LIVING AND FEELING MATHEMATICS LEARNING

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Abstract: School mathematics has often been taught in a rather mechanical manner, and frequently outside the context of everyday life. Frequent over-emphasis on arithmetic, manipulation of algebraic expressions, and pure memorisation of facts and theorems have left many students feeling that mathematics is mechanical, abstract, and unsuitable for the common person's consumption. This is ironical, since mathematics has developed out of the pure necessity of a routine of daily life - counting. Learning need not be fun at every stage. However, as teachers, we could create contexts at each stage of learning so that students' learning of mathematics does not become merely a mechanical process, but consists of experiences that they could live and feel. In this paper, I will share experiences in my mathematics classrooms that evoke emotions of some strength, so that mathematics will become part of the repository of unforgettable memories in students' lives. This is, in fact, in line with the findings of brain research which reveal the important role that emotions play in learning.

Introduction

Reading the article *Climbing Around on the Tree of Mathematics* by Kennedy (1995) provided me a golden opportunity to reflect on my learning of mathematics. I am one of the lucky few who have managed to climb the 'Tree of Mathematics' high enough to reach some branches and twigs. This has brought a new level of appreciation of the subject and developed in me a passion for it that I had never felt before.

Some years ago, I had decided that after graduating with a degree in Pure Mathematics, I would begin a career in teaching so that I could share my love of the subject with younger learners. I conducted my first 'evangelistic' outreach' in a secondary school where I taught. That has since been a lesson itself to me more than it was to the students I thought I was reaching out to.

On that day, as I sought to prove a theorem in trigonometry to my class, I had virtually danced across the blackboard, scribbling on it some of the most elegant algebraic manipulations, and exuding such a level of passion and interest in the subject that I thought even passers-by would not have failed to be moved. Upon completing my exhilarating performance, I stood facing the class with a wide smile on my face and asked unabashedly, "Isn't that beautiful!?". The class smiled brightly, but I soon discovered that it was not because they appreciated or understood my proof. Rather, they were merely enjoying my excitement – an excitement brought about by my passion – and that I had actually failed to deliver my love of the subject to them.

As I then reflected on the experience, I realised that I had not simply learned to appreciate mathematics from teachers who had shared their love for the subject. Rather, I had developed a keenness for mathematics through some of the learning experiences that my teachers had provided. As in other areas of my life, these experiences were those that were filled with the most unforgettable moments, those which evoked strong emotions in me, like joy and surprises.

However, mathematics has often been taught in a rather mechanical manner, and frequently outside the context of everyday life. Frequent over-emphasis on arithmetic, manipulation of algebraic expressions, and pure memorisation of facts and theorems have left many students feeling that mathematics is mechanical, abstract, and unsuitable for the common person's consumption. This is ironical, since mathematics has developed out of the pure necessity of a routine of daily life – counting.

(Of course, the 'Tree of Mathematics' has, over the years, grown also because it has been nourished by the ideas and concepts of many mathematicians who have worked out of sheer appreciation of the beauty of mathematics. And there are many others who have found use in these ideas and concepts to solve some of life's problems in elegant fashion, such as in the use of matrix algebra in quantum physics.)

I am not implying that I should make learning always fun, and I do know that not all stages of learning can be fun. The point here is to create contexts at every stage of learning so that students' learning of mathematics does not become merely a mechanical process, but will also consist of contextual experiences that the students could live and feel. Fennema & Franke (1992, p.160) noted that "[K]nowledge acquired through activities set in a context which enable a learner to connect the knowledge to his or her broader culture is generalizable and useful". Furthermore, Sylwester (1995, p. 77) pointed out that memory is contextual.

