Optimal Orientations Of G Vertex-multiplications

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A thesis submitted to the
National Institute of Education,
Nanyang Technological University, Singapore
in partial fulfilment of the requirement for the degree of
Master of Science

2019

Acknowledgements

First and foremost, I would like to thank my family and relatives, particularly my parents, Mr Jason Wong Kok Luen and Mdm Helen Kok Chek Soon. They have always been putting in their best in me, and providing utmost understanding and support. Nothing would have been possible without them.

Also, I would like to thank my supervisor, A/Prof Tay Eng Guan, for giving me an opportunity to conduct meaningful research with him. His invaluable guidance and mathematical knowledge has motivated me through the project. I would also like to take the chance to thank all my lecturers and teachers, who had inspired and helped in one way or another. Special thanks goes to Dr Tan Ban Pin for being my Honours supervisor, and Dr Ng Kah Loon for introducing Graph Theory to me. Furthermore, I would like to thank the National Institute of Education, Nanyang Technological University of Singapore, for the generous support of scholarship.

Last but not least, I give many thanks to all my friends.

Contents

Terminology and Notation

$\{1, 2, \ldots, n\}$	$\mathbb{N}_n$
$\{(1, v), (2, v) \ldots, (n, v)\}$	$(\mathbb{N}_n, v)$, $v \in V(G)$
Complement of the set A	$\bar{A}$
Length of a shortest cycle	girth
Subgraph of G induced by $A \subseteq V(G)$	$\langle A \rangle_G$
The smallest integer more than or equal to x, $x \in \mathbb{R}$	$\lceil x \rceil$
The greatest integer less than or equal to x, $x \in \mathbb{R}$	$\lfloor x \rfloor$
Complete graph of order n	K_n
Complete n-partite graph with partite sizes $p_1, p_2, \ldots, p_n$	$K(p_1, p_2, \ldots, p_n)$
Null graph of order n	O_n
Path of order n	P_n
n-cube graphs	Q_n
$\{(i, v) \mid (j, u) \to (i, v),\ i = 1, 2, \ldots, s_v\}$ in a digraph D	$O_D^v((j, u))$
$\{(i, v) \mid (i, v) \to (j, u),\ i = 1, 2, \ldots, s_v\}$ in a digraph D	$I_D^v((j, u))$

Often in graphs, it is more convenient to use the congruence classes of i modulo n, denoted by $[i]_n$, $n \in \mathbb{N}$, where $i = 1, 2, \ldots, n$, instead of the commonly used $i = 0, 1, \ldots n - 1$. Hence, we adopt this notion throughout this thesis.

Summary

In Chapter 1, we introduce the fundamentals and applications of optimal orientations. We then survey some results relevant to our research. An extension of complete n-partite graphs, G vertex-multiplications, will be introduced formally. Let $s_i \geq 2$ for all $1 \leq i \leq n$. It is known that the G vertex-multiplications, $G(s_1, s_2, \ldots, s_n)$, can be classified into three classes $\mathscr{C}_j$, where

$$\mathscr{C}_j = \{G(s_1, s_2, \ldots, s_n) |\ \bar{d}(G(s_1, s_2, \ldots, s_n)) = d(G) + j\},$$

for $j = 0, 1, 2$.

In Chapter 2, we present our findings on a special case of G vertex-multiplications, which is commonly known as complete tripartite graphs. Particularly, we prove a conjecture by Rajasekaran and Sampathkumar in Section 2.2; for any integers $q \geq p \geq 3$, if $\bar{d}(K(2, p, q)) = 2$, then $q \leq \binom{p}{\lfloor p/2 \rfloor}$. Also, Rajasekaran and Sampathkumar proved $\bar{d}(K(p, p, q)) = 2$ for $p \geq 4$, $4 \leq q \leq 2p$. In Section 2.3, we improve the upper bound of q significantly, especially for large p.

In Chapter 3, we focus on the G vertex-multiplications of trees. Precisely, we investigate some special cases of vertex-multiplications of trees with diameter 4.

1. Literature Review

1.1. Introduction To Optimal Orientations

1.1.1. Fundamentals

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. In this thesis, we consider graphs G with no loops nor parallel edges, unless otherwise stated. For any vertices $v, x \in V(G)$, the *distance* from v to x, $d_G(v, x)$, is defined as the length of a shortest path from v to x. For $v \in V(G)$, its *eccentricity* $e_G(v)$ is defined as $e_G(v) := \max\{d_G(v, x)|\ x \in V(G)\}$. The *diameter* of G, denoted by $d(G)$, is defined as $d(G) := \max\{e_G(v)|\ v \in V(G)\}$ while the *radius* of G, denoted by $r(G)$, is defined as $r(G) := \min\{e_G(v)|\ v \in V(G)\}$.

The above notions are defined similarly for a digraph D. For any vertices $v, x \in V(D)$, the *distance* from v to x, $d_D(v, x)$, is defined as the length of a shortest directed path from v to x. A vertex x is said to be *reachable* from another vertex v if $d_D(v, x) < \infty$. For $v \in V(D)$, its *eccentricity* $e_D(v)$ is defined as $e_D(v) := \max\{d_D(v, x)|\ x \in V(D)\}$. The *diameter* of D, denoted by $d(D)$, is defined as $d(D) := \max\{e_D(v)|\ v \in V(D)\}$ while the radius of D, denoted by $r(D)$, is defined as $r(D) := \min\{e_D(v)|\ v \in V(D)\}$. The *outset* and *inset* of a vertex $v \in V(D)$ are defined to be $O_D(v) := \{x \in V(D)|\ v \to x\}$ and $I_D(v) := \{y \in V(D)|\ y \to v\}$ respectively. If there is no danger of confusion, we shall omit the subscript for the above notations.

An *orientation* D of a graph G is a digraph obtained from G by assigning a direction to every edge $e \in E(G)$. An orientation D of G is said to be *strong* if every two vertices in $V(D)$ are mutually reachable. An edge $e \in E(G)$ is a *bridge* if $G - e$ is disconnected. Robbins' well-known One-way Street Theorem [10] states the following.

Theorem 1.1.1 *(Robbins [44])*
Let G be a connected graph. Then, G has a strong orientation if and only if G is bridgeless.

Roberts [45], Boesch and Tindell [3] and Chung et al [8] constructed efficient algorithms for finding a strong orientation of a bridgeless connected graph. Furthermore, Boesch and Tindell [3] generalised Robbins' One-way Street Theorem for mixed multigraphs (which allow edges to be directed or undirected). In the same article, Boesch and Tindell [3] investigated another extension of Robbins' Theorem, using the notion of $\rho(G)$ given below. Independently, Chvátal and Thomassen [9], and Roberts [46] also considered the same notion $\rho(G)$. Given a connected and bridgeless graph G, let $\mathscr{D}(G)$ be the family of strong orientations of G. Define

$$\rho(G) := \min\{d(D) |\ D \in \mathscr{D}(G)\} - d(G).$$

The **orientation number** of G is defined as

$$\bar{d}(G) := \min\{d(D) |\ D \in \mathscr{D}(G)\}.$$

An orientation $D \in \mathscr{D}(G)$ is an **optimal orientation** of G if $d(D) = \bar{d}(G)$.

Clearly, $d(G) \leq \bar{d}(G)$. Here, we are interested to find the optimal orientation(s) that minimise the increase in diameter, $\rho(G)$. The general problem of finding the orientation number of a connected and bridgeless graph is very difficult. Moreover, Chvátal and Thomassen [9] proved that it is NP-hard to determine if a graph admits an orientation of diameter 2. Since $\rho(G)$ is easily computed with a given $\bar{d}(G)$, we will express all following results using only one of the two terms, $\bar{d}(G)$ and $\rho(G)$, whichever describes the situation better. Noting that every strong orientation of a cycle is a directed cycle, the following easy example serves to familiarise readers with the notion of $\bar{d}(G)$.

Example 1.1.2

i) $\bar{d}(C_n) = n - 1$ *for any cycle* C_n, $n \geq 3$.

ii) $\bar{d}(G) \geq g - 1$ *for any connected and bridgeless graph* G *with girth* $g \geq 3$.

1.1.2. Applications

Diameter problems arise in network optimisation in a natural way. One real-life application of optimal orientations lies in traffic systems. Robbins [44] provided the following example. Consider the scenario of a two-way street system, which can be modeled by a graph G. Each vertex of G correspond to a street intersection and an edge joins two vertices if and only if travelling from one intersection to another without going through a third intersection is possible. Now, road repairs take place on some days, and it is required to transform the two-way street system into a one-way system. Travelling from one vertex to every other vertex is possible if and only if a strong orientation F is assigned. For obvious reasons, we seek a 'best' orientation F that minimises the additional distance. Roberts and Xu [47–50] discussed three functions as criterias of optimality for one to minimise.

(i) $D(F) = \max\{d(u,v) |\ u, v \in V(F)\}$.

(ii) $L(F) = \sum_{u \in V(F)} \max\{d(u,x) |\ x \in V(F)\}$.

(iii) $A(F) = \sum_{u,v \in V(F)} d(u,v)$.

Clearly, minimising the first function $D(F)$ is the same as evaluating $\bar{d}(G)$. In other words, minimising the diameter of the assigned orientation is a way of optimising the one-way street system.

Another application of optimal orientations concerns an adaptation of the Gossip Problem on a graph G. Due to Boyd, the Gossip Problem is stated by Hajnal et al. [14] as follows.

"There are n ladies, and each one of them knows an item of scandal which is not known to any of the others. They communicate by telephone, and whenever two ladies make a call, they pass on to each other, as much scandal as they know at that time. How many calls are needed before all ladies know all the scandal?"

The problem inspired the study of information broadcast by conference calls, telephone calls, letters and computer networks. The adaptation of the problem that relates to our topic of interest is the half-duplex model, where all points simultaneously relay information to all other points. Furthermore, information are consolidated at no cost and all links are concurrently used in only one direction at a time. In this problem, Fraigniaud and Lazard [10] showed that the time taken for the gossip to be completed is bounded below by $d(G)$ and above by $\min\{2d(G), \bar{d}(G)\}$. Some classes of graphs discussed in [10] include complete graphs, cycles, cartesian product of cycles and cartesian product of paths. In addition, hypercube graphs and de Brujin graphs, which are of interest in the area of communication networks, were also considered.

1.2. Relevant Results

1.2.1. Extremal Problems

In 1966, Goldberg [11] investigated an extremal problem concerning the diameter of a strong digraph with n vertices and $n+q$ arcs. As mentioned in Example 1.1.2, $\bar{d}(C_n) = n-1$ for $n \geq 3$. Hence, excluding C_n, Goldberg proved the following theorem.

Theorem 1.2.1 *(Goldberg [11])*
Suppose G is a bridgeless connected graph with order n and size $n+q$, where $n \geq 4$ and $q \geq 1$. Then, $\bar{d}(G) \geq \lceil \frac{2(n-1)}{q+1} \rceil$.

