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Mathematics Teachers Conference 2008

Theme : Mathematical Problem Solving

29th May 2008 National Institute of Education 1 Nanyang Walk Singapore

Jointly organized by:

- Association of Mathematics Educators
- Mathematics and Mathematics Education Academic Group, National Institute of Education, NTU

Keynote Lectures

Primary
- **Using good questions to enhance learning** by Professor Peter Sullivan (Monash University, Australia)
- **Infusing thinking skills in mathematical problem solving** by Juliana Ng (Master Teacher, Singapore)

Secondary
- **Problem solving as a vehicle for teaching mathematics: A Japanese perspective** by Professor Yoshinori Shimizu (University of Tsukuba, Japan)
- **Curiosity in mathematics and mathematical problem solving** by Dr Toh Tin Lam (MME/NIE)

Junior College
- **Using extended problem solving tasks in classroom assessment: A story from the Japanese senior high school classroom** by Professor Yoshinori Shimizu (University of Tsukuba, Japan)
- **Problem solving and mathematical modeling** by A/P Ang Keng Cheng (MME/NIE)

Dialogue with Distinguished Mathematics Educators

International perspective on mathematical problem solving in the school curriculum
A/P Berinderjeet Kaur (Chairperson)
Prof Peter Sullivan (Panelist, Australia)
Prof Yoshinori Shimizu (Panelist, Japan)
Prof A. Takahashi (Panelist, USA)
Prof Catherine Vistro-Yu (Panelist, Philippines)
A/P Wong Khoon Yoong (Panelist, Singapore)

Concurrent Workshops (Participants to attend only one)

Primary

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- P 2 - **Thinking skills in primary mathematics** by Juliana Ng (Master teacher, Singapore)
- P 3 - **Teaching mathematics through problem solving for primary grades** by Prof A. Takahashi (DePaul University, USA)
- P 4 - **Newspaper as a source of mathematical problems** by Dr Yeap Ban Har (MME/NIE)
- P 5 - **Engaging primary pupils in open-ended mathematical problem-solving** by Mdm Foo Kum Fong (Master teacher, Singapore)
- P 6 - **Solving mathematical problems by investigation** by Mr Joseph Yeo Boon Wooi (MME/NIE)

Secondary

- S1 - **Teaching mathematics through problem solving for lower secondary grades** by Prof A. Takahashi (DePaul University, USA)
- S2 - **Innovate to keep problem solving alive!** by Prof Catherine Vistro-Yu (Atenio de Manila University, Philippines)
- S3 - **Some heuristics for problems in school geometry** by Mr Chua Puay Huat (MME/NIE)
- S4 - **Teaching geometrical problem solving** by Mr Leong Yew Hoong (MME/NIE)
- S5 - **Students' concept image in problem solving** by Mdm Teo Soh Wah (MME/NIE)
- S6 - **Integrating open-ended tasks in the lower secondary mathematics lessons** by Dr Joseph Yeo Kai Kow (MME/NIE)
- S7 - **Mathematical problems for the secondary classroom** by Dr Jaguthsing Dindyal (MME/NIE)

Junior College

- J 1 - **Problem solving using statistics** by A/P Yap Sook Fwe (MME/NIE)
- J 2 - **Enhancing problem solving with the TI-Nspire** by Dr Ng Wee Leng (MME/NIE)
- J 3 - **Problem solving strategies for JC mathematics** by Dr Lee Tuo Yeong (MME/NIE)

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- **abstracts of keynote lectures and workshops**
- **registration details**

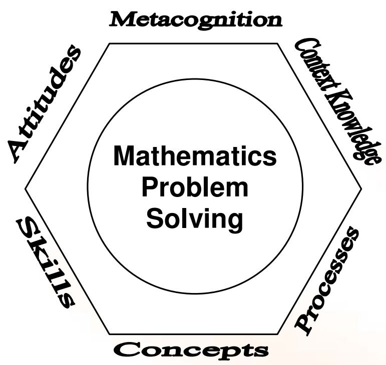
You are advised to register early as places are limited.
Registration closes on: Saturday 12th April 2008.