So, I have decided to provide contextual experiences in my mathematics classroom that will evoke emotions of some strength, so as to aid students in their learning and memory. In fact, in Sprenger's (1999, p.60) book on applying brain research on learning and memory to situations that teachers face daily, commented that "emotional engagement appears to be a key to learning". In her (1999, p. 54) analysis of the various types of memory, she pointed out that "[E]motional memory takes precedence over any other kind of memory. The brain always gives priority to emotions". Freeman (1995, p.89) also pointed out that while the frontal lobes allow us to elaborate on the details of our goals and plans, it's emotions that generate them and drive their execution. Thus, as aptly put across by Jensen (1998) – "Good learning does not avoid emotions, it embraces them."

I wish to share some ideas for providing contextual experiences in the classroom learning of mathematics that evoke a certain amount of emotions. Not all the ideas presented here may be original, as good gospels have travelled far and wide over the years. I apologise, therefore, if I should fail to give due credit to the originators of some of these ideas.

The ideas are grouped into the following stages in students' learning:

- Introducing new concepts
- Correcting students' misconceptions
- Applying learned concepts
- Previewing advanced concepts

Introducing New Concepts

In many secondary schools, students encounter the use of indices to represent large numbers. The exponential function $y=a^x$ is also presented to the students in relation to the use of indices and its properties are examined. In particular, students are often told that the exponential function causes the y-values to grow much faster than the x-values. To illustrate the point, the common practice is to tabulate a series of x-values with the corresponding y-values, such as in Figure 1.

x	$y=2^x$
1	2
2	4
3	8
4	16
5	32
10	1024
20	1048576
30	1073741824
40	1.0995×10^{12}
50	1.1239×10^{15}

Figure 1

A problem with this approach is that it is unable to represent huge values with all their integers on a calculator, so that these values become merely approximated in standard form. As a result, this approach is unable to develop number sense among students.

One way of getting round this problem is to associate large numbers represented in standard form with distances that students could relate to, such as the radius of the earth, the distance between the earth and the moon, or the distance between the earth and the sun. Similarly, the teaching of the rapid growth of the y-values in the exponential function as x increases could be illustrated with such ideas.

For example, students could be asked to first fold a piece of writing paper into equal halves, then to cut along the folded line, and repeating the procedure with the two equal halves of the paper. They are told to repeat the procedure for a total of 7 cuts, and then to stack all the pieces of paper into a pile. Following that, each student is told to complete the table in Figure 2. The students will soon discover that the total number of pieces of paper formed from the x number of cuts is 2^x.

Number of cuts (x)	Total number of pieces of paper formed from the cut(s)	$y=2^x$
1		
2		
3		
4		
5		
6		
7		

Figure 2

What follows seldom fails to jolt the most reticent students to pay closer attention to the activity. First, the students are asked to measure the thickness of the newly formed pile of paper. Then, they are instructed to calculate the thickness of such a pile of paper with 50 cuts, based on the data they have collected on the pile with 7 cuts. And finally, they are asked to compare the thickness of such a pile of paper with some of the known distances presented earlier with numbers represented in standard form – which happens to be in the region of the distance between the earth and the moon! The students' usual reaction to this is one of amazement and utter disbelief.

Before this lesson on indices, I would often have problems with students not doing their homework. Now, when I tell my students that not doing their homework will only increase it exponentially, they no longer respond with a smug smile on their faces but a shocking realisation that sends a cold chill down their spine. This conveys to me clearly that the little exercise has left not just an understanding of the nature of exponential functions, but has allowed the students to internalise its meaning as well. In fact, Roald Dahl (1993, p.151), in his autobiography Boy and Going Solo, commented that "[T]he only time I can remember him (his mathematics teacher) vaguely touching upon mathematics was when he whisked a square of tissue paper" to illustrate the same point discussed here.

Correcting Students' Misconceptions

Students' misconceptions about what is taught are a constant source of irritation to their teachers. When not effectively corrected, they often become a more powerful influence on students' learning than the teacher's teaching. Let me illustrate this point with two cases.