The bound in Theorem 1.2.1 is sharp for digraphs. However, it is not necessary that the sharpness follows for orientations of graphs. If $q = 1$, then cycles with one chord are examples that hit the bound. Let $G(n, q)$, where $q \geq 2$ and $n \geq 3(q+1)$, be the identification of $C_{x_1}, C_{x_2} \ldots, C_{x_{q+1}}$ at one vertex, such that $n = \sum_{i=1}^{q+1} x_i - q$

and the x_i's differ by at most 1. Then, the orientation numbers of $G(n, q)$ are close to Goldberg's bound. If $n \equiv k\ (mod(q+1))$, then it can be shown that

$$\bar{d}(G(n,q)) = \begin{cases} \lceil \frac{2(n-1)}{q+1} \rceil + 1, & \text{if } 3 \leq k \leq \frac{q+3}{2}, \\ \lceil \frac{2(n-1)}{q+1} \rceil, & \text{otherwise.} \end{cases}$$

Henceforth, the following were conjectured and posed.

Conjecture 1.2.2 *Let G be a bridgeless connected graph with order n and size $n+q$, where $q \geq 2$ and $n \geq 3(q+1)$. If $n \equiv k\ (mod(q+1))$, where $3 \leq k \leq \frac{q+3}{2}$, then $\bar{d}(G) \geq \lceil \frac{2(n-1)}{q+1} \rceil + 1$.*

Problem 1.2.3 *If G has higher connectivity, can Goldberg's bound be improved ? What if G is hamiltonian?*

On a related note, Ng [40] examined the orientation numbers of graphs obtained by adding exactly n edges between K_n and C_n, adding 2 edges between 2 arbitrary graphs G_1 and G_2 with orientation numbers d_1 and d_2 respectively, adding n edges between n cycles, adding edges between K_p and K_q, adding edges among K_p, K_q and K_r, and adding edges between K_p and O_q.

Bondy and Murty (see [9]) suggested to investigate quantitative variations of Robbins' theorem. Specifically, they conjectured the existence of a function f, for which every bridgeless graph G of diameter d admits an orientation of diameter at most $f(d)$. In 1978, Chvátal and Thomassen [9] obtained bounds on $f(d)$, as stated below.

$$\frac{1}{2}d^2 \leq f(d) \leq 2d^2 + d, \text{ for } d \geq 2.$$

It follows that $\bar{d}(G) \leq 2d^2+d$. We will see later (from Theorem 1.2.4) that $f(1) = 3$, as every graph with diameter 1 is a complete graph. In addition, they showed that $f(2) = 6$, with the Petersen graph as an extremal graph. In 2007, Kwok, Liu and West [32] proved $9 \leq f(3) \leq 11$.

1.2.2. Complete Graphs And Complete n-partite Graphs

In the subsequent subsections, we shall share existing results on some special classes of graphs. Plesnik [41], Boesch and Tindell [3], and Maurer [36] independently evaluated the orientation number of complete graphs K_n, $n \geq 3$. The orientaton number of complete bipartite graphs $K(p,q)$ was obtained by Šoltés [52] and Gutin [12] independently.

Theorem 1.2.4 *(Plesnik [41], Boesch and Tindell [3], Maurer [36])*
For $n \geq 3$,

$$\bar{d}(K(n)) = \begin{cases} 2, & \text{if } n \neq 4, \\ 3, & \text{if } n = 4. \end{cases}$$

Theorem 1.2.5 *(Šoltés [52], Gutin [12])*
For $q \geq p \geq 2$,

$$\bar{d}(K(p,q)) = \begin{cases} 3, & \text{if } q \leq \binom{p}{\lfloor p/2 \rfloor}, \\ 4, & \text{if } q > \binom{p}{\lfloor p/2 \rfloor}. \end{cases}$$

In his proof, Gutin made use of Sperner's Lemma, a celebrated result in combinatorics. Let $T, S \subseteq \mathbb{N}_n := \{1, 2, \ldots, n\}$, where $n \in \mathbb{Z}^+$. T and S are said to be *independent* if $T \nsubseteq S$ and $S \nsubseteq T$. If S and T are independent, we may say that S is independent of T or T is independent of S. An *antichain* $\mathscr{A}$ of $\mathbb{N}_n$ is a collection of pairwise independent subsets of $\mathbb{N}_n$, i.e. for all $S, T \in \mathscr{A}$, S and T are independent. If $X_i \subseteq \mathbb{N}_n$ for $i = 1, 2, \ldots, k$, and $X_1 \subset X_2 \subset \ldots \subset X_k$, we say that $X_1, X_2, \ldots, X_k$ forms a *chain*. Sperner's Lemma essentially says that the maximum size of any antichain of $\mathbb{N}_n$ is $\binom{n}{\lfloor n/2 \rfloor}$. There are several ways to prove Sperner's Lemma, each with its unique elegance. We state the lemma formally, with a proof by Lubell [35] using maximal chains.

Lemma 1.2.6 ***(Sperner [51])*** *(Proof adapted from [1], [2])*

Let $n \in \mathbb{Z}^+$ and $\mathscr{A}$ be an antichain of $\mathbb{N}_n$. Then,

$$|\mathscr{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

with equality holding if and only if all members in $\mathscr{A}$ have the same size, $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$. (The two sizes coincide if n is even.)

Proof: Firstly, consider the maximal chains $\emptyset = X_0 \subset X_1 \subset ... \subset X_n = \mathbb{N}_n$, where $X_i \subseteq \mathbb{N}_n$ and $|X_i| = i$ for $i = 0, 1, ..., n$. There are $n!$ such chains as each chain is obtained by adding one by one the elements of $\mathbb{N}_n$. Secondly, for each set $S \in \mathscr{A}$ where $|S| = s$, there are exactly $s!(n-s)!$ such chains that contain S, i.e. $S = X_i$ for some $1 \leq i \leq n$. Thirdly, each chain contains at most one element of $\mathscr{A}$.

Now, let m_k be the number of k-sets in $\mathscr{A}$. Then, $|\mathscr{A}| = \sum\limits_{k=0}^{n} m_k$. It follows that the number of chains passing through some member of $\mathscr{A}$ is $\sum\limits_{S \in \mathscr{A}} s!(n-s)! = \sum\limits_{k=0}^{n} m_k k!(n-k)!$, which cannot exceed the total number of chains $n!$. In other words,

$$\begin{aligned} \sum_{k=0}^{n} m_k k!(n-k)! &\leq n! \\ \sum_{k=0}^{n} \frac{m_k}{\binom{n}{k}} &\leq 1 \qquad (1.1) \\ \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \sum_{k=0}^{n} m_k &\leq 1 \\ \frac{|\mathscr{A}|}{\binom{n}{\lfloor n/2 \rfloor}} &\leq 1 \end{aligned}$$

For even n, it is clear from (1.1) that equality holds if and only if $m_{n/2} = \binom{n}{n/2}$, i.e. $\mathscr{A} = \binom{\mathbb{N}_n}{n/2}$, where $\binom{\mathbb{N}_n}{k}$ denotes the collection of all k-subsets of $\mathbb{N}_n$. Also, if n is odd and $\mathscr{A} = \binom{\mathbb{N}_n}{\lfloor n/2 \rfloor}$ or $\mathscr{A} = \binom{\mathbb{N}_n}{\lceil n/2 \rceil}$, then it is clear that $|\mathscr{A}| = \binom{n}{\lfloor n/2 \rfloor}$. Now, let n be odd and $|\mathscr{A}| = \binom{n}{\lfloor n/2 \rfloor}$. Then, equality must hold at all stages, particularly, each chain contains exactly one element of $\mathscr{A}$. And, from (1.1), $\mathscr{A}$ contains only sets of size $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$. We want to prove that either $\mathscr{A} = \binom{\mathbb{N}_n}{\lfloor n/2 \rfloor}$ or $\mathscr{A} = \binom{\mathbb{N}_n}{\lceil n/2 \rceil}$. Suppose $\mathscr{A}$ contains some but not all $\lceil \frac{n}{2} \rceil$-sets. So, there exist some $X \in \mathscr{A}$ and $Y \notin \mathscr{A}$ and $|X| = |Y| = \lceil \frac{n}{2} \rceil$. By a suitable relabelling, if necessary, we may assume that $X = \{a_1, a_2, \ldots, a_{\lceil n/2 \rceil}\}$ and $Y = \{a_i, a_{i+1}, \ldots, a_{i+\lceil n/2 \rceil}\}$. Then, there must exist a largest integer $j < i$ such that $X^* = \{a_j, a_{j+1}, \ldots, a_{j+\lceil n/2 \rceil}\} \in \mathscr{A}$ and $Y^* = \{a_{j+1}, a_{j+2}, \ldots, a_{j+1+\lceil n/2 \rceil}\} \notin \mathscr{A}$. Now, $X^* \cap Y^* \subset X^* \in \mathscr{A}$ implies $X^* \cap Y^* \notin \mathscr{A}$. Furthermore, note that $|X^* \cap Y^*| = \lfloor \frac{n}{2} \rfloor$ and $|Y^*| = \lceil \frac{n}{2} \rceil$. So, the chain containing $X^* \cap Y^* \subset Y^*$ does not have any elements of $\mathscr{A}$, a contradiction.

$\square$

Remark 1.2.7

a) A strength of Lubell's proof is the inequality (1.1)*, also known as the Lubell-Yamamoto-Meshalkin(LYM) Inequality. In fact, this inequality is stronger than Sperner's Lemma itself as it tells us that if we aim to construct a large antichain, then we should choose sets of size about $\frac{n}{2}$. It was discovered independently by Lubell [35], Yamamoto [54] and Meshalkin [38].*

b) Sperner's Lemma launched the remarkable rise of a distinctive and important area in discrete mathematics and combinatorial optimization, known as Sperner's Theory. A number of generalisations of Sperner's Lemma and LYM Inequality has been explored. They consist of many natural and fundamental questions about families of subsets, with emphasis on size, intersection and containment. Though many of such problems remain unsolved, elegant techniques have since emerged to be found useful.

For complete n-partite graphs, where $n \geq 3$, Plesnik [42] and Gutin [13] proved independently that the orientation number is either 2 or 3. This result was also obtained separately by Koh and Tan [18].

Theorem 1.2.8 *(Plesnik [42], Gutin [13], Koh and Tan [18])*

For all positive integers $n \geq 3$ and $p_1, p_2, \ldots, p_n$, $2 \leq \bar{d}(K(p_1, p_2, \ldots, p_n)) \leq 3$.

In the same paper, Koh and Tan [18] established the following sufficient condition for $\bar{d}(K(p_1, p_2, ..., p_n)) = 3$. Furthermore, they raised the example that $\bar{d}(K(2, 1, 1)) = 3$, highlighing that the condition (1.2) in Theorem 1.2.9 is sufficient but not necessary.

Theorem 1.2.9 *(Koh and Tan [18])*

Let $n \geq 3$ and $p_1, p_2, \ldots, p_n$ be positive integers. Denote $h = \sum_{k=1}^{n} p_i$. If

$$p_i > \binom{h - p_i}{\lfloor (h - p_i)/2 \rfloor} \tag{1.2}$$

for some $i = 1, 2, \ldots, n$, then $\bar{d}(K(p_1, p_2, ..., p_n)) = 3$.

In another paper, Koh and Tan [19] constructed large families of complete n-partite graphs, which can be optimally oriented. An idea primarily employed by them was that of a co-pair. A pair $\{p, q\}$ of integers is called a *co-pair* if $1 \leq p \leq q \leq \binom{p}{\lfloor p/2 \rfloor}$ or $1 \leq q \leq p \leq \binom{q}{\lfloor q/2 \rfloor}$. Specifically, if $\{p, q\}$ is a co-pair, then $p = 1$ if and only if $q = 1$.

