exists $c \in C$ such that $x \sqsubseteq c$. Because C is a Scott-closed subset of P, C is a lower subset in P. But $x \in P$ so that $x \in C$. Thus, we have $(\downarrow a) \cap P \subseteq C$. $\qquad\square$

From the above proof, we distill an important fact:

Corollary 7.17. *Let M be a stably H-algebraic lattice and $P = K(M)$. Then*

$$\phi : M \to H(P), \ x \mapsto (\downarrow x) \cap P$$

is an order-isomorphism, whose inverse is given by

$$\varepsilon : H(P) \to M, \ C \mapsto \bigsqcup_M C.$$

8 Dcpo's with sober Scott topology

Definition 8.1. A stably H-algebraic dcpo P with $K(P) = P$ will be called strongly H-algebraic.

Theorem 8.2. *Let P be a poset. The following statements are equivalent.*

(i) ΣP is sober.

(ii) P is strongly H-algebraic.

Proof. (i) $\implies$ (ii): Since P is sober, every irreducible subset X of P has its supremum equal to the unique point x for which $X = \mathrm{cl}(\{x\})$. Thus, $\theta : H(P) \longrightarrow P, \ \downarrow x \mapsto x$ is an isomorphism. Because P is sober, P is dominated. So, by Theorem 7.16, P is stably H-algebraic. By Corollary 7.2, $\eta : P \longrightarrow K(H(P)), \ x \to \downarrow x$ is an isomorphism. Since ΣP is sober, $K(H(P)) = H(P)$ and consequently, $\theta \circ K(\eta)$ is the identity map, i.e., $K(P) = P$. So, P is strongly H-algebraic.

(ii) $\implies$ (i): Since P is stably H-algebraic and contains only H-compact elements, i.e., $P = K(P)$, it follows from Corollary 7.17 that the map $\phi : P \longrightarrow H(K(P)), \ x \mapsto \downarrow x \cap P$ is an isomorphism. Thus, for each $F \in H(P)$, it holds that $F = \downarrow x$ for some $x \in P$, i.e., $F = \mathrm{cl}\{x\}$. So $(P, \sigma(P))$ is sober. $\square$

Corollary 8.3. *Let P be a dominated dcpo. Then the following statements are equivalent:*

1. ΣP is sober.

2. $\Sigma H(P)$ is sober.

Proof. (1) $\implies$ (1): Assume that P is sober with respect to its Scott topology. Then $H(P) \cong P$ by the proof of Theorem 8.2. So the Scott topology on $H(P)$ is sober.

(2) $\implies$ (1): Assume that $H(P)$ is sober with respect to its Scott topology. Since P is dominated, $K(H(P)) \cong P$ by Corollary 7.2. Since $H(P)$ is sober with respect to its Scott topology, by Theorem 8.2 $H(P)$ contains nothing but all its H-compact elements, i.e, $K(H(P)) = H(P)$. Thus $H(P) \cong P$, and hence P is sober with respect to its Scott topology. $\square$

Corollary 8.4. *The following statements are equivalent for a dominated dcpo*
P.

1. ΣP *is not sober.*

2. $\Sigma H^k(P)$ *is not sober for each* $k \in \mathbb{N}$.

Proof. (2) $\implies$ (1): Obvious.
(1) $\implies$ (2): This is easily verified by induction. The base case is just (1).
For the inductive step, suppose that $\Sigma H^{k+1}(P)$ is sober. Then $K(H^{k+1}(P)) \cong$
$H^k(P)$ by Corollary 7.2. Since $H^{k+1}(P)$ is sober, $K(H^{k+1}(P)) = H^{k+1}(P)$.
Thus, $H^k(P) \cong H^{k+1}(P)$ implying that $\Sigma H^k(P)$ is sober, contradicting the
inductive hypothesis. $\qquad\square$

Example 8.5. (Johnstone's construction)
Let $P = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ where $(m, n) \sqsubseteq (m', n')$ if

$$\text{either } (m = m' \ \& \ n \sqsubseteq n') \text{ or } (n' = \infty \ \& \ n \sqsubseteq m').$$

Since P itself is an irreducible Scott-closed set whose supremum does not
exists, it fails to be a stably H-algebraic dcpo by Definition 7.15. So, by
Theorem 8.2, P cannot be sober with respect to its Scott topology.