The Problem with Units

The importance of units in word problems cannot be overemphasised. Students' sheer sloppiness with units often results in their giving inaccurate and ridiculous answers. Figure 3 shows a conversion of units for area, typically used by teachers to get his/her students to appreciate how different it is from conversion of units for length.

$$100 \text{ cm} = 1 \text{ m}$$
$$\therefore (100 \text{ cm})^2 = (1 \text{ m})^2$$
i.e. $100 \text{ cm} \times 100 \text{ cm} = 1 \text{ m} \times 1 \text{ m}$
i.e. $10\,000 \text{ cm}^2 = 1 \text{ m}^2$

Figure 3

No matter how often this is emphasised by the well-intentioned teacher, who would appear at times to be just too nagging, students would quite happily write "100 cm^2 = 1 m^2. I therefore decided to present the situation in Figure 4 to my students.

	12 eggs = 1 dozen ....(1)
(1) × 2	24 eggs = 2 dozens ....(2)
(1) × 🗁	6 eggs = 🗁 dozen ....(3)
(2) × (3)	144 eggs = 1 dozen ???

Figure 4

The last line in Figure 4 always left a number of the students scratching their heads. However, when the earlier argument on conversion of cm^2 to m^2 was presented side by side with the above argument on dozens of eggs, it took little time for the students to realise that the problem lay with the units. Well, egg^2 and $dozen^2$ certainly does not make sense, but the experience of seeing something that is so simple and yet so easily misconstrued by just a few steps of mathematical manipulations usually remains with the students long beyond that moment in the classroom. A comment, like "You can't even count a dozen eggs!" is enough to make them realise their folly in the use of units!

The Problem with Dividing by a Zero in Algebraic Manipulations

This problem is another area that causes many headaches among teachers teaching students to solve algebraic equations, be them quadratic, trigonometric or derived from word problems. The result of dividing by a zero in algebraic manipulations often narrows down the solution set that students can obtain, and in some cases, leads to their giving ridiculous and unreasonable solutions.

To deal with this problem, I tell my students the story of the intelligent beaver:

‘Once upon a time, there lived an intelligent beaver. One day, he encountered a huge elephant and said, "I could prove mathematically that you and I have the same mass!" Of course the elephant laughed incredulously, and challenged the beaver to prove his case. The beaver very calmly proceeded with his proof.’ (Figure 5)

Let the mass of the elephant be E kg, and the mass of the beaver be b kg.

Let	$d = E - b$
Then,	$(E - b) \times d = (E - b) \times (E - b)$
i.e.	$Ed - bd = E^2 - Eb - bE + b^2$
i.e.	$bE - b^2 - bd = E^2 - Eb - Ed$
i.e.	$b \times (E - b - d) = E \times (E - b - d)$
Therefore	$b = E$!'

Figure 5

Now of course the elephant was shocked, but I think the level of amazement is even higher among the students to whom I told the story. Together with PowerPoint slides for illustration, the story seldom fails to leave a lasting impression on the students' mathematics learning experiences. In fact, some students even go on to tell the story to their parents, of course in the hope of fooling them with the fallacy and proving their own 'superiority' in mathematics. While mischievous, these students have nevertheless brought the learning of mathematics beyond the boundaries of the classroom into the living room, where they relive and communicate their mathematics learning experiences.

Applying Learned Concepts

Most mathematics teachers actively teach the application of learned concepts. However, I am more concerned with applying mathematical concepts to contexts which students could readily relate to, or as tools for constructing models to help students visualise these contexts better. It is this latter aspect that my next example will illustrate.

Figure 6 shows a number series which most secondary school students encounter. To help students in relaxing the infinite series sums to unity, the point has always been beautifully illustrated with cutting a unit area paper into halves, then into quarters, then into eighths, and so on.

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$$

Figure 6

Now I remember I once had a student who did not seem to perform in my class, and had, in fact, even given up trying. When I approached him to offer my help, he told me bluntly, "Mr Lee, don't waste your time. I'm just stupid!" How and when did he develop such a low self-concept, I wondered. I sought to find out by asking "How do you know?". He explained that unlike his classmates, he just could not understand what went on in most of the classes he attended.