Theorem 1.2.10 *(Koh and Tan [19])*

Let $G = K(p_1, q_1, p_2, q_2, \ldots, p_k, q_k)$, where $k \geq 2$ and $\{p_i, q_i\}$ is a co-pair for each $i = 1, 2, \ldots, k$. Then $\bar{d}(G) = 2$ if $(k, p_1, p_2) \neq (2, 1, 1)$.

Theorem 1.2.11 *(Koh and Tan [19])*

Let $G = K(p_1, q_1, p_2, q_2, \ldots, p_k, q_k, r)$ where $k \geq 2$ and $\{p_i, q_i\}$ is a co-pair for each $i = 1, 2, \ldots, k$. Suppose $\{r, p_j\}$ is a co-pair for each $j = 1, 2, \ldots, k$. Then, $\bar{d}(G) = 2$.

Though there have been significant results achieved for complete n-partite graphs, a characterisation of $K(p_1, p_2, \ldots, p_n)$ with $\bar{d}(K(p_1, p_2, \ldots, p_n)) = 2$ remains elusive. Koh and Tan further asserted that it is very difficult to determine whether the orientation number of a given $K(p_1, p_2, ..., p_n)$ is 2 or 3.

Problem 1.2.12 *Characterise the complete multipartite graphs $G = K(p_1, p_2, \ldots, p_n)$, where $n \geq 3$, according to whether $\bar{d}(G) = 2$ or 3.*

1.2.3. Cartesian Product Of Graphs

Optimal orientations of cartesian product of graphs have received substantial attention in recent decades. Research on cartesian products was motivated by the fact that the rectangular grid structure of many city traffic systems could be modeled by the graph $P_m \times P_n$. Independently, Roberts and Xu [47–50], and Koh and Tan [17] examined $\rho(P_m \times P_n)$. We summarise their results below.

Theorem 1.2.13

$$\rho(P_m \times P_n) = \begin{cases} 0, & \textit{if } m \geq 3, n \geq 6, (m,n) \neq (3,6), \\ 1, & \textit{if } m = 2, n \neq 3,5 \textit{ or } (m,n) = (3,3),(4,4), \\ 2, & \textit{if } (m,n) = (2,3),(2,5). \end{cases}$$

A generalisation of the above result is the cartesian product of two trees. Koh and Tay [28] derived the following result, concerning trees with diameter at least 4. Koh and Lee [16] further investigated the case, where one of the trees has diameter 2 or 3.

Theorem 1.2.14 *(Koh and Tay [28])*
Let T_i be a tree, where $d(T_i) \geq 4$, for $i = 1,2$. Then, $\rho(T_1 \times T_2) = 0$.

Koh and Tay [24, 26] also worked out $\rho(C_m \times C_n)$ for some special cases of m and n. Separately, Konig et. al. [31] and Chew [7] investigated the same problem. Their results are summarised as follows.

Theorem 1.2.15 *For $m \geq 3$ and $n \geq 3$, the values of $\rho(C_m \times C_n)$, where (?) indicates a conjecture, are summarised in the table below.*

m \ n	**3**	**4**	**5**	**0 $(mod\ 4)$**	**1 $(mod\ 4)$**	**2 $(mod\ 4)$**	**3 $(mod\ 4)$**
3	1	1	2	1	2	1	2
4	-	0	1	0	1	0	1
5	-	-	2(?)	1	2(?)	0	1
0 $(mod\ 4)$	-	-	-	0	0	0	0
1 $(mod\ 4)$	-	-	-	-	1	0	1
2 $(mod\ 4)$	-	-	-	-	-	0	0
3 $(mod\ 4)$	-	-	-	-	-	-	1

Table 1.1: Summary of $\rho(C_m \times C_n)$.

$\rho(G \times H)$ was determined by Cai and Xu [5], Koh and Tay [20–23], and Koh and Lee [15] for the Cartesian products of some other pairs of graphs G and H. These included $C_{2n} \times P_k$, $C_{2n+1} \times P_k$, $K_m \times P_n$, $K_m \times K_n$, $K_m \times C_{2n+1}$ and $K_m \times C_{2n}$

The n-cubes $Q_n := \overbrace{K_2 \times K_2 \times \ldots \times K_2}^{n \text{ terms}}$, $n \geq 2$, were the first cartesian products involving more than two graphs to be studied. Šoltés [52] proved that $\rho(Q_n) \leq 1$ for $n \geq 4$ and McCanna [37] settled the problem completely in 1988.

Theorem 1.2.16 *(McCanna [37])*

$$\rho(Q_n) = \begin{cases} 1, & \textit{if } n = 2, \\ 2, & \textit{if } n = 3, \\ 0, & \textit{if } n \geq 4. \end{cases}$$

McCanna invoked the use of the following lemma, due to Thomassen, in his proof.

Lemma 1.2.17 *Suppose G is a bipartite graph which admits an orientation of diameter at most k, where $k \geq 3$, and every vertex is in a cycle of length at most k.*

Then, the graph $G \times P_2$ admits an orientation of diameter at most $k+1$ such that every vertex is in a cycle of length at most k.

This lemma was extended by Koh and Tay [28] and applied to prove results on cartesian products involving at least three graphs. They derived that $\rho(G_1 \times G_2 \times \ldots \times G_s) = 0$, where G_1 is a bipartite graph with some weak conditions imposed, and $\{G_i |\ 2 \leq i \leq s\}$ is some combination of cycles, paths, complete graphs, complete bipartite graphs, trees, and graphs of diameter 2. We describe their results as follows.

In [25], four families of graphs were defined. Let $\mathscr{G}$ be the set of all bipartite graphs G with $d(G) \geq 4$, and which admit orientations of diameter $d(G)$ where every vertex is contained in a cycle of length at most $d(G)$.

Let $\mathscr{G}^*$ be the set of all bipartite graphs G which admit orientations F of diameter $d(G)$, where $d(G) \geq 4$, and in F,

(i) every vertex is contained in a cycle of length at most $d(G)$, and

(ii) if v is adjacent to w, then there exists a $v-w$ walk of length at least 3 and at most $d(G)$.

Let $\mathscr{J}$ be the set of all graphs G which admit an orientation H such that for all vertices $v, w \in V(H)$, one of the following holds:

(i) $d_H(v, w) \leq d(G)$, or

(ii) $d_H(w, v) \leq d(G)$, or

(iii) there exist vertices x_{vw} and y_{vw} such that $d_H(v, x_{vw}) + d_H(w, x_{vw}) \leq d(G)$ and $d_H(y_{vw}, v) + d_H(y_{vw}, w) \leq d(G)$.

Let $\mathscr{J}^*$ be the set of all graphs G which admit an orientation H such that for all vertices $v, w \in V(H)$, one of the following holds:

(i) $d_H(v, w) \leq d(G)$, or

(ii) $d_H(w, v) \leq d(G)$, or

(iii) there exists a vertex x_{vw} such that $d_H(v, x_{vw}) + d_H(w, x_{vw}) \leq d(G)$, or

(iv) there exists a vertex y_{vw} such that $d_H(y_{vw}, v) + d_H(y_{vw}, w) \leq d(G)$.

It is clear from their definitions that $\mathscr{G}^* \subseteq \mathscr{G}$ and $\mathscr{J} \subseteq \mathscr{J}^*$. Koh and Tay [25] proved that many graphs belong to at least one of $\mathscr{G}, \mathscr{G}^*$, $\mathscr{J}$ and $\mathscr{J}^*$. With their extension of Lemma 1.2.17, they derived the following results.

Theorem 1.2.18 *(Koh and Tay [25])*
If $G \in \mathscr{G}$ and $H_i \in \mathscr{J}$, $1 \leq i \leq n$, then $\rho(G \times \prod_{i=1}^{n} H_i) = 0$.

Theorem 1.2.19 *(Koh and Tay [25])*
If $G \in \mathscr{G}^*$ and $H_i \in \mathscr{J}^*$, $1 \leq i \leq n$, then $\rho(G \times \prod_{i=1}^{n} H_i) = 0$.

Parallel to the study of cartesian products of more than two graphs, Konig [31] obtained the following result concerning cycles.

Theorem 1.2.20 *(Konig [31])*
Let $r \geq 3$. If there exist p and q such that $1 \leq p < q \leq r$, $i_p \geq 6$, $i_q \geq 6$ and $\rho(C_{i_p} \times C_{i_q}) = 0$, then $\rho(C_{i_1} \times C_{i_2} \times \ldots \times C_{i_r}) = 0$.

1.2.4. Join Of Graphs

The join of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is defined to be the graph obtained by adding the set of edges $\{uv|\ u \in V(G_1), v \in V(G_2)\}$ between G_1 and G_2. In other words, $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv|\ u \in V(G_1), v \in V(G_2)\}$. Note that a wheel of order n is isomorphic to $C_{n-1} + O_1$, while a fan of order n is isomorphic to $P_{n-1} + O_1$. Ng [39] proved the following results on wheels and fans.

Theorem 1.2.21 *(Ng [39])*
Let W_n be the wheel of order $n \geq 4$. Then,

$$\bar{d}(W_n) = \begin{cases} 3, & \text{if } n = 4, 5, \\ 4, & \text{if } n \geq 6. \end{cases}$$

Theorem 1.2.22 *(Ng [39])*

Let F_n be the fan of order $n \geq 4$. Then,

$$\bar{d}(F_n) = \begin{cases} 3, & \text{if } n = 4, 5, \\ 4, & \text{if } n \geq 6. \end{cases}$$

These results were extended by Ng [39] and Lee [33].

Theorem 1.2.23 *(Ng [39])*

For $k \geq 2$ and $n \geq 3$,

$$\bar{d}(C_n + O_k) = \begin{cases} 2, & \text{if } (n, k) = (4, 2), \\ 3, & \text{otherwise.} \end{cases}$$

Theorem 1.2.24 *(Lee [33])*

For $k \geq 2$ and $n \geq 3$, $\bar{d}(P_n + O_k) = 3$.

Lee [33] further generalised Theorem 1.2.24 to evaluate the orientation number of the join of any tree with an empty graph.

Theorem 1.2.25 *(Lee [33])*

If T_n is a tree of order n, then $\bar{d}(T_n + O_k) = 3$ for each $n \geq 3$ and $k \geq 2$.

1.3. G Vertex-multiplications

1.3.1. A Fundamental Classification

In 2000, Koh and Tay [27] introduced a new family of graphs, G vertex-multiplications, and extended the results on complete n-partite graphs. Let G be a given connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. For any sequence of n positive integers (s_i), a G vertex-multiplication, denoted by $G(s_1, s_2, \ldots, s_n)$, is the graph with vertex set $V^* = \bigcup_{i=1}^{n} V_i$ and edge set E^*, where V_i's are pairwise disjoint sets with $|V_i| = s_i$, for $i = 1, 2, \ldots, n$ and for any $u, v \in V^*$, $uv \in E^*$ if and only if $u \in V_i$ and $v \in V_j$ for some $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ such that $v_i v_j \in E(G)$. For instance, if $G \cong K_n$, then the graph $G(s_1, s_2, \ldots, s_n)$ is a complete n-partite graph with partite sizes $s_1, s_2, \ldots, s_n$. Also, we say G is a parent graph of graph $G(s_1, s_2, \ldots, s_n)$.