An Extended Singapore Mathematics Curriculum Framework

Wong Khoon Yoong
National Institute of Education

The Singapore Mathematics Curriculum Framework in the shape of the "Pentagon" was first developed by a review committee in 1988-1989. That review committee decided that mathematics problem solving should be the main aim of the school mathematics curriculum, in line with the ministry's thrust and international trends about problem solving and investigations at the time, as spelt out by the National Council of Teachers of Mathematics (1980) from the US and the Cockcroft's report (1982) from UK. The committee also teased out from available research five key factors that would help students become good solvers of various types of problems, ranging from simple exercises to open-ended investigative tasks. These five factors were then placed along the sides of a pentagon, so that the pictorial framework will display the essential features of the curriculum. Specific topics were then selected to define the coverage of the Concepts and Skills factors. (See Wong, 1991, for a description of the process undertaken to design this framework.)

Over the past fifteen years, this framework has remained virtually unchanged even after the reviews conducted in 2000 and 2006. However, the 2006 revision has placed stronger emphasis on real-life applications and mathematical modelling than in the past. An ensuing pedagogical issue is: what additional factor(s) is required to enable students to solve these types of problems? I wish to propose here that the new factor should be "context knowledge." With this additional factor, a hexagonal framework is obtained, as shown below.



"Context knowledge" refers to knowledge outside of mathematics that students need to bring into the problem solving process in order to understand the context or "story line" of the problem and to make sense of the solutions. This knowledge is not necessary if the problem is free of contexts or involves only neutral contexts. Several examples illustrate this type of knowledge.

(1) Neutral contexts. Example: Ali has five pens. His friend, John, has three more pens than him. How many pens does John have? The contexts of "pens" and "friends" are neutral as these can be replaced by pencils, marbles, apples, brothers, and so on, and the problem structure remains the same.

(2) Basic real-life experience. Example: Mary wants to cut a piece of string into six parts. How many cuts will she make? The problem solver needs to know the relationship between the number of cuts and the number of strings obtained. It is also assumed that a clumsy person might take more than one trial to make one cut, and that the person does not fold the string in any particular way before cutting! Similar context knowledge is also called into play to avoid giving fractional answers such as 12.3 buses when whole numbers should be used and knowing that a rabbit has four legs and a chicken has two legs (usually), and the arms of a clock move clockwise (those who grow up using only digital clocks might not know this).

A slightly harder problem is: A shopkeeper adjusted the weighing scale so that the zero mark is incorrect. One bag of apples weighs 2 kg and one bag of pears weighs 3 kg. When both are weighed together, the reading is 4.8 kg. What is the true weight of each bag of fruit? Many teachers in my inservice classes assumed that the error is proportional to the weight; they make the wrong assumption about how a weighing scale works. Teachers who assign real-world problems must ensure that they use authentic data. I have come across problems that mention: the length of a pendulum is 21 meters; a car took 12 hours to travel from Toa Payoh to Johor Baru (massive jam, car break down?); exposing pupils to such improbable contexts may have led to the phenomenon called "suspension of meaning" when pupils solve school mathematics problems (Verschaffel, Greer & De Corte, 2000).

If the curriculum aim is to help pupils solve real-life problems, we need to identify the types and levels of real-world knowledge that pupils should have about the given problems. The mathematics items in the Program for International Student Assessment (PISA) (http://www.math.hawaii.edu/~tom/hctm/PISA_Highlights.pdf) are supposed to test this form of mathematics literacy.

(3) Conflicting contexts. Example: There were five birds on a tree branch. A hunter fired a shot and one bird dropped dead. How many birds were left? In the old days, the "mathematical" answer was 4, but a real context answer might be zero. Pupils who give the latter answer may be penalised! When pupils are confronted with such situations where their "real-world answers" are not accepted when compared to the "mathematical" answers, they may develop the "suspension of meaning" belief as mentioned above.

(4) Domain-specific knowledge. Many modelling problems require some knowledge of domains outside mathematics, for example, engineering, sciences, economics, and medicine. At school levels, the domain-specific knowledge may be given by an explanation or a formula, and the students are expected to work on the mathematics only. As a simple example: How high does a stone rise when it is thrown vertically at 2 ms^{-1}?

In the proposed hexagonal framework, Concepts form the foundation of the problem solving process. Metacognition is the over-arching factor that coordinates and monitors the other five factors (Wong, 2002).