Example 8.6. (Kou's construction)
Let

$$X = (0, 1] = \{x \in \mathbb{R} \mid 0 < x \le 1\}$$

and

$$P_0 = \{(k, a, b) \in \mathbb{R}^3 \mid 0 < k < 1, \ 0 < b \le a \le 1\}.$$

Define $P = P_0 \cup X$, ordered by

- $x_1 \sqsubseteq x_2$ iff $x_1 = x_2$ for $x_1, \ x_2 \in X$;

- $(k_1, a_1, b_1) \sqsubseteq (k_2, a_2, b_2)$ iff $k_1 \le k_2$, $a_1 = a_2$ and $b_1 = b_2$;

- $(k, a, b) \sqsubseteq x$ iff $a = x$ or $kb \le x < b$.

It can be shown that P is an irreducible Scott closed set whose supremum
does not exist, and thus fails to be ISC-complete. By Theorem 8.2, P cannot
be sober with respect to its Scott topology.

Example 8.7. (Isbell's construction)
In [15]J, Isbell constructed a complete lattice L for which the Scott topol-
ogy is not sober. Since every complete lattice is dominated, it follows that
Isbell's construction is an example of a dominated dcpo with a non-sober

Scott topology. We can analyse why L is not sober, exploiting Theorem 8.2. Suppose that L is sober. Then every element of L is H-compact. However, according to Isbell's argument, there is an irreducible subset X of L such that $\bigsqcup_L X \notin X$. Consequently, X is not principally generated. So the element $\bigsqcup_L X$, in particular, is not H-compact.

Let's say more. Because L is not sober, L cannot be isomorphic to $H(L)$. Indeed, since L is complete and thus dominated, $H(L)$ cannot be sober with respect to its Scott topology by Corollary 8.3. Furthermore, $H(L)$ is stably H-algebraic by Theorem 7.16. Thus, $H(L)$ fails to be sober for the only reason that it has some element which is not H-compact.

9 Some categorical aspects

In this section, we explain how Theorem 7.16 can be better understood in a categorical setting. More precisely, we aim to establish a categorical equivalence between the category of dominated dcpo and that of stably H-algebraic dcpo's. To achieve this, we begin by showing that there is a categorical adjunction between a certain luff subcategory of posets and the category of $\flat$-stable, H-algebraic ISC-complete dcpo's. Then, we restrict this adjunction to the desired equivalence.

In this section, whenever we write $\operatorname{cl}_P(A)$ (or sometimes $\operatorname{cl}(A)$ if there is no confusion) for any subset A of a poset P, we mean the closure of A taken with respect to the Scott topology on P. Suppose M is a sub-poset of P and $A \subseteq M$. Then we write $\operatorname{cl}_M(A)$ to mean the closure of A taken with respect to the Scott topology on M, distinguishing it from $\operatorname{cl}_P(A)$.

Consider $\mathbf{POS_d}$ the category whose objects are posets and whose morphisms are the Scott-continuous maps (i.e., monotone maps preserving suprema of directed sets). This is the luff subcategory of the category of posets and monotone maps.

Let $\mathbf{ISCD}$ be the category whose objects are ISC-complete dcpo's and whose morphisms are the Scott-continuous mappings which preserve the suprema of irreducible Scott-closed sets. Denote by $\mathbf{SHAD}$ the full subcategory of $\mathbf{ISCD}$ whose objects are the stably H-algebraic dcpo's.

Definition 9.1. A mapping $h : P \longrightarrow Q$ between dcpo's is said to *preserve the relation* $\lhd$ if for any x, $y \in P$, $x \lhd y$ implies $f(x) \lhd f(y)$.