For $i = 1, 2, \ldots, n$, we denote the xth vertex in V_i by (x, v_i), or simply (x, i). i.e. $V_i = \{(x, i) | \ x = 1, 2, \ldots, s_i\}$. Hence, two vertices (x, i) and (y, j) in V^* are adjacent in $G(s_1, s_2, \ldots, s_n)$ if and only if $i \neq j$ and $v_i v_j \in E(G)$. For convenience, we write $G^{(s)}$ in place of $G(s, s, \ldots, s)$ for any positive integer s, and it is understood that the number of s's is equal to the order of G, n. Thus, $G^{(1)}$ is simply the graph G itself.

Koh and Tay [27] derived the following fundamental classification on G vertex-multiplication.

Theorem 1.3.1 *(Koh and Tay [27])*
Let G be a connected graph of order $n \geq 3$. If $s_i \geq 2$ for $i = 1, 2, \ldots, n$, then $d(G) \leq \bar{d}(G(s_1, s_2, \ldots, s_n)) \leq d(G) + 2$.

In view of Theorem 1.3.1, all graphs of the form $G(s_1, s_2, \ldots, s_n)$, with $s_i \geq 2$

for all $1 \leq i \leq n$, can be classified into three classes $\mathscr{C}_j$, where

$$\mathscr{C}_j = \{G(s_1, s_2, \ldots, s_n) |\ \bar{d}(G(s_1, s_2, \ldots, s_n)) = d(G) + j\},$$

for $j = 0, 1, 2$. Hence, in this thesis, we shall assume $s_i \geq 2$ for $i = 1, 2, \ldots, n$, unless otherwise stated. The following results were also proven in the same paper.

Lemma 1.3.2 *(Koh and Tay [27])*
Let μ_i, λ_i be integers such that $\mu_i \leq \lambda_i$ for $i = 1, 2, \ldots, n$. If the graph $G(\mu_1, \mu_2, \ldots, \mu_n)$ admits an orientation F in which every vertex v lies on a cycle of length not exceeding m, then $\bar{d}(G(\lambda_1, \lambda_2, \ldots, \lambda_n)) \leq max\{m, d(F)\}$.

Theorem 1.3.3 *(Koh and Tay [27])*
If $d(G) \geq 4$ and $s_i \geq 4$ for $i = 1, 2, \ldots, n$, then $G(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$.

Corollary 1.3.4 *(Koh and Tay [27])*
If $d(G) = 3$ and $s_i \geq 4$ for $i = 1, 2, \ldots, n$, then $G(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$.

By virtue of Theorem 1.3.3 and Corollary 1.3.4, any search for graphs of the form $G(s_1, s_2, \ldots, s_n) \in \mathscr{C}_2$ should be confined to graphs G with $d(G) \leq 2$ or on the sequence (s_i), where $s_j \leq 3$ for some j. Using graphs G with $d(G) = 2$, Koh and Tay [27] constructed G vertex-multiplications, $G(s_1, s_2, \ldots, s_n)$, that belong to $\mathscr{C}_2$. However, parent graphs of diameter at least 3 whose vertex-multiplications belong to $\mathscr{C}_2$ have not been found. Hence, Koh and Tay conjectured the following.

Conjecture 1.3.5 *(Koh and Tay [27])*
If G is a graph such that $d(G) \geq 3$ and $s_i \geq 2$ for $i = 1, 2, \ldots, n$, then $G(s_1, s_2, \ldots, s_n) \notin \mathscr{C}_2$.

1.3.2. Tree Vertex-multiplications

Koh and Tay [30] further investigated tree vertex multiplications. Since trees with diameter at most 2 are parent graphs of complete bipartite graphs and are completely solved, they considered trees of diameter at least 3. It was shown that any tree, with diameter 3 or 4, does not belong to the class $\mathscr{C}_2$.

Theorem 1.3.6 *(Koh and Tay [30])*
If T is a tree of order n and $d(T) = 3$ or 4, then $T(s_1, s_2, ..., s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$.

Theorem 1.3.7 *(Koh and Tay [30])*
Let T be a tree with diameter 4 and its only central vertex be u.
(a) If $deg_T(u) = 2$, then $T(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$.
(b) If $deg_T(u) \geq 3$, then $T^{(2)} \in \mathscr{C}_1$.

Furthermore, a vertex-multiplication of a tree with diameter at least 6 belongs to the class $\mathscr{C}_0$.

Theorem 1.3.8 *(Koh and Tay [30])*
If T is a tree of order n and $d(T) \geq 6$, then $T(s_1, s_2, ..., s_n) \in \mathscr{C}_0$.

On a related note, Buckley and Lewinter [4] proved the characterisation of graphs with a diameter-preserving spanning tree (d.p.s.t.) in 1988.

Theorem 1.3.9 *(Buckley and Lewinter [4])*
A connected graph G has a d.p.s.t. if and only if either
(1) $d(G) = 2r(G)$, or
(2) $d(G) = 2r(G) - 1$ and G contains a pair of adjacent central vertices u and v that have no common eccentric vertex.

Together with Theorem 1.3.8, the next proposition follows easily.

Proposition 1.3.10 *Let G be a graph, where $d(G) \geq 6$. If*
(1) $d(G) = 2r(G)$, or
(2) $d(G) = 2r(G) - 1$ and G contains a pair of adjacent central vertices u and v that have no common eccentric vertex,
then $G(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$.

Proof:

By Theorem 1.3.9, there exists a d.p.s.t. T of G, where $d(T) = d(G) \geq 6$. It follows from Theorem 1.3.8 that $\bar{d}(T(s_1, s_2, ..., s_n)) = d(T)$. i.e. there exists an orientation D of $T(s_1, s_2, ..., s_n)$ such that $d(D) = d(T)$. Note that $T(s_1, s_2, ..., s_n)$ is a spanning subgraph of $G(s_1, s_2, ..., s_n)$. Define an orientation F of $G(s_1, s_2, ..., s_n)$ such that D is a subdigraph of F, and any unspecified edges may be oriented arbitrarily. So, $d(F) \leq d(D) = d(G)$. Since $d(F) \geq d(G)$, we have $\bar{d}(G(s_1, s_2, ..., s_n)) = d(G)$.

□

In a similar line of thought, we can prove the following proposition using Theorem 1.3.6.

Proposition 1.3.11 *Let G be a graph. If*

(1) $r(G) = 2$ and $d(G) = 4$, or

(2) $r(G) = 2$ and $d(G) = 3$ and G contains a pair of adjacent central vertices u and v that have no common eccentric vertex,

then $G(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$.

1.3.3. Cycle Vertex-multiplications

In 2004, Ng [40] examined vertex-multiplications of cycles. In particular, he considered $C_n^{(s)}$ for $n \geq 3$. We quote a summary of the results in the table below.

n	Cases of $C_n(s_1, s_2, \ldots, s_n)$	$\in \mathscr{C}_0, \mathscr{C}_1, \mathscr{C}_2$?
3	$C_3^{(s)}$, $s \geq 2$	$\mathscr{C}_1$
4	$C_4(s_1, s_2, s_3, s_4)$, $s_i \geq 2$	$\mathscr{C}_1$ if $\{s_1 + s_3, s_2 + s_4\}$ is a co-pair, $\mathscr{C}_2$ otherwise
5	$C_5^{(s)}$, $s = 3, 4$	$\mathscr{C}_1$
6	$C_6^{(s)}$, $s = 3, 4$	$\mathscr{C}_1$ if $s = 3$ $\mathscr{C}_0$ if $s = 4$
7	$C_7^{(s)}$, $s = 3, 4$	$\mathscr{C}_1$ if $s = 3$ $\mathscr{C}_0$ if $s = 4$
8	$C_8^{(3)}$ $C_8(s_1, s_2, \ldots, s_8)$, $s_i \geq 4$	$\mathscr{C}_1$ $\mathscr{C}_0$
9	$C_9^{(3)}$ $C_9(s_1, s_2, \ldots, s_9)$, $s_i \geq 4$	$\mathscr{C}_1$ $\mathscr{C}_0$
≥ 10	$C_n(s_1, s_2, \ldots, s_n)$, $s_i \geq 3$	$\mathscr{C}_0$

Table 1.2: Orientation numbers of $C_n(s_1, s_2, \ldots, s_n)$.

2. Complete Tripartite Graphs

2.1. Existing Results And Motivation

Given any positive integers, $n, p_1, p_2, \ldots, p_n$, let K_n denote the complete graph of order n and $K(p_1, p_2, \ldots, p_n)$ denote the complete n-partite graph having p_i vertices in the ith partite set for $i = 1, 2, \ldots, n$, where $p_1 \le p_2 \le \ldots \le p_n$. The n partite sets are denoted by V_i, $i = 1, 2, \ldots, n$. i.e. $|V_i| = p_i$ for $i = 1, 2, \ldots, n$. Furthermore, i_j denotes the jth vertex in V_i for $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, p_i$. Thus, $K_n \cong K(p_1, p_2, \ldots, p_n)$, where $p_1 = p_2 = \ldots = p_n = 1$. Note that complete n-partite graphs are G vertex-multiplications of complete graphs with order n. In this chapter, we shall focus on complete n-partite graphs, particularly, complete tripartite graphs. We start by listing some existing results of concern.

Plesnik [42], Gutin [13], and Koh and Tan [18] independently proved that the orientation number of a complete multipartite graph is 2 or 3. Some sufficient and necessary conditions were also established in the same papers. However, a complete characterisation remains elusive.

Theorem 2.1.1 *(Plesnik [42], Gutin [13], Koh and Tan [18])*
For all positive integers $n \ge 3$ and $p_1, p_2, \ldots, p_n$, $2 \le \bar{d}(K(p_1, p_2, \ldots, p_n)) \le 3$.

Theorem 2.1.2 *(Gutin [13], Koh and Tan [18])*
For all integers $n \ge 3$ and $p \ge 2$, $\bar{d}(K(\overbrace{p, p, \ldots, p}^{n\ terms})) = 2$.

Theorem 2.1.3 *(Koh and Tan [18])*

Let $n \geq 3$ and $p_1, p_2, \ldots, p_n$ be positive integers. Denote $h = \sum_{k=1}^{n} p_i$. If

$$p_i > \binom{h - p_i}{\lfloor (h - p_i)/2 \rfloor},$$

for some $i = 1, 2, \ldots, n$, then $\bar{d}(K(p_1, p_2, ..., p_n)) = 3$.

Along a similar line of research, Rajasekaran and Sampathkumar investigated special cases of complete tripartite graphs.

Theorem 2.1.4 *(Rajasekaran and Sampathkumar [43])*

For $q \geq p \geq 2$, $\bar{d}(K(1, p, q)) = 3$.

Theorem 2.1.5 *(Koh and Tan [19])*

For $q \geq p \geq 2$, if $q \leq \binom{p}{\lfloor p/2 \rfloor}$, then $\bar{d}(K(2, p, q)) = 2$.

Theorem 2.1.6 *(Rajasekaran and Sampathkumar [43])*

For $q \geq 3$, $\bar{d}(K(2, 2, q)) = 3$.

Theorem 2.1.7 *(Rajasekaran and Sampathkumar [43])*

For $q \geq 4$, $\bar{d}(K(2, 3, q)) = 3$.