In recent years, the Singapore framework has become well known as more and more visiting educators arrive here to find out how Singapore manages to stay at the top of international mathematics studies such as TIMSS. It is instructive for us to compare these six factors with the types of knowledge mentioned by international educators. This comparison is shown in the table below.

Singapore (extended)	Kilpatrick et al. (2001)	Schoenfeld (2008)	Mayer (2006)
Concepts	Conceptual	Knowledge base	Conceptual knowledge
Skills	Procedural fluency		Procedural knowledge
Processes	Adaptive reasoning	Problem solving strategies, heuristics	Strategic knowledge
Context knowledge			Factual knowledge
Attitudes	Productive disposition	Beliefs, values, orientations	Metacognitive knowledge, including monitoring, beliefs, self-efficacy
Metacognition	Strategic competence	Metacognition, monitoring, self-regulation	

Although the terms may have slightly different meanings, the fact that they deal with a similar set of factors is a strong indication that teachers should help their pupils develop this set of competency in a relational way in order to become good problem solvers.

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Questions from Teachers

Jaguthsing Dindyal

National Institute of Education

Reasoning is the basis of all mathematics. While teaching mathematics, one important aspect of the teacher's role is to develop the reasoning ability of students. However, this is not an easy task, particularly as the teacher has to adopt a relational approach for teaching the subject. Given the fact that students try to model their teacher's reasoning, it is extremely important for the teacher to have a very good understanding of the subject. Teachers have to nurture their students' reasoning and they can confidently do so only if they themselves question some of the practices about the teaching of mathematics. Amongst others, mathematics teachers usually ask three types of questions which can be broadly classified as those pertaining to the curriculum, the content and the pedagogy.

Each of the three types of questions from teachers is briefly described below with original questions from a group of secondary school teachers who participated in a workshop (original wording from teachers). It is not the purpose of this paper to answer these questions, but rather to highlight their importance in teachers' on-going professional development.

Curriculum Questions

Questions pertaining to the curriculum refer to those that ask about the reasons for including or excluding particular topics in secondary school mathematics as well as those that pertain to the reasons why we have to teach particular topics. Questions about the mathematical content are considered separately. The questions listed below are among such questions (original wording from teachers):

- Why the need to include standard deviation in the new syllabus?
- Why do we need to learn and teach trigonometry since it is hardly used in real life?
- With technology, do we still need students to plot and

draw graphs? Should efforts be used on interpreting the data?

- Why do we need to teach vectors as pupils may not be able to relate it to daily life?
- Why do students learn logarithms? What is the use?
- Why study algebra? What can algebra do that the model method cannot?
- Why do we teach the box and whisker plot?
- Why introduce standard deviation in statistics? It is not easy for students to understand this topic.
- Why do students always have problems with algebra?
- Why are some important reasonings or sections of part of the syllabus, e.g. functions or other topics removed from the syllabus when it is beneficial to the student to see the linkage within that topic?
- Why study angle properties of circle?
- Why is sketching of quadratic graphs in E. Mathematics? Even A. Mathematics students have problems with it.
- Why is 'A' level stuff moved in A. Mathematics?

Content Questions

Questions pertaining to the content refer to those questions that teachers generally ask about particular aspect of the mathematical content that they may not be very familiar with. These may include questions about particular concepts, results, techniques or conventions used in mathematics. Some such questions are listed below (original wording from teachers)

- For differentiation of trigonometric functions, why is the angle in radians?
- Why exterior angle of a polygon is (180° - interior angle) and not (360° - interior angle)?
- Why the drawing of constant speed in a speed-time graph is empty in an acceleration-time graph?
- Why is the second derivative not always the best thing to use to check if it is a maximum or minimum value?
- Why zero is neither positive nor negative?
- Why zero is even?
- Why 1 is not a prime number?
- Why ${}^{n}C_0$ = 1 and a° = 1?
- Why $\frac{0}{0} \neq 0, 1, \infty$?
- Why do we use $\subset, \supseteq, \supset$, for sets?
- Why is $0.\dot{9} = 1$?
- Why the tangent and normal of a circle is perpendicular?
- $\sqrt{\left(-\frac{7}{3}\right)^2} = \sqrt{\frac{49}{9}}$, $\sqrt{\left(\frac{7}{3}\right)^2} = \sqrt{\frac{49}{9}}$. Shouldn't $\sqrt{\frac{49}{9}} = \pm\frac{7}{3}$?