Proposition 9.2. *Let* $f : P \to Q$ *be a morphism in* $\mathbf{POS_d}$. *Define the mapping*
$$h : H(P) \longrightarrow H(Q), \ X \mapsto \operatorname{cl}(f(X)).$$
Then h is a morphism in $\mathbf{ISCD}$.

Proof. Note that if $g : X \longrightarrow Y$ is a continuous map between topological spaces X and Y, then for any irreducible set A of X, $g(A)$ is irreducible in Y. It follows from this general fact that the above-mentioned h is well-defined.

To show that h is Scott-continuous, we proceed as follows. Define $\overline{h} : \Gamma(P) \longrightarrow \Gamma(Q)$ by

$$\overline{h}(F) = \mathrm{cl}\,(h(F)), \ F \in \Gamma(P).$$

Clearly, $\overline{h}$ is left adjoint to $f^{-1} : \Gamma(Q) \longrightarrow \Gamma(P)$ and so $\overline{h}$ preserves joins of arbitrary suprema. Since $H(P)$ and $H(Q)$ are sub-dcpo's of P and Q respectively, the restriction of $\overline{h}$ on $H(P)$ preserves the directed suprema, i.e., h is Scott-continuous.

Lastly, we prove that h preserves the suprema of irreducible Scott-closed sets. Let $\mathcal{C} \in H(H(P))$. Then its supremum is given by $\bigcup \mathcal{C}$ by Lemma 4.3. Now,

$$
\begin{aligned}
\bigsqcup_{H(Q)} \{h(C) \mid C \in \mathcal{C}\} \subseteq \ & h(\bigsqcup_{H(P)} \mathcal{C}) \\
= \ & h(\bigcup \mathcal{C}) \\
= \ & \mathrm{cl}\,(f(\bigcup_{C \in \mathcal{C}} C)) \\
= \ & \mathrm{cl}\,(\bigcup_{C \in \mathcal{C}} f(C)) \\
= \ & \mathrm{cl}\,(\bigcup_{C \in \mathcal{C}} \mathrm{cl}\,(f(C))) \\
= \ & \bigsqcup_{\Gamma(Q)} \{h(C) \mid C \in \mathcal{C}\}.
\end{aligned}
$$

Since $\bigsqcup_{\Gamma(Q)} \{h(C) \mid C \in \mathcal{C}\}$ is an irreducible Scott closed set, it holds that

$$\bigsqcup_{\Gamma(Q)} \{h(C) \mid C \in \mathcal{C}\} = \bigsqcup_{H(Q)} \{h(C) \mid C \in \mathcal{C}\}.$$

It follows that

$$h(\bigsqcup_{H(P)} \mathcal{C}) = \bigsqcup_{H(Q)} h(\mathcal{C}).$$

$\square$

Remark 9.3. The preceding result enables us to extend the assignment $P \mapsto H(P)$ to a functor $H : \mathbf{POS_d} \longrightarrow \mathbf{ISCD}$.

Let us now consider the restriction of H on the subcategory **DP** whose objects are the dominated dcpo's and whose morphisms are the Scott-continuous mappings. Given a dominated dcpo P, the dcpo $H(P)$ is an stably H-algebraic dcpo. In the opposite direction, given a stably H-algebraic dcpo M, the poset $K(M)$ is actually a dominated dcpo (which in fact is a sub-dcpo of M). If $f : A \longrightarrow B$ is a morphism in **SHAD**, since f is Scott-continuous, and $K(A)$ and $K(B)$ are sub-dcpo's of A and B respectively, then f restricts to a morphism $K(f) : K(A) \longrightarrow K(B)$ in **DP**. In summary, we have the following pair of functor:

$$H : \mathbf{DP} \longrightarrow \mathbf{SHAD}, \ K : \mathbf{SHAD} \longrightarrow \mathbf{DP}.$$

Before we prove that $H \dashv K$, we need the following technical proposition, whose corollary is useful.

Proposition 9.4. *Let A and X be topological spaces such that $A \subseteq X$ and*

$$\mathrm{id}_A : A \hookrightarrow X$$

is continuous w.r.t. the respective topologies. Then, for any closed subset F of X, whenever $B \subseteq A$ is such that $B \subseteq F$ then $\mathrm{cl}_A(B) \subseteq F$.