2.2. A Conjecture On $K(2, p, q)$

Based on Theorems 2.1.6, 2.1.7, and and an unpublished paper "The orientation number of the complete tripartite graph $K(2, 4, p)$", Rajasekaran and Sampathkumar conjectured that the converse of Theorem 2.1.5 holds for complete tripartite graphs $K(2, p, q)$, $q \geq p \geq 5$. Ng [40] showed for $q \geq p$, $\bar{d}(K(1, 1, p, q)) = 2$ implies $q \leq \binom{p}{\lfloor p/2 \rfloor}$. Since an orientation D of $K(2, p, q)$, with $d(D) = 2$, is a spanning subdigraph of $K(1, 1, p, q)$, the conjecture follows from Ng's assertion. In this section, we provide an independent proof of the conjecture. The following two lemmas will be found useful in our proof.

Lemma 2.2.1 ***(Duality)***

Let D be an orientation of a graph G. Let $\tilde{D}$ be the orientation of G such that $uv \in E(\tilde{D})$ if and only if $vu \in E(D)$. Then, $d(\tilde{D}) = d(D)$.

Proof: Suppose not. Then, there exist some vertices $u, v \in V(\tilde{D})$ such that $d_{\tilde{D}}(u, v) > d(D)$. By definition of $\tilde{D}$, $d_D(v, u) = d_{\tilde{D}}(u, v)$. It follows that $d_D(v, u) > d(D)$, a contradiction.

□

Lemma 2.2.2 *Let $G = K(p_1, p_2, \ldots, p_n)$, $n \geq 3$, and D be an orientation of G. Suppose there exist vertices i_s and j_t for some i, j, s and t, where $i \neq j$, $1 \leq i, j \leq n$, $1 \leq s \leq p_i$ and $1 \leq t \leq p_j$, such that*

(i) $O(i_s) \cap (V(G) - V_j) = O(j_t) \cap (V(G) - V_i)$, or

(ii) $I(i_s) \cap (V(G) - V_j) = I(j_t) \cap (V(G) - V_i)$.

Then, $d(D) \geq 3$.

Proof: Suppose (i). W.l.o.g., we assume $j_t \rightarrow i_s$. It follows that $d_D(i_s, j_t) > 2$ and $d(D) \geq 3$. (ii) follows now from the Duality Lemma.

□

Theorem 2.2.3 *For any integers $q \geq p \geq 3$, if $\bar{d}(K(2, p, q)) = 2$, then $q \leq \binom{p}{\lfloor p/2 \rfloor}$.*

Proof:

Since $\bar{d}(K(2, p, q)) = 2$, there exists an orientation D of $K(2, p, q)$ such that $d(D) = 2$.

Case 1. $V_1 \rightarrow V_2$.

It follows from $d_D(3_i, 1_j) \leq 2$, for every $i = 1, 2, \ldots, q$, and $j = 1, 2$, that $V_3 \rightarrow V_1$. Also, since $d_D(2_i, 3_j) \leq 2$ for every $i = 1, 2, \ldots, p$, and $j = 1, 2, \ldots, q$, we have $V_2 \rightarrow V_3$. However, $d_D(3_i, 3_j) \geq 3$ for any $1 \leq i, j \leq q$, $i \neq j$, which contradicts $d(D) = 2$.

Similarly, by the Duality Lemma, we cannot have $V_2 \rightarrow V_1$.

Case 2. $1_i \to V_2 \to 1_{3-i}$ for exactly one of $i = 1, 2$.

W.l.o.g., we may assume that $1_1 \to V_2 \to 1_2$. It follows from $d_D(1_2, 3_i) \leq 2$ and $d_D(3_i, 1_1) \leq 2$ for every $i = 1, 2, \ldots, q$ that $1_2 \to V_3 \to 1_1$. Now, for any $i \neq j$, $1 \leq i, j \leq q$, $d_D(3_i, 3_j) \leq 2$ and thus, $O(3_i) \cap V_2$ and $O(3_j) \cap V_2$ are independent. By Sperner's Lemma, $q \leq \binom{p}{\lfloor p/2 \rfloor}$.

Case 3. $1_i \to V_2$ for exactly one of $i = 1, 2$.

W.l.o.g., let $i = 1$. Furthermore, we assume that $\emptyset \neq O(1_2) \cap V_2 \subset V_2$ in view of Cases 1 and 2. Hence, let $|O(1_2) \cap V_2| = k$, where $0 < k < p$. Since $d_D(u, 3_j) \leq 2$ for every $u \in O(1_2) \cap V_2$ and every $j = 1, 2, \ldots, q$, it follows that $O(1_2) \cap V_2 \to V_3$. It also follows from $d_D(3_j, 1_1) \leq 2$ for every $j = 1, 2, \ldots, q$, that $V_3 \to 1_1$.

Partition V_3 into L_1 and L_2 such that $L_1 := \{v \in V_3 | \ v \to 1_2\}$ and $L_2 := \{v \in V_3 | \ 1_2 \to v\}$. Note that $L_1 \to V_1$. Since for each $v \in L_1$, $d_D(2_j, v) \leq 2$ for all $j = 1, 2, \ldots, p$, we have $V_2 \to L_1$. Thus, $|L_1| \leq 1$, otherwise if $u, v \in L_1$, then $d_D(u, v) \geq 3$. Also, $|L_2| \leq \binom{p-k}{\lfloor (p-k)/2 \rfloor}$. Otherwise, by Sperner's Lemma, there exist $3_i, 3_j \in L_2$ such that $O(3_i) \cap V_2 \subseteq O(3_j) \cap V_2$ for some $i \neq j$ and $1 \leq i, j \leq q$, which implies $d_D(3_i, 3_j) > 2$. Hence, $q = |V_3| = |L_1| + |L_2| \leq 1 + \binom{p-k}{\lfloor (p-k)/2 \rfloor} \leq 1 + \binom{p-1}{\lfloor (p-1)/2 \rfloor} \leq \binom{p}{\lfloor p/2 \rfloor}$.

Similarly, the case where $V_2 \to 1_i$ for exactly one of $i = 1, 2$, follows from the Duality Lemma.

Case 4. $\emptyset \neq O(1_i) \cap V_2 \subset V_2$ for $i = 1, 2$.

Partition V_2 into the sets $K_A := \{v \in V_2 | \ A \to v \to (V_1 - A)\}$, where $A \subseteq V_1$. Similarly, partition V_3 into the sets $L_A := \{v \in V_3 | \ A \to v \to (V_1 - A)\}$, where $A \subseteq V_1$.

Since $d_D(u, 2_j) \leq 2$ for any $u \in V_3$ and $j = 1, 2, \ldots, p$, it follows that $L_\emptyset \to K_\emptyset$, $L_{\{1_1\}} \to K_{\{1_1\}} \cup K_\emptyset$, $L_{\{1_2\}} \to K_{\{1_2\}} \cup K_\emptyset$ and $L_{V_1} \to V_2$. Similarly, since $d_D(u, 3_j) \leq 2$ for any $u \in V_2$ and $j = 1, 2, \ldots, q$, it follows that $K_\emptyset \to L_\emptyset$, $K_{\{1_1\}} \to L_{\{1_1\}} \cup L_\emptyset$,

$K_{\{1_2\}} \to L_{\{1_2\}} \cup L_{\emptyset}$ and $K_{V_1} \to V_2$.

Invoking Sperner's Lemma on each L_A, $A \subseteq V_1$, we have $|L_{\emptyset}| \leq 1$, $|L_{\{1_1\}}| \leq \binom{|K_{\{1_2\}}|}{\lfloor |K_{\{1_2\}}|/2 \rfloor}$, $|L_{\{1_2\}}| \leq \binom{|K_{\{1_1\}}|}{\lfloor |K_{\{1_1\}}|/2 \rfloor}$ and $|L_{V_1}| \leq 1$. Otherwise, there would exist $3_i, 3_j \in L_A$ such that $O(3_i) \cap V_2 \subseteq O(3_j) \cap V_2$ for some $i \neq j$ and $1 \leq i, j \leq q$, implying $d_D(3_i, 3_j) > 2$.

Subcase 4.1. $|K_{V_1}| = 0$.

For $i = 1, 2$, $K_{\{1_i\}} \neq \emptyset$, since $O(1_i) \cap V_2 \neq \emptyset$ by assumption. From Lemma 2.2.2, it follows that $L_{\{1_1\}} = L_{\{1_2\}} = \emptyset$. So, $q = |V_3| = |L_{\emptyset}| + |L_{V_1}| \leq 1 + 1 < \binom{p}{\lfloor p/2 \rfloor}$.

Subcase 4.2. $|K_{V_1}| > 0$.

Then, $L_{V_1} = \emptyset$ by Lemma 2.2.2. Recall that $|K_{\emptyset}| + |K_{\{1_1\}}| + |K_{\{1_2\}}| + |K_{V_1}| = p$. By Lemma 2.2.2, for each $i = 1, 2$, if $K_{\{1_i\}} \neq \emptyset$, then $L_{\{1_i\}} = \emptyset$. Hence, if $K_{\{1_1\}} \neq \emptyset$ and $K_{\{1_2\}} \neq \emptyset$, then $q = |V_3| = |L_{\emptyset}| \leq 1$. If $K_{\{1_1\}} = \emptyset$ and $K_{\{1_2\}} \neq \emptyset$, then $q = |L_{\emptyset}| + |L_{\{1_1\}}| \leq 1 + \binom{|K_{\{1_2\}}|}{\lfloor |K_{\{1_2\}}|/2 \rfloor} \leq 1 + \binom{p-1}{\lfloor (p-1)/2 \rfloor}$. By symmetry, if $K_{\{1_1\}} \neq \emptyset$ and $K_{\{1_2\}} = \emptyset$, it also follows that $q \leq 1 + \binom{p-1}{\lfloor (p-1)/2 \rfloor}$. Lastly, if $K_{\{1_1\}} = K_{\{1_2\}} = \emptyset$, it follows that $q = |L_{\emptyset}| + |L_{\{1_1\}}| + |L_{\{1_2\}}| \leq 1 + 1 + 1$. Therefore, $q \leq max\{1 + \binom{p-1}{\lfloor (p-1)/2 \rfloor}, 3\} \leq \binom{p}{\lfloor p/2 \rfloor}$.

$\square$

In the proof of Theorem 2.2.3, we can be more conscientious and make deductions about the optimal orientations of $K(2, p, q)$ if $1 + \binom{p-1}{\lfloor (p-1)/2 \rfloor} < q \leq \binom{p}{\lfloor p/2 \rfloor}$.

Corollary 2.2.4 *For any integers $p \geq 4$ and $1 + \binom{p-1}{\lfloor (p-1)/2 \rfloor} < q \leq \binom{p}{\lfloor p/2 \rfloor}$, let D be an optimal orientation of $K(2, p, q)$, where $d(D) = 2$. Then, in D,*

(i) $1_i \to V_2 \to 1_{3-i} \to V_3 \to 1_i$ for exactly one of $i = 1, 2$.

(ii) $\{O(3_i) \cap V_2 |\ i = 1, 2, \ldots, q\}$ is a family of independent subsets of $\{2_1, 2_2, \ldots, 2_p\}$.

In particular, there are at most two optimal orientations (up to isomorphism) in the case where $q = \binom{p}{\lfloor p/2 \rfloor}$.