- Why is there a need to measure the spread of data in standard deviation from the mean? Since there are other measures of central tendency (i.e. mode and median), why are they not used instead?
- Why when corresponding angles are equal in triangles, they are similar?
- How do we determine and design interesting shapes that tessellate?
- Why prime numbers can only be positive?
- Why is $\sqrt{2}$ an irrational number? How to prove?
- Why $\int_b^a y\, dx = -\int_a^b y\, dx$?
- Does the graph of $y = x^2 + 2x + 7$ have a maximum value for the range -2≤ $x \le 5$?
- Why is the product of 2 negative numbers positive?

Pedagogy Questions

Pedagogy questions are those that teachers ask about the approaches for teaching particular topics in secondary mathematics. Included among these are questions about misconceptions in mathematics. A list of such questions is given below (original wording from teachers):

- Even after correcting/explaining students' misconceptions, they get it right once or twice and then they still make the same mistakes? Why and how to change this?
- How geometry can be applied to real life situation?
- $(x + y)(x – y) = x^2 - y^2$ by expansion, so $x^2 - y^2 = (x + y)(x – y)$. Another way to prove without memorization?
- How to explain $a^2 - b^2$ is not $(a – b)^2$?
- Why do most teachers always start teaching a topic by the formula followed by examples? Other ways of approach?

Why Would Teachers Ask Such Questions?

Mathematics teachers in secondary schools come with different backgrounds in mathematics. Although they have all undergone a formal training for teaching mathematics, many teachers still have some difficulties with the subject after competing their training. These questions are asked by both novice teachers and more experienced teachers. Some of the questions may seem very trivial. However, for the teachers these questions are legitimate and represent stumbling blocks that they wish to overcome to be better at teaching the subject. Teachers must ask more and more questions like these to have a better understanding of the subject they have to teach at school level. Asking such questions should be seen as a natural process of growing into the profession and not be equated with being ineffective or lacking in some way. While professional development can be a good avenue for discussing these questions, discussions within the department at the school level should not be ignored. Teachers who ask such questions and seek answers to them are professionally in a better position to teach mathematics to school students.

Dealing with Generalising Tasks

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Introduction

Expressing generality, a notion fundamental and central to mathematics learning, is most apparent in the teaching of number patterns. A typical task on number patterns involves such skills as identifying a numerical pattern, extending the pattern to the next few terms, calculating the value of specified terms, as well as using symbols to articulate the functional relationship that defines the pattern. With symbols being used as a means for expressing generality, generalising tasks are thus regarded as important and helpful for developing algebraic thinking and reasoning (Blanton & Kaput, 2005; Carpenter & Levi, 2000; Mason, 1996). Despite its importance for algebraic learning, generalisation seems to remain rather elusive for many students however. For instance, the GCE "O" level examiners' reports for 1996 and 2004 examinations highlighted students' difficulties in expressing generality (University of Cambridge Local Examinations Syndicate, 1997; Cambridge International Examinations, 2005). In the 2004 examination, finding the nth term of a sequence whose first five terms are 1, 4, 7, 10 and 13 was a challenge to several students. The wrong answer $n+3$ was fairly common. The failure to correctly find the functional relationship defining the pattern is quite a concern given that generalising tasks, which have been a feature in the Singapore secondary school mathematics textbooks for over a decade, are not totally new to students. Consequently, there is a need to understand more about how secondary school students deal with generalising tasks. Therefore, this article aims to shed light on how secondary school students make generalisations in ways that are meaningful to them based on a numerical linear generalising task. This task is so called because it involves a linear pattern that lists the numbers sequentially instead of being set in a pictorial context.

The Generalising Task

The numerical linear generalising task that was given to some Secondary One students from different schools is presented in *Figure 1*.

> The first five terms of a sequence are given as follows:
>
> 1, 4, 7, 10, 13, …
>
> (a) What is the 10th term in this sequence? Show how you obtained your answer.
> (b) What is the 100th term in this sequence? Show how you obtained your answer.
> (c) Write down a rule for finding the nth term in this sequence if you were told what n is. Show how you obtained your answer.