I paused and reflected before asking him whether he remembered the above number series which was shared in his class. He said he did, but he could not understand everything that was taught about it. I went on to explain that as in the numbers in the series, one's first encounters with new concepts (such as the term "🗁" in the series) can leave much unanswered. By planning one's daily schedule to allow for time to re-look at materials covered in class, one could then obtain a better understanding of the newly acquired knowledge (such as the next term, "🗏", in the series). Further clarification with the teachers concerned can give one an even larger picture of the knowledge (which could correspond to the next term, "🖮", in the series).

I told the student, in the same way that I have always impressed upon all my students, that learning is a life-long process, reflected by the infinite number of terms in the series. He should realise that while there may exist perfection (which is the unity term), it is a long and tedious process. Each revision of previously acquired knowledge will take him a small step further, seeming insignificant though it might be (as reflected by the sharply decreasing value of each subsequent term), toward excellence. These steps are taken by constantly revising acquired knowledge or, in other words, through hard work, over time, or through maturation.

I cannot claim to be a good counsellor. However, after sharing the story with the student, he went on to graduate from high school with impressive grades, and he has kept in regular contact with me, especially in the area of mathematics. The number series is not just an abstract representation to him and to me anymore. Rather, it is a model to help both of us to each live a life of learning.

Previewing Advanced Concepts

Kennedy (1995) has pointed out that the 'Tree of Mathematics' "has grown to the point at which it is much too big to know". While climbing the Tree, many students have given up as they "become discouraged every time they saw how far away they were from the foliage that was to be their goal". However, as teachers who have enjoyed the view from the tree at some height, we could always share with students glimpses of the beauty that the view from up there offers. This could serve to encourage our students to persevere in their endeavour to reach the top, a virtue which I am sure we would like our students to acquire.

I remember one morning, at a school’s assembly, the student responsible for reading aloud all the announcements to the school that morning, proclaimed dryly, “There is no announcement today". I thought what a golden opportunity this was to share with my students the idea of Gödel's Incompleteness Theorem – an advanced theorem in logic which helps to explain many so-called paradoxical statements encountered by students.

During my maths lesson, I then posed my students the question, "Was there an announcement at assembly this morning?". There were of course some enthusiastic hands, and contradictory answers were offered. I further illustrated the point by writing on the blackboard an example of a commonly observed graffiti (Figure 7) and asking the students to decide how they would response to the message.

Please ignore this sign.

Figure 7

Whatever the students' decisions, the message often leaves them with much to wonder about, because of its paradoxical nature. I then shared with the students that it is a case where all their decisions could be explained with Gödel's Incompleteness Theorem, in the area of mathematical logic.

I am not suggesting that, as a result of my lesson, the students had acquired some concrete knowledge in the area of mathematical logic. However, the lesson situated problems that confound the students in contexts that they could relate to. By doing so, it provided the students with a preview of what mathematics could offer them as they climb the 'Tree of Mathematics'.

Conclusion

In conclusion, just as Kennedy (1995) suggested to use 'technological ladders' to help students access the 'Tree of Mathematics', I would like also to build 'emotional contextual ladders' as an aid to students climbing the 'Tree of Mathematics'. Undoubtedly, build these 'emotional contextual ladders' will require a good deal of effort and many experiences, both in mathematics and in education. However, given that there are professional channels, many of which have been made possible by the advancement in technology, for sharing of ideas, many ideas generated by individual teachers could be harvested, and even compiled for appropriate use by other teachers.

At the same time, just as in the application of any other strategies, appropriate and moderate use of contexts that evoke emotions is important. "[E]tremes of emotions are generally counterproductive to school goals … engage emotions appropriately … as part of the learning, not as an add-on" (Jensen, 1998, pp. 79-80).

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