Proof:

Case 1 of the proof of Theorem 2.2.3 shows that it is impossible for $V_1 \to V_2$ or $V_2 \to V_1$. Since $q > 1 + \binom{p-1}{\lfloor (p-1)/2 \rfloor}$ and $p \geq 4$, Cases 3 and 4 are also impossible. This leaves us with the result of Case 2, i.e. $1_i \to V_2 \to 1_{3-i} \to V_3 \to 1_i$ for exactly one of $i = 1, 2$.

Now, for any i, j where $i \neq j$ and $1 \leq i, j \leq q$, $3_i, 3_j \in V_3$, $d(3_i, 3_j) = 2$ if and only if $O(3_i) \cap V_2 \not\subseteq O(3_j) \cap V_2$. Thus, (ii) follows.

Furthermore, if $q = \binom{p}{\lfloor p/2 \rfloor}$, then $|O(3_i) \cap V_2| = \lfloor \frac{p}{2} \rfloor$ or $\lceil \frac{p}{2} \rceil$ by Sperner's Lemma. Thus, there are at most two optimal orientations (up to isomorphism) D.

$\square$

Theorem 2.2.3 completes the characterisation of complete tripartite graphs $K(2, p, q)$ with $\bar{d}(K(2, p, q)) = 2$. Together with Theorems 2.1.5 and 2.1.6, we have the following theorem. Interestingly, this characterisation has the same bounds for q as the general bipartite graph $K(p, q)$. (See Theorem 1.2.5)

Theorem 2.2.5 *For any integers $q \geq p \geq 2$, $\bar{d}(K(2, p, q)) = 2$ if and only if $q \leq \binom{p}{\lfloor p/2 \rfloor}$.*

In the remainder of this section, we describe our attempt to generalise the technique used in Theorem 2.2.3 for complete tripartite graphs $K(p, q, r)$, $3 \leq p \leq q \leq r$. To this end, we first investigate optimal orientations of complete multipartite graphs with at least 3 partite sets. Proposition 2.2.6 and Corollary 2.2.7 are two commonly known results.

Proposition 2.2.6 *Suppose T is a multipartite tournament, with partite sets V_i, $i = 1, 2, \ldots, n$, where $n \geq 3$. Then, $d(T) = 2$ if and only if the following two conditions are satisfied in T.*

(i) If $u, v \in V_i$ for some $i = 1, 2, \ldots, n$, then u and v lie on a directed C_4.

(ii) If $u \in V_i$, $v \in V_j$ for some $1 \leq i < j \leq n$, then u and v lie on a directed C_3.

Proof:

($\Rightarrow$)

(i) Since $d(T) = 2$, it follows that $d(u,v) = 2$ for any $u,v \in V_i$, for some $i = 1,2,\ldots,n$. Thus, there exist a $u-v$ path and $v-u$ path, each of length 2, which form a directed C_4.

(ii) Now, let $u \in V_i$, $v \in V_j$ for some $1 \le i < j \le n$. W.l.o.g., assume $v \to u$. Since $d(u,v) \le 2$, it follows that there exists a $u-v$ path of length 2. Hence, u and v lie on a directed C_3.

($\Leftarrow$)

It follows from (i) and (ii) that $d(u,v) \le 2$ for any two vertices u and v in $V(T)$.

$\square$

Corollary 2.2.7 *Suppose T is a multipartite tournament, with partite sets V_i, $i = 1,2,\ldots,n$, where $n \ge 3$. If $d(T) = 2$, then every arc in T lies on a directed C_3.*

The next lemma counts the number of subsets independent of a given set S.

Lemma 2.2.8 *Let $\emptyset \ne S \subset \mathbb{N}_p$. If $I(S) = \{T \subset \mathbb{N}_p | \ T \not\subseteq S$ and $S \not\subseteq T\}$, then $|I(S)| = 2^p - 2^{|S|} - 2^{p-|S|} + 1$. Furthermore, the maximum is attained when $|S| = \lfloor \frac{p}{2} \rfloor$ or $|S| = \lceil \frac{p}{2} \rceil$.*

Proof:

Case 1. $|S| = 1$.

Let $S \ne T \subset \mathbb{N}_p$. If $|T| = 1$, then T and S are independent. There are $\binom{p}{1} - 1$ such subsets. Suppose $2 \le |T| \le p-1$, then T and S are independent if and only if $S \not\subseteq T$. There are $\binom{p}{|T|} - \binom{p-|S|}{|T|-|S|}$ such subsets. Therefore, $|I(S)| = \binom{p}{1} - 1 + \sum\limits_{i=2}^{p-1} [\binom{p}{i} - \binom{p-1}{i-1}] = \sum\limits_{i=1}^{p-1} \binom{p}{i} - \sum\limits_{i=1}^{p-1} \binom{p-1}{i-1} = 2^p - \binom{p}{0} - \binom{p}{p} - [2^{(p-1)} - \binom{p-1}{p-1}] = 2^p - 2 - 2^{(p-1)} + 1$.

Case 2. $|S| = p-1$.

Let $S \ne T \subset \mathbb{N}_p$. If $|T| = p-1$, then T and S are independent. There are $\binom{p}{p-1} - 1$ such subsets. Suppose $1 \le |T| \le p-2$, then T and S are independent if and only if $T \not\subseteq S$. There are $\binom{p}{|T|} - \binom{|S|}{|T|}$ such subsets. Therefore, $|I(S)| = \binom{p}{p-1} - 1 + \sum_{i=1}^{p-2} [\binom{p}{i} - \binom{p-1}{i}] = \sum_{i=1}^{p-1} \binom{p}{i} - \sum_{i=1}^{p-1} \binom{p-1}{i} = 2^p - \binom{p}{0} - \binom{p}{p} - [2^{(p-1)} - \binom{p-1}{0}] = 2^p - 2^{(p-1)} - 2 + 1$.

Case 3. $1 < |S| < p - 1$.

Let $S \neq T \subset \mathbb{N}_p$. If $|T| = |S|$, then T and S are independent. There are $\binom{p}{|S|} - 1$ such subsets. If $1 \leq |T| \leq |S| - 1$, then T and S are independent if and only if $T \not\subseteq S$. There are $\binom{p}{|T|} - \binom{|S|}{|T|}$ such subsets. If $|S| + 1 \leq |T| \leq p - 1$, then T and S are independent if and only if $S \not\subseteq T$. There are $\binom{p}{|T|} - \binom{p-|S|}{|T|-|S|}$ such subsets. Therefore, $|I(S)| = \binom{p}{|S|} - 1 + \sum_{i=1}^{|S|-1} [\binom{p}{i} - \binom{|S|}{i}] + \sum_{i=|S|+1}^{p-1} [\binom{p}{i} - \binom{p-|S|}{i-|S|}] = \sum_{i=1}^{p-1} \binom{p}{i} - \sum_{i=1}^{|S|-1} \binom{|S|}{i} - \sum_{i=|S|+1}^{p-1} \binom{p-|S|}{i-|S|} - 1 = 2^p - \binom{p}{0} - \binom{p}{p} - [2^{|S|} - \binom{|S|}{0} - \binom{|S|}{|S|}] - [2^{p-|S|} - \binom{p-|S|}{0} - \binom{p-|S|}{p-|S|}] - 1 = 2^p - 2^{|S|} - 2^{p-|S|} + 1$.

Lastly, let $f(x) := 2^x + 2^{p-x}$. Since $f'(x) = (2^x - 2^{p-x})(ln\ 2)$, it is easy to see that $f(x)$ has only one minimum point for all $x \in \mathbb{R}$. Furthermore, $f(x)$ attains its minimum value of $2^{\frac{p}{2}} + 2^{\frac{p}{2}}$, when $x = \frac{p}{2}$. Therefore, the function $g(|S|) = 2^p - 2^{|S|} - 2^{p-|S|} + 1$ attains its maximum value when $|S| = \lfloor \frac{p}{2} \rfloor$ or $|S| = \lceil \frac{p}{2} \rceil$.

$\square$

Discussion 2.2.9

Suppose p, q, r are integers where $3 \leq p \leq q \leq r$ and $\bar{d}(K(p,q,r)) = 2$. Our objective is an upper bound of r, in terms of p and q.

Let D be an orientation of $K(p,q,r)$, where $d(D) = 2$. Partition V_2 into the sets $K_A := \{v \in V_2 |\ A \rightarrow v \rightarrow (V_1 - A)\}$, where $A \subseteq V_1$. Similarly, partition V_3 into the sets $L_A := \{v \in V_3 |\ A \rightarrow v \rightarrow (V_1 - A)\}$, where $A \subseteq V_1$.

Let $A, B \subseteq V_1$. Suppose there exist $u \in L_A$ and $v \in K_B$. If $B \subseteq A$, then $O(u) \cap V_1 = V_1 - A \subseteq V_1 - B = O(v) \cap V_1$. Since $d_D(u, v) \leq 2$, it follows that $u \rightarrow v$. This holds for each $u \in L_A$ and each $v \in K_B$. Hence, we have $L_A \rightarrow K_B$.

In particular, $L_{V_1} \to K_B$ for every $B \subseteq V_1$ and $L_A \to K_\emptyset$ for every $A \subseteq V_1$. i.e. $L_{V_1} \to V_2$ and $V_3 \to K_\emptyset$.

On the other hand, if $A \subseteq B$, then $O(v) \cap V_1 = V_1 - B \subseteq V_1 - A = O(u) \cap V_1$. Since $d_D(v,u) \leq 2$, it follows that $v \to u$, thus, $K_B \to L_A$. In particular, $K_{V_1} \to V_3$ and $V_2 \to L_\emptyset$. Furthermore, for any $C \subseteq V_1$, at least one of L_C and K_C is empty by Lemma 2.2.2.

Let $A \subseteq V_1$ and consider two vertices v and w in L_A. Then, v and w have common in-vertices and out-vertices except possibly the vertices in the sets K_B, where $B \subseteq V_1$ and, A and B are independent. Since $d_D(v,w) \leq 2$ and $d_D(w,v) \leq 2$, it follows that $O(v)$ and $O(w)$ are independent. Therefore, $|L_A| \leq \binom{\mathscr{L}_A}{\lfloor \mathscr{L}_A/2 \rfloor}$, where $\mathscr{L}_A := \sum_{A \not\subseteq B,\ B \not\subseteq A} |K_B|$. In particular, $|L_\emptyset| \leq 1$ and $|L_{V_1}| \leq 1$.

Now, $r = |V_3| = \sum_{A \subseteq V_1} |L_A| \leq |L_\emptyset| + |L_{V_1}| + \sum_{\emptyset \neq A \subset V_1} |L_A| \leq 2 + \sum_{\emptyset \neq A \subset V_1} \binom{\mathscr{L}_A}{\lfloor \mathscr{L}_A/2 \rfloor}$.

Case 1. Exactly one element K_{B_0} of $\{K_B |\ B \subseteq V_1\}$ is nonempty.