Figure 1. The numerical linear generalising task

It comprises two questions that ask for what Stacey (1989) calls a near and far generalisation to the 10th and 100th terms, and a third on finding a general rule for the nth term of the sequence. Item (a) is viewed as a near generalisation question because its answer can be obtained from step-by-step listing of the successive terms. Item (b) is a far generalisation question because the listing of terms is no longer a practical approach for obtaining a large term like the 100th term. In Item (c), the rule for finding the nth term is described as functional because it produces an output (i.e., the value of the term) when given any input (i.e., the ordinal number indicating the term's position), thus encapsulating the idea of a function.

Readers should take note that the first five terms of the sequence may not be sufficient to predict its subsequent terms. This is because the sequence could develop in many ways such as 1, 4, 7, 10, 13, 1, 4, 7, 10, 13, … or 1, 4, 7, 10, 13, 13, 10, 7, 4, 1, 1, 4, 7, 10, 13, …. However, the sequences in such generalising tasks are usually assumed to follow the simplest and most sensible hypothesis. So in this present linear generalising task, the next term is assumed to be obtained by adding 3 to the immediate term preceding it. In fact, all the participating Secondary One students seemed to be aware of this assumption and calculated the subsequent terms in this expected way. Based on this hypothesis, the 10th term is 28 and will be denoted by $T(10)$ in this article, the 100th term is 298 and denoted by $T(100)$, and the nth term is $T(n)=3n-2$. As the intention of this article is to mainly focus on the strategies used by these students in their dealing with the task, no statistical data will be reported herein to overwhelm the readers.

Students' Strategies

Four methods were commonly used by the Secondary One students to find $T(10)$ and $T(100)$.

Method 1	By listing	Students list the next term by adding 3 to the immediate term preceding it. That is, $T_{n+1}=T_n+3$.
Method 2	By looking for a pattern	Students search for a consistent structure that gives rise to the pattern among the given terms.
Method 3	By formula	Students directly apply the formula for finding the nth term of an arithmetic progression, $T(n) = a + (n - 1)d$, where *a* is the first term and *d* the common difference of the progression.
Method 4	By direct proportion	Students take a multiple of a smaller term, thinking that $T(mn) = m \times T(n)$, where $m > n$.

The near generalisation question was straightforward to students, with all successfully finding $T(10)$. The Listing method was most commonly used although the Looking for a Pattern and Formula methods were also seen in some cases. A typical working by listing, which shows how the sequence is continued using the recursive rule of $T_{n+1} = T_n + 3$, is presented below:

$$\begin{array}{cccccccccc} 1, & 4, & 7, & 10, & 13, & 16, & 19, & 22, & 25, & 28 \\ & +3 & +3 & +3 & +3 & +3 & +3 & +3 & +3 & +3 \end{array}$$

In contrast, the far generalisation question of finding $T(100)$ proved to be challenging for many students. Many students who succeeded in determining $T(10)$ by listing failed to correctly generalise $T(100)$, despite having identified the recursive rule in the previous question. This observation resonates with the finding that being able to recognise the pattern does not automatically lead to the generalisation of the pattern (English & Warren, 1995; Ursini, 1991). A few wrong answers were detected in an analysis of the students' workings for this question but only the two most common mistakes will be discussed here. The first one involved finding $T(100)$ from $T(10)$ by using the Direct Proportion method. A sizeable number of students calculated it by multiplying their near generalisation answer by 10, thinking that $T(100) = 10 \times T(10)$ .Their calculation was frequently accompanied by the following working:

$$10^{\text{th}} \text{ term} = 28$$
$$100^{\text{th}} \text{ term} = 28 \text{ x } 10 = 280$$

The other mistake, also somewhat involving direct proportion, stemmed from the following flawed reasoning as clearly explained by a student:

> *For every 10 numbers, you need to add 3 nine times. So when you reach the 100th term, you need to add 3 ninety times.*

Students thinking in this way would add 270 (that is, ninety times of 3) to the first term 1 to derive the wrong answer of 271 for $T(100)$, in tandem with the way $T(10) = 28$ is calculated by adding nine times of 3 to 1.