Proof. Just note that $B \subseteq \mathrm{id}_A{}^{-1}(F) = A \cap F \subseteq F$ and that $\mathrm{id}_A{}^{-1}(F)$ is closed in A. $\qquad \square$

Corollary 9.5. *Let A be a sub-dcpo of P and $E \subseteq A$. If $E \subseteq\ \downarrow b$ for some $b \in P$, then $\mathrm{cl}_A(E) \subseteq\ \downarrow b$.*

Proof. Apply the preceding proposition to the fact that id_A is a Scott-continuous map and the choice of $F := \mathrm{cl}_P(\{b\}) =\ \downarrow b$. $\qquad \square$

Proposition 9.6. *Let $\mathcal{D}$ be a directed family of irreducible Scott closed subsets of a poset P. Then*

$$\bigsqcup_{H(P)} \mathcal{D} = \mathrm{cl}_P \left(\bigcup \mathcal{D} \right).$$

Proof. Immediate from the fact that $H(P)$ is a sub-dcpo of $\Gamma(P)$ for any poset. $\qquad \square$

Theorem 9.7. DP *is equivalent to* **SHAD**.

Proof. For each dominated dcpo P, let $\eta_P : P \longrightarrow K(H(P))$ be the mapping defined by $\eta_P(x) = \downarrow x$ for all $x \in P$. η_P is an isomorphism in **DP** by Corollary 7.2. Suppose that Q is a stably H-algebraic dcpo and $f : P \longrightarrow K(Q)$ is a morphism in **DP**. Define $\overline{f} : H(P) \longrightarrow Q$ by

$$\overline{f}(X) = \bigsqcup_Q (\mathrm{cl}_{K(Q)}(f(X)))^\flat, \ X \in H(P).$$

Note that $f(X)$ is an irreducible set of $K(Q)$ for each $X \in H(P)$ and so its closure in $K(Q)$ is irreducible. It follows that $\overline{f}$ is well-defined since Q is $\flat$-stable and ISC-complete. Also, it is clear that for every $x \in P$,

$$K(\overline{f}) \circ \eta_P(x) = \bigsqcup_Q (\mathrm{cl}_{K(Q)}(f(\downarrow x)))^\flat$$
$$= f(x).$$

Thus, $f = K(\overline{f}) \circ \eta_P$.

We have to now prove that $\overline{f}$ is

1. Scott-continuous, and

2. preserves the suprema of irreducible Scott-closed sets.

To prove (1), we take any directed family of irreducible Scott-closed subsets of P, say $\mathcal{D}$ and aim to show that

$$\overline{f}(\bigsqcup_{H(P)} \mathcal{D}) = \bigsqcup_Q \overline{f}(\mathcal{D}).$$

Since f is clearly monotone,

$$\overline{f}(\bigsqcup_{H(P)} \mathcal{D}) \supseteq \bigsqcup_Q \overline{f}(\mathcal{D}).$$

It remains for us to show the reverse inclusion. Since f is Scott-continuous, for any $E \subseteq P$,

$$\mathrm{cl}_{K(Q)}(f(E)) \subseteq \mathrm{cl}_{K(Q)}(f(\mathrm{cl}_P(E)))$$
$$\subseteq \mathrm{cl}_{K(Q)}(\mathrm{cl}_{K(Q)}(f(E)))$$
$$= \mathrm{cl}_{K(Q)}(f(E)),$$

and so $\mathrm{cl}_{K(Q)}(f(E) = \mathrm{cl}_{K(Q)} f(\mathrm{cl}_P(E))$. Now, f is Scott-continuous from P to $K(Q)$ and $\bigsqcup_{H(P)} \mathcal{D} = \mathrm{cl}_P(\bigcup \mathcal{D})$ by Proposition 9.6, we have:

$$\overline{f}(\bigsqcup_{H(P)} \mathcal{D}) = \bigsqcup_Q (\mathrm{cl}_{K(Q)} f(\mathrm{cl}_P(\bigcup \mathcal{D})))^\flat$$