Then, $|K_{B_0}| = q$. Let $|B_0| = b$. Thus,

$$\begin{aligned}
r &\leq 2 + \sum_{\emptyset \neq A \subset V_1} \binom{\mathscr{L}_A}{\lfloor \mathscr{L}_A/2 \rfloor} \\
&= 2 + \sum_{\substack{\emptyset \neq A \subset V_1 \\ A \subseteq B_0 \text{ or } B_0 \subseteq A}} 1 + \sum_{\substack{\emptyset \neq A \subset V_1 \\ A \not\subseteq B_0,\ B_0 \not\subseteq A}} \binom{|K_{B_0}|}{\lfloor |K_{B_0}|/2 \rfloor} \\
&= 2 + 2^p - 2 - (2^p - 2^b - 2^{p-b} + 1) + (2^p - 2^b - 2^{p-b} + 1)\binom{q}{\lfloor q/2 \rfloor} \\
&= (2^p + 1)\binom{q}{\lfloor q/2 \rfloor} - 1 + (2^b + 2^{p-b})\left[1 - \binom{q}{\lfloor q/2 \rfloor}\right] \\
&\leq (2^p + 1)\binom{q}{\lfloor q/2 \rfloor} - 1 + (2^{\lfloor p/2 \rfloor} + 2^{p - \lfloor p/2 \rfloor})\left[1 - \binom{q}{\lfloor q/2 \rfloor}\right] \\
&= (2^p - 2^{\lfloor p/2 \rfloor} - 2^{\lceil p/2 \rceil} + 1)\binom{q}{\lfloor q/2 \rfloor} + 2^{\lfloor p/2 \rfloor} + 2^{\lceil p/2 \rceil} - 1.
\end{aligned}$$

By Lemma 2.2.8, the number of subsets of V_1 independent of B_0 is $2^p - 2^b - 2^{p-b} + 1$. Since the number of elements in $2^{V_1} - \{\emptyset, V_1\}$ not independent of B_0 is $2^p - 2 - (2^p - 2^b - 2^{p-b} + 1)$, the second equality follows. The last inequality follows from the fact that $f(x)$ (as defined in Lemma 2.2.8) attains the minimum value of $2^{\frac{p}{2}} + 2^{\frac{p}{2}}$, when $x = \frac{p}{2}$.

Case 2. At least two elements of $\{K_B \mid B \subseteq V_1\}$ are nonempty.

Suppose $|K_{B_1}| + |K_{B_2}| + \ldots + |K_{B_t}| = q$, where $|B_i| = b_i$ for $i = 1, 2, \ldots, t$, for some $t \geq 2$. For $j = 0, 1, 2, \ldots, t$, denote the number of elements in $2^{V_1} - \{\emptyset, V_1\}$ that are independent of exactly j elements of $\{B_i \mid i = 1, 2, \ldots, t\}$ by $E(j)$. For $j = 1, 2, \ldots, t$, let $\omega(B_{i_1} B_{i_2} \ldots B_{i_j})$ denote the number of elements in $2^{V_1} - \{\emptyset, V_1\}$ that are independent of exactly the sets $B_{i_1}, B_{i_2}, \ldots, B_{i_j}$. Then,

$$\begin{aligned} r \leq & 2 + \sum_{\emptyset \neq A \subset V_1} \binom{\mathscr{L}_A}{\lfloor \mathscr{L}_A/2 \rfloor} \\ \leq & 2 + E(0) + \sum_{i=1}^{t} \omega(B_i) \binom{|K_{B_i}|}{\lfloor |K_{B_i}|/2 \rfloor} + \sum_{i_1 < i_2} \omega(B_{i_1} B_{i_2}) \binom{|K_{B_{i_1}}| + |K_{B_{i_2}}|}{\lfloor (|K_{B_{i_1}}| + |K_{B_{i_2}}|)/2 \rfloor} \\ & + \ldots + \sum_{i_1 < i_2 < \ldots < i_t} \omega(B_{i_1} B_{i_2} \ldots B_{i_t}) \binom{|K_{B_{i_1}}| + |K_{B_{i_2}}| + \ldots + |K_{B_{i_t}}|}{\lfloor (|K_{B_{i_1}}| + |K_{B_{i_2}}| + \ldots + |K_{B_{i_t}}|)/2 \rfloor}. \end{aligned} \tag{2.1}$$

It will be ideal if we can simplify (2.1) and derive an upper bound in terms of p and q, as in Case 1. The difficulty here can be broken down into two issues.

1. Does there exist a general expression (or good upper bound) for $\omega(B_{i_1} B_{i_2} \ldots B_{i_j})$ for $j \geq 2$? From Lemma 2.2.8, we have $\omega(B_i) = 2^p - 2^{b_i} - 2^{p-b_i} + 1$. Since the number of elements in $2^{V_1} - \{\emptyset, V_1\}$ that are independent of $B_{i_1}, B_{i_2}, \ldots, B_{i_j}$ is at most the number of elements that are independent of any one of them, it follows that $\omega(B_{i_1} B_{i_2} \ldots B_{i_j}) \leq \omega(B_{i^*})$, where i^* be such that $b_{i^*} = \underset{1 \leq i \leq t}{min}\{|b_i - \frac{p}{2}|\}$. However, to evaluate $\omega(B_{i_1} B_{i_2} \ldots B_{i_j})$ accurately, we need an analogue of Lemma 2.2.8. This is formally phrased as follows.

Problem 2.2.10 *If $\emptyset \neq B_i \subset \mathbb{N}_p$ for $i = 1, 2, \ldots, j$, where $j \geq 2$, determine the number of elements in $2^{V_1} - \{\emptyset, V_1\}$ that are independent of each $B_{i_1}, B_{i_2}, \ldots, B_{i_j}$.*

It is important to note that for $1 \leq x < y \leq j$, B_{i_x} and B_{i_y} may or may not be independent. Though there is extensive literature available on generalisation of

Sperner families, we did not find any paper which tackles Problem 2.2.10 directly.

2. The other significant group of terms in (2.1) are the terms $\binom{|K_{B_{i_1}}|+|K_{B_{i_2}}|+\ldots+|K_{B_{i_j}}|}{\lfloor(|K_{B_{i_1}}|+|K_{B_{i_2}}|+\ldots+|K_{B_{i_j}}|)/2\rfloor}$, where $j \geq 2$. Since $|K_{B_1}| + |K_{B_2}| + \ldots + |K_{B_t}| = q$, where each $K_{B_i} \neq \emptyset$, it follows easily that $\binom{|K_{B_{i_1}}|+|K_{B_{i_2}}|+\ldots+|K_{B_{i_j}}|}{\lfloor(|K_{B_{i_1}}|+|K_{B_{i_2}}|+\ldots+|K_{B_{i_j}}|)/2\rfloor} \leq \binom{q-(t-j)}{\lfloor[q-(t-j)]/2\rfloor}$ for $j = 1, 2, \ldots, t$. However, this is not a good bound for large q. Hence, for fixed $B_1, B_2, \ldots, B_t$, we need to optimally allocate q vertices to K_{B_i}, $i = 1, 2, \ldots, t$ such that the maximum value of (2.1) is attained. This required optimal allocation is the second difficult issue.

2.3. Sufficient Conditions For $\bar{d}(K(p,p,q)) = 2$

Apart from $K(2,p,q)$, Rajasekaran and Sampathkumar also studied complete tripartite graphs of the form $K(p,p,q)$.

Theorem 2.3.1 *(Rajasekaran and Sampathkumar [43])*
For $p \geq 4$, $4 \leq q \leq 2p$, $\bar{d}(K(p,p,q)) = 2$.

In this section, we provide some sufficient conditions on p and q so that $\bar{d}(K(p,p,q)) = 2$. Our result (see Theorem 2.3.17) improves significantly the upper bound $2p$ of q given in Theorem 2.3.1, especially when p increases. We begin by solving a combinatorics problem, which will be of assistance later.

Suppose $p \geq 4$ is a composite integer, say $p = kd$ for some $k, d \in \mathbb{Z}^+$, $1 < k, d < p$, i.e. k and d are non-trivial divisors of p. Let there be $2d$ groups of k distinct elements each. Now, what is the number of ways to select p elements, such that some but not all are chosen from each group?

Definition 2.3.2 *Suppose $p \geq 4$ is an integer such that $p = kd$ for some non-trivial divisors $k, d \in \mathbb{Z}^+$. Denote a solution $(x_1, x_2, \ldots, x_{2d})^*$ if $x_1, x_2, \ldots, x_{2d}$ satisfies*

$$x_1 + x_2 + \ldots + x_{2d} = p, \text{ and} \tag{2.2}$$
$$1 \leq x_i \leq k-1, \text{ for } i = 1, 2, \ldots, 2d.$$

Define $\Phi^*(p,d) := \sum_{(x_1,x_2,\ldots,x_{2d})^*} \binom{k}{x_1}\binom{k}{x_2}\cdots\binom{k}{x_{2d}}$.

Definition 2.3.3 *Suppose $p \geq 4$ is an integer such that $p = kd$ for some non-trivial divisors $k, d \in \mathbb{Z}^+$. For any non-negative integers i, j, define $[i, j]$ to be the set of solutions $(x_1, x_2, \ldots, x_{2d})$ satisfying*

$$x_1 + x_2 + \ldots + x_{2d} = p,$$

$$x_{s_m} = 0, \ \textit{for } m = 1, 2, \ldots, i, \ \textit{where } \{s_1, s_2, \ldots, s_i\} \subseteq \{1, 2, \ldots, 2d\},$$

$$x_{t_n} = k, \ \textit{for } n = 1, 2, \ldots, j, \ \textit{where } \{t_1, t_2, \ldots, t_j\} \subseteq \{1, 2, \ldots, 2d\}, \ \textit{and}$$

$$1 \le x_r \le k - 1, \ \textit{for } r \in \{1, 2, \ldots, 2d\} - (\{s_1, s_2, \ldots, s_i\} \cup \{t_1, t_2, \ldots, t_j\}).$$

Furthermore, we denote $\Phi(p, d, [i, j]) := \sum_{(x_1, x_2, \ldots, x_{2d}) \in [i,j]} \binom{k}{x_1}\binom{k}{x_2} \ldots \binom{k}{x_{2d}}$.

Observation 2.3.4

(a) $\Phi(p, d, [i, j]) \ge 0$ *for* $0 \le i, j \le d$.

(b) For each $[i, j]$ *defined above,* $0 \le i, j \le d$.

(c) $\Phi(p, d, [d, d]) = \binom{2d}{d}$.

(d) $\Phi(p, d, [i, d]) = \Phi(p, d, [d, i]) = 0$ *for* $0 \le i \le d - 1$.

(e) If p *is even, then* $\Phi^*(p, \frac{p}{2}) = 2^p$.

Proof:

(a) This follows directly from the definition of $\Phi(p, d, [i, j])$.

(b) Suppose $i > d$. Hence, $\sum_{h=1}^{2d} x_h = \sum_{m=1}^{i} x_{s_m} + \sum_{h \ne s_m} x_h \le 0 + (2d - i)k < dk = p$, a contradiction. Similarly, if $j > d$, then $\sum_{h=1}^{2d} x_h \ge \sum_{n=1}^{j} x_{t_n} = jk > dk = p$, a contradiction.

(c) $\Phi(p, d, [d, d]) = \binom{2d}{d,d,0}\binom{k}{0}^d\binom{k}{k}^d = \binom{2d}{d}$.