An analysis of the successful students' workings for the far generalisation question found that the Looking for a Pattern and Formula methods were predominantly used as compared to the Listing method. This finding was perhaps not surprising given that $T(100)$ was large enough to deter students from finding it by listing, which was also not at all an efficient method to calculate it. Among the students who looked for a pattern, a majority of them found $T(100)$ by determining the number of times the common difference has to be added to either the first term or the fifth term of the sequence. For instance, the common difference 3 has to be added 99 times to the first term 1 to get $T(100)$. Figure 2 shows a typical working of how a student had calculated $T(100)$ in this manner. The same reasoning applies to finding $T(100)$ from the fifth term 13, with 3 being added 95 times to it to yield such an expression as $T(100) = (95 \text{ x } 3) + 13$. A few students used another variation of this reasoning to compute $T(100)$ from $T(10)$, instead of the first or fifth term, thus producing $T(100) = (90 \text{ x } 3) + 28$. Besides this line of reasoning, some students offered another way of seeing how each term in the given sequence was being constructed. They noticed that each term is 2 less than 3 times the ordinal number indicating its position. That is, the first term is 2 less than 3 times 1 and can be written as $3 \times 1 - 2$, the second term is 2 less than 3 times 2 or, symbolically, $3 \times 2 - 2$, and continuing in this manner, $T(10) = 3\text{x}10 - 2$ and $T(100) = 3\text{x}100 - 2$.

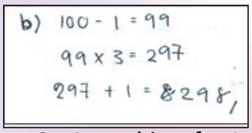


Figure 2. A working for Item (b)

The Formula method was used only by a group of students from the same school. *Figure 3* shows how a student could have used the formula for determining the n^{th} term of an arithmetic progression to correctly answer the near and far generalisation questions.

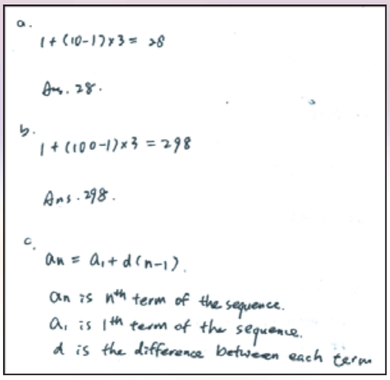
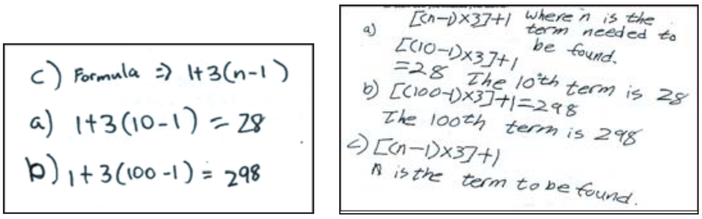


Figure 3. Use of formula: explicit

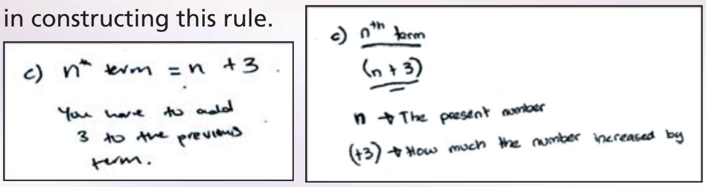
Working such as this, where the use of the Formula method was explicitly shown, was very few. In most instances, it was not apparent whether the Formula method was actually used. But the intuition of its use comes from the students' unconventional way of dealing with the generalising task. Many of them were able to write down the correct functional rule for Item (c) way before Items (a) and (b) were even answered (see *Figure 4*). So how did they construct the correct functional rule at the beginning when the *near* and *far generalisation* questions were supposed to give them a sense of the underlying pattern to be used subsequently for deriving the rule?



(a) (b)

Figure 4. Use of formula: inexplicit

When it came to finding the expression of generality in Item (c), numerous students who had used the Looking for a Pattern and Formula methods in Item (b) proceeded on to produce the correct general rule. It is particularly notable that many using the Looking for a Pattern method were able to make a smooth transition from arithmetic to algebra. For instance, students who answered $T(100) = (95 \text{ x } 3) + 13$. could express the functional rule as $(\text{n} - 5) \text{ x } 3 + 13$; those who gave $T(100) = 3 \times 100 - 2$ wrote down $3 \times n - 2$. There were some unexpected answers though. A few students accurately worked out $T(100) = (99 \times 3) + 1$ and yet expressed the n^{th} term as $(n \times 3) + 1$. But not so surprising was the sighting of the wrong answer $n + 3$ on several occasions. However, not all students explained how they made such a generalisation but the following justifications provided by two students (*see Figure 5*) should shed light on some students' thinking processes in constructing this rule.