$$= \bigsqcup_Q (\mathrm{cl}_{K(Q)}(f(\bigcup \mathcal{D})))^\flat$$

$$= \bigsqcup_Q \mathrm{cl}_{K(Q)}(f(\bigcup \mathcal{D}))$$

$$= \bigsqcup_Q \mathrm{cl}_{K(Q)}(\bigcup_{D \in \mathcal{D}} f(D)).$$

On the other hand, since $\mathcal{D}$ is a directed family, we have:

$$\bigsqcup_Q \overline{f}(\mathcal{D}) = \bigsqcup_Q \{\bigsqcup_Q (\mathrm{cl}_{K(Q)} f(D))^\flat \mid D \in \mathcal{D}\}$$

$$= \bigsqcup_Q \left(\bigcup_{D \in \mathcal{D}} (\mathrm{cl}_{K(Q)} f(D))^\flat \right)$$

$$= \bigsqcup_Q \left(\bigcup_{D \in \mathcal{D}} \mathrm{cl}_{K(Q)}(f(D)) \right)$$

Denote by b the element $\bigsqcup_Q \bigcup_{D \in \mathcal{D}} \mathrm{cl}_{K(Q)}(f(D))$ and by X the subset $\bigcup_{D \in \mathcal{D}} f(D)$ of the subdcpo $K(Q)$. Because $X \subseteq \bigcup_{D \in \mathcal{D}} \mathrm{cl}_{K(Q)}(f(D))$, it follows that $X \subseteq \downarrow b$. By Lemma 9.5, $\mathrm{cl}_{K(Q)}(X) \subseteq \downarrow b$. So it follows that

$$\bigsqcup_Q \mathrm{cl}_{K(Q)}(\bigcup_{D \in \mathcal{D}} f(D)) = \bigsqcup_Q X \sqsubseteq b = \bigsqcup_Q \left(\bigcup_{D \in \mathcal{D}} \mathrm{cl}_{K(Q)}(f(D)) \right),$$

as desired.

To prove (2), we must show that

$$\overline{f}(\bigsqcup_{H(P)} \mathcal{C}) = \bigsqcup_Q \overline{f}(\mathcal{C})$$

holds for every $\mathcal{C} \in H(H(P))$. Firstly, we have:

$$\overline{f}(\bigsqcup_{H(P)} \mathcal{C}) = \bigsqcup_Q (\mathrm{cl}_{K(Q)}(f(\bigsqcup_{H(P)} \mathcal{C})))^\flat$$

$$= \bigsqcup_Q (\mathrm{cl}_{K(Q)}(f(\bigcup \mathcal{C})))^\flat \qquad (\text{since } \bigsqcup_{H(P)} \mathcal{C} = \bigcup \mathcal{C})$$

$$= \bigsqcup_Q \mathrm{cl}_{K(Q)}(f(\bigcup \mathcal{C}))$$

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Now with the same argument as of (1), we deduce that $\overline{f}(\bigsqcup_{H(P)} \mathcal{C} = \bigsqcup_Q \overline{f}(\mathcal{C}))$.

We now show that $\overline{f}$ is the unique **SHAD**-morphism for which

$$K(\overline{f}) \circ \eta_P = f.$$

Suppose we have another **SHAD**-morphism g such that

$$K(g) \circ \eta_P = f.$$

Let $X \in H(P)$ be arbitrary. Then, we have:

$$
\begin{aligned}
g(X) &= g(\bigsqcup_{H(P)} \downarrow_{H(P)} \{C \in K(H(P)) \mid C \subseteq X\}) \\
&= \bigsqcup_Q g(\downarrow_{H(P)} \{C \in K(H(P)) \mid C \subseteq X\}) \quad \text{(since } g \text{ is an \textbf{ISCD}-morphism)} \\
&= \bigsqcup_Q \{g(C) \mid C \in K(H(P)) \wedge C \subseteq X\} \\
&= \bigsqcup_Q \{g(\downarrow x) \mid x \in X\} \quad \text{(since } K(H(P)) = \{\downarrow x \mid x \in P\} \text{ by Theorem 7.1)} \\
&= \bigsqcup_Q \{f(x) \mid x \in X\} \\
&= \bigsqcup_Q f(X) \\
&= \bigsqcup_Q \mathrm{cl}_{K(Q)}(f(X)) \\
&= \overline{f}(X).
\end{aligned}
$$

Note that the last-to-second equality holds by virtue of Lemma 9.5.