(a) (b)

Figure 5. Students' justifications for Item (c)

The justifications revolve around how the students perceived the sequence is being extended. They spotted the rhythm of the sequence: add three to get the next term. Thus the idea of extending the sequence recursively by adding three to the preceding term led them to think that the rule for the n^{th} term was $n+3$. So what were the students' interpretations of the letter n? In Figure 5(a), n is used to denote the "*previous term*" – presumably the $(n-1)^{th}$ term – whereas it is the "*present number*" – presumably the n^{th} term – in Figure 5(b). Clearly, students were ignorant of the meaning associated with n. Not only did students not know how the letter n is used, they also did not seem to realise it was the functional rule, and not the recursive rule, that Item (c) was asking for. This is because they failed to recognise that $n+3$ did not produce the first five terms of the given sequence when the respective integral values of 1 to 5 were used in place of n.

Implications for Teaching

When students are asked to determine a general rule defining the pattern, they ought to be able to interpret from the question whether the recursive rule or the functional rule is required. The recursive rule, allowing the next term to be computed by adding the constant difference between any two consecutive terms to the immediate term preceding it, is particularly useful for making a near generalisation. This is why many students used it to find $T(10)$ by listing. However, teachers should highlight to students that the recursive rule is limited in the sense that while it can easily predict the next term of any given term, it is not effective for determining the far generalisation and the n^{th} term. Thus recognising the recursive rule does not always lead to the far generalisation and later the formulation of the functional rule (Tall, 1992).

To be able to make the correct generalisation for both the far generalisation question as well as the functional rule, recognising how the pattern grows recursively is therefore not sufficient. Students also ought to be able to recognise the underlying structure which gives rise to the pattern. In the case of this numerical linear generalising task, the structure is about the link between the term and the ordinal number indicating its position. But before any pattern exploration and recognition can even begin, the use of the letter n must first be clearly explained. Teachers can point out to students that n is used to represent the ordinal number indicating the term's position. With this explanation, students can then be instructed to look for a pattern between the terms and their corresponding ordinal numbers. By doing this, teachers are actually helping students to concentrate on the right variables during their search for the correct structure that can later be generalised to become the functional rule.

To develop algebraic reasoning, students ought to be able to explain their generalisations by representing the pattern verbally and symbolically (Mason, 1996). So to provide such opportunities to students, it is essential for teachers to get students to describe what they see in the pattern. However, the description should focus not only on the outcome obtained from the recursive rule (that is, 1, 4, 7, 10, 13, …), but, more importantly, also on the actions performed to arrive at these outcomes. For instance, the second term 4 is obtained by adding a three to 1 and can be written as $1+1\times 3$; the third term 7 by adding another three to the second term and is thus $1+2\times 3$. This mode of description should continue for a few more terms.Then to consolidate the pattern they see, teachers can encourage students to test the pattern on $T(10)$ and other near generalisation terms. Subsequently, it is hoped that students are able to abstract the pattern for far generalisation terms as well as to generalise it to $1+(n-1)\times 3$ for the n^{th} term. Developing the general rule through this manner can accentuate the dual role of the notation $1+(n-1)\times 3$: as a process of adding three to the preceding term to obtain the next, and as a mathematical object representing this process. It is hoped that the notation will then make sense and be endowed with meaning to students.

Finally, since different expressions for the same functional rule can be devised in this generalising task (e.g. $1+(n-1)\times 3$, $(n-5)\times 3+13$, $3\times n-2$), teachers can nurture flexibility in thinking by getting students to look for different ways of seeing the same pattern or, alternatively, to see "through" someone else's expression and say how it has come about. To foster algebraic reasoning, students can also be encouraged to explain the structural equivalence of these different expressions. This provides a platform for students to discuss and reason why the different-looking expressions actually represent the same generality without having to revisit the entire sequence. Since the expressions represent the same rule, albeit looking differently, they must always produce the same answers. So one simple way to illustrate equivalence is to substitute positive integral values of n into the expressions and verify that the outcomes are the same. A more sophisticated method, without recourse to substitution, is to apply the rules of algebra. This is where students can show the equivalence of different expressions through algebraic manipulation.

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Fastest Fingers Graphing Calculator Competition 2007

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August is hardly a good month to disrupt lectures for a junior college, especially for JC2 Higher-2 Mathematics. With the constraint of time and the need to gear students up academically for the impending examinations, it was indeed a rare sight to witness the entire cohort gathered in the school hall for a 'competition'! Look closely, and you would find three teams, each of three students, operating their graphing calculators (GC) in a bid to solve the posted question as a team, and rushing to send in their answers via the Texas Instrument (TI) Navigator system.

The Fastest Fingers Graphing Calculator Competition was conceptualized to popularize the use of the GC among Meridian Junior College students. The winds of change brought by the new Advanced Level curriculum heralded fresh approaches and perspectives towards Mathematics and Mathematics education. As trailblazers who are constantly seeking out new opportunities to engage our students, Meridian's Mathematics teachers conducted class-based selections, inter-class competitions, and finally three teams emerged. Speed and accuracy were the key considerations in the Finals conducted on 1 August 2007, as students pitted their skills against each other, hoping to out-wit and out-do their opponents in order to clinch the top group prize of $100.

Besides the three participating teams, the audience was not neglected. In fact, some of the most exciting moments were seen off-stage, where students clamoured for a chance to provide their answers to specially-crafted audience round questions to win attractive GC pouches and decorative stickers from TI. A further testament to the success of this competition came in the form of the students' feedback. Out of 156 JC2 students surveyed, 93% gave the approval that this competition provided a different learning platform for them, and that it should be institutionalized as an annual event. Generally, students appreciated the subtle yet effective manner to promote the use and usefulness of the graphing calculator.

With plans to extend this competition to all pre-university institutions in Term 3 of 2008, it is believed that the Fastest Fingers GC Competition will indeed establish a new frontier in engaging all junior college students in becoming experts in their use of this new tool in Mathematics education.

Secondary Students' Understanding of the Distributive Property

Ida Mok Ah Chee
Reported by Agnes Lee Ser Hua
Nan Hua High School

Algebraic Workings of a Secondary Student

$$(t+1)^2 = t^2 + 1^2$$

$$5(x^2)\,(2xy) = (5 \times x^2)(5 \times 2xy) = 5x^2\,(10xy)$$

The above mistakes may seem familiar to all Secondary teachers teaching Algebra to a group of students in a Mathematics class.

To find out the underlying reasons for the above mistakes made by the students, more than twenty Mathematics educators attended the talk "Secondary Students' Understanding of the Distributive Property", at the Singapore Polytechnic on the 5 March by Dr Ida Mok Ah Chee, an Associate Professor at the University of Hong Kong.

The study first began from observations of common mistakes made by students in the learning of Algebra. The distributive property was chosen as a key focus in tracing students' progression of algebraic thinking for several reasons:

(1) The Errors on distributive rule are commonly found in students' work.
(2) The distributive rule is linked to multiplication and addition, and the application rule is found in a wide range of Algebraic topics.

Although the instruments were based on a study of secondary students in Hong Kong (12-18), the algebraic mistakes were common to students from all parts of the world. The instruments were conceptualized based on the framework of the Structure of the Observed Learning Outcome (SOLO) taxonomy for diagnostic purpose. It was used as a tool to help the Mathematics educators find out how students learn Algebra.

The instrument consists of four task-based interviews. The tasks share a common underpinning stem and show increasing complexity. The interviews were carried out on a one-to-one basis. The duration of each interview, on an average, is about twenty minutes depending on the student's response.

The interviews provided opportunities for the teacher to find out:

- the depth of the student's understanding in the learning of Algebra
- how students' accounted for the generalization of the Distribution Law;
- the transition from Arithmetic to Algebra, the underlying reasons for certain mistakes.

To obtain more information about Dr Ida Ah Chee Mok's research, you may contact her at the following email address: iacmok@hku.hk.

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