In summary, one has $H \dashv K$ where the unit η and conunit ϵ are natural isomorphisms such that

$$\eta : \mathrm{id}_{\textbf{DP}} \cong KH \text{ and } \epsilon : HK \cong \mathrm{id}_{\textbf{SHAD}},$$

and hence **DP** is equivalent to **SHAD**. $\qquad\square$

10 Conclusion

As explained earlier on, sobriety is more than a desirable topological property which appears in different branches of mathematics. There are at least two

important examples worth mentioning which we have not cited in the introduction. The first one is the famous Hofmann-Mislove Theorem [13]. For any T_0 space X and $K \subseteq X$ a compact subset, the subset $\{U \in \mathcal{O}X \mid K \subseteq U\}$ forms an Scott-open filter of the frame of opens $\mathcal{O}X$. The Hofmann-Mislove Theorem asserts that for a sober space X, every Scott-open filter of $\mathcal{O}X$ arises in this way. The second is a result that provides a convenient construction for continuous distributive lattice. Given any continuous distributive lattice L, there exists a locally compact sober space X (namely the spectrum) such that L is order-isomorphic to $\mathcal{O}X$ [22]. In both examples, sobriety emerges as an indispensable concept. Because of its importance in topology, it is natural to ask: What kind of dcpo P yields a sober Scott topology $\sigma(P)$?

In this paper, we first obtain an order-theoretic characterization of dcpo's Q which is isomorphic to $H(P)$ for some dominated dcpo P. Then, we successfully formulate and prove a necessary and sufficient condition for a dcpo to have sober Scott topology. Relying on Theorem 8.2 and the functor H, we managed to manufacture an infinite chain of pairwise non-isomorphic dcpo's whose Scott topologies are not sober. There is, however, one major drawback of our main result, i.e., Theorem 8.2. Notice that this characterization makes use of notions that involve directly the irreducible Scott-closed sets, e.g., $\lhd$, H-algebraicity, H-compactness and H-coherence. Because of this, in the process of establishing or refuting strong H-algebraicity one has to identify the 'shape' of an irreducible Scott-closed set in the given poset. In practice, such an identification can be very difficult as irreducibility and Scott-closedness are more of topological attributes, rather than order-theoretic ones. Our examples of dcpo's with non-sober Scott topologies are built from Isbell's construction. At present, we are still unable to manufacture fresh (and relatively simple) counterexamples without the use of existing ones.

Many open problems in domain theory centre around the lattice-theoretic properties of the frame of opens $\sigma(P)$ for a dcpo P. One of these asks for the order-theoretic characterization of those complete lattices L whose frame of Scott opens $\sigma(L)$ is continuous. It is known that $\sigma(L)$ is continuous implies that ΣL is sober [9]. Moreover, for any topology X, it is known that the frame of opens $\mathcal{O}X$ is continuous if and only if X is core-compact. Additionally, in the presence of sobriety, core-compactness is equivalent to local compactness [14]. When applied to the Scott topology, this result implies that $\sigma(L)$ is continuous for a complete lattice if and only if ΣL is locally compact and sober. In view of our present result, the aforementioned open problem reduces to obtaining an order-theoretic characterization of those complete lattices L whose Scott topology is locally compact. Locally compact spaces (and their generalizations) have attracted increasing attention over the past 60 years in several areas of mathematics: C^*-algebra [3], locale theory [6],

Abstract Stone Duality [26], the theory of stably compact spaces [19, 18, 2] and even potential theory [8]. With the aim to contribute in this area, it will be a meaningful enterprise to investigate local compactness in connection to the Scott topology.

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