(d) Observe that $[i, d] = \emptyset$ for every i, $0 \le i \le d - 1$. For if $x_{t_n} = k$, for $n = 1, 2, \ldots, d$, then $\sum_{h=1}^{2d} x_h = p$ implies $x_r = 0$ for all $r \ne t_n$, i.e. $i = d$, a contradiction. Similarly, $[d, i] = \emptyset$ for every i, $0 \le i \le d - 1$. For if $x_{s_m} = 0$, for $m = 1, 2, \ldots, d$, then $\sum_{h=1}^{2d} x_h = p$ implies $x_r = k$ for all $r \ne s_m$, i.e. $i = d$, a contradiction. Hence, $\Phi(p, d, [i, d]) = \Phi(p, d, [d, i]) = 0$ for all i, $0 \le i \le d - 1$.

(e) Let $p = 2d$. Then, $k = 2$ and $(x_1, x_2, \ldots, x_{2d}) = (1, 1, \ldots, 1)$ is the only solution satisfying (2.2). Hence, $\Phi^*(p, \frac{p}{2}) = \binom{k}{1}^{2d} = 2^p$.

$\square$

Now, our aim is to determine $\Phi^*(p,d)$, which can be seen as $\Phi^*(p,d) = \Phi(p,d,[0,0])$. In the proof of our next result, we make use of the following combinatorial identities which we state without proof.

Lemma 2.3.5 *For non-negative integers x_i, n_i, n, k, r, $n \geq 1$, $r \leq k \leq n$ and $x_i \leq n_i$ for $i = 1, 2 \ldots, r$,*

(a) $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$.

(b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n} = 0$.

(c) $\sum\limits_{x_1+x_2+\ldots x_r=p} \binom{n_1}{x_1}\binom{n_2}{x_2}\ldots\binom{n_r}{x_r} = \binom{n_1+n_2+\ldots+n_r}{p}$. *(Generalised Vandermonde's identity)*

Lemma 2.3.6 *Suppose $p \geq 4$ is an integer such that $p = kd$ for some non-trivial divisors $k, d \in \mathbb{Z}^+$. Then, for $0 \leq i, j \leq d$,*

$$\Phi(p,d,[i,j]) = \sum_{s=i}^{d}\sum_{t=j}^{d}\left[(-1)^{(s-i)+(t-j)}\binom{2d}{s,t,2d-(s+t)}\binom{(2d-(s+t))k}{(d-t)k}\binom{s}{i}\binom{t}{j}\right].$$

Proof: Let μ, λ be any two integers such that $i \leq \mu \leq d$ and $j \leq \lambda \leq d$. We proceed using a double counting method. Suppose $\alpha := \binom{k}{\bar{x}_1}\binom{k}{\bar{x}_2}\ldots\binom{k}{\bar{x}_{2d}}$, where $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2d})$ is an element of $[\mu, \lambda]$. We shall show that each α contributes the same count to both sides of the equality.

Case 1. $\mu = i$ and $\lambda = j$.

On the left side, α is counted exactly once. The expression $\binom{2d}{s,t,2d-(s+t)}\binom{k}{0}^s\binom{k}{k}^t\binom{(2d-(s+t))k}{(d-t)k}$ represents choosing s and t groups from all $2d$ groups of k elements to select 0 and k elements, respectively, from each group, after which $(d-t)k$ elements are selected from the remaining $(2d-(s+t))k$ elements to form a total of $p = dk$ selected elements.

Thus, on the right, α is counted exactly once in the first term $(-1)^{(i-i)+(j-j)}\binom{2d}{i,j,2d-(i+j)}$ $\binom{(2d-(i+j))k}{(d-j)k}\binom{i}{i}\binom{j}{j} = \binom{2d}{i,j,2d-(i+j)}\binom{k}{0}^i\binom{k}{k}^j\binom{(2d-(i+j))k}{(d-j)k}$ and contributes a zero count in the subsequent terms $\binom{2d}{s,t,2d-(s+t)}\binom{(2d-(s+t))k}{(d-t)k}$ if $s > i$ or $t > j$. Thus, α is counted once on each side.

By definition of α, α is counted by the term, $\Phi(p,d,[i,j])$, on the left if and only if $[\mu,\lambda]=[i,j]$. Therefore, α has a zero count on the left side for the following three cases. It suffices to show that α contributes to a count of zero on the right in each of the following cases as well.

Case 2. $\mu = i$ and $\lambda > j$.

The λ groups of k in α could be assembled with j groups of k from $\binom{2d}{i,j,2d-(i+j)}$ and $\lambda - j$ groups of k that are possible from $\binom{(2d-(i+j))k}{(d-j)k}$, i.e. α is counted $\binom{\lambda}{j}$ times in $\binom{2d}{i,j,2d-(i+j)}\binom{(2d-(i+j))k}{(d-j)k}$. So, on the right, α is counted

$$\begin{aligned}
\binom{\lambda}{j} &\text{ times in } \binom{2d}{i,j,2d-(i+j)}\binom{(2d-(i+j))k}{(d-j)k},\\
\binom{\lambda}{j+1} &\text{ times in } \binom{2d}{i,j+1,2d-(i+j+1)}\binom{(2d-(i+j+1))k}{(d-(j+1))k},\\
&\vdots\\
\binom{\lambda}{\lambda} &\text{ times in } \binom{2d}{i,\lambda,2d-(i+\lambda)}\binom{(2d-(i+\lambda))k}{(d-\lambda)k}.
\end{aligned}$$

and none in the subsequent terms $\binom{2d}{s,t,2d-(s+t)}\binom{(2d-(s+t))k}{(d-t)k}$ if $s > i$ or $t > \lambda$. So, α has a total count of $\sum_{s=i}^{i}\sum_{t=j}^{\lambda}[(-1)^{(s-i)+(t-j)}\binom{\lambda}{t}\binom{s}{i}\binom{t}{j}] = (-1)^{(i-i)}\binom{i}{i}\sum_{t=j}^{\lambda}(-1)^{(t-j)}\binom{\lambda}{t}\binom{t}{j}$ $= \sum_{t=j}^{\lambda}(-1)^{(t-j)}\binom{\lambda}{j}\binom{\lambda-j}{t-j} = \binom{\lambda}{j}\sum_{t=j}^{\lambda}(-1)^{(t-j)}\binom{\lambda-j}{t-j} = \binom{\lambda}{j}(0) = 0$, where Lemmas 2.3.5(a) and 2.3.5(b) were invoked in the second and fourth equalities respectively. Thus, α has a zero count on each side.

Case 3. $\mu > i$ and $\lambda = j$.

Similar to Case 2.

Case 4. $\mu > i$ and $\lambda > j$.

On the right, α is counted $\binom{\mu}{s}\binom{\lambda}{t}$ times in the term $\binom{2d}{s,t,2d-(s+t)}\binom{(2d-(s+t))k}{(d-t)k}$, $i \le s \le \mu$ and $j \le t \le \lambda$ and 0 times if $\mu < s \le d$ or $\lambda < t \le d$. In other words, on the right, α is counted

$$\begin{aligned}
&\sum_{s=i}^{\mu}\sum_{t=j}^{\lambda}[(-1)^{(s-i)+(t-j)}\binom{\mu}{s}\binom{\lambda}{t}\binom{s}{i}\binom{t}{j}\\
&=\sum_{s=i}^{\mu}\left\{(-1)^{(s-i)}\binom{\mu}{s}\binom{s}{i}\sum_{t=j}^{\lambda}[(-1)^{(t-j)}\binom{\lambda}{t}\binom{t}{j}]\right\}\\
&=\sum_{s=i}^{\mu}\left\{(-1)^{(s-i)}\binom{\mu}{s}\binom{s}{i}\sum_{t=j}^{\lambda}[(-1)^{(t-j)}\binom{\lambda}{j}\binom{\lambda-j}{t-j}]\right\}\\
&=\sum_{s=i}^{\mu}\left\{(-1)^{(s-i)}\binom{\mu}{s}\binom{s}{i}\binom{\lambda}{j}\sum_{t=j}^{\lambda}[(-1)^{(t-j)}\binom{\lambda-j}{t-j}]\right\}\\
&=\sum_{s=i}^{\mu}\left\{(-1)^{(s-i)}\binom{\mu}{s}\binom{s}{i}\binom{\lambda}{j}(0)\right\}\\
&=0.
\end{aligned}$$

times, where Lemmas 2.3.5(a) and 2.3.5(b) were invoked in the second and fourth equalities above respectively. Thus, α contributes a count of zero on each side.

$\square$

Corollary 2.3.7 *Suppose $p \geq 4$ is an integer such that $p = kd$ for some non-trivial divisors $k, d \in \mathbb{Z}^+$. Then,*

(i) $\Phi^*(p,d) = \sum_{s=0}^{d}\sum_{t=0}^{d}[(-1)^{(s+t)}\binom{2d}{s,t,2d-(s+t)}\binom{(2d-(s+t))k}{(d-t)k}]$.

(ii) $\binom{2p}{p} = \sum_{i=0}^{d}\sum_{j=0}^{d}\sum_{s=i}^{d}\sum_{t=j}^{d}[(-1)^{(s-i)+(t-j)}\binom{2d}{s,t,2d-(s+t)}\binom{(2d-(s+t))k}{(d-t)k}\binom{s}{i}\binom{t}{j}]$.

(iii) $\Phi(p,d,[i,j]) = \Phi(p,d,[j,i])$ *for* $0 \leq i, j \leq d$.

Proof:

(i) This follows from the fact that $\Phi^*(p,d) = \Phi(p,d,[0,0])$.

(ii) By generalised Vandermonde's identity, $\binom{2p}{p} = \sum_{i=0}^{d}\sum_{j=0}^{d}\Phi(p,d,[i,j])$.

(iii) Since $\binom{2d}{s,t,2d-(s+t)} = \binom{2d}{t,s,2d-(s+t)}$ and $\binom{(2d-(s+t))k}{(d-t)k} = \binom{(2d-(s+t))k}{(d-s)k}$, it follows that
$\Phi(p,d,[j,i]) = \sum_{s=j}^{d}\sum_{t=i}^{d}[(-1)^{(s-j)+(t-i)}\binom{2d}{s,t,2d-(s+t)}\binom{(2d-(s+t))k}{(d-t)k}\binom{s}{j}\binom{t}{i}]$
$= \sum_{t=i}^{d}\sum_{s=j}^{d}[(-1)^{(t-i)+(s-j)}\binom{2d}{t,s,2d-(s+t)}\binom{(2d-(s+t))k}{(d-s)k}\binom{t}{i}\binom{s}{j}] = \Phi(p,d,[i,j])$.

$\square$

Now, we are ready to construct an orientation F of $K(p,p,q)$, which resembles the definition of $\Phi^*(p,d)$ (see (2.2)). We divide each of V_1 and V_2 into d groups of size k. Then, orientate F such that for all $1 \leq i \leq q$, $|O(3_i)| = p$ and $O(3_i)$ contains some but not all vertices of each group. This distinctive property will aid in ensuring $d(F) = 2$.

Proposition 2.3.8 *Suppose $p \geq 4$ is an integer such that $p = kd$ for some non-trivial divisors $k, d \in \mathbb{Z}^+$. Then, $\bar{d}(K(p,p,q)) = 2$, if $2k+2 \leq q \leq \max\limits_{d}\{\Phi^*(p,d)\} + 2$, where the maximum is taken over all positive divisors d of p satisfying $1 < d < p$.*

Proof: Partition $V_1 \cup V_2$ into $X_1, X_2, \ldots, X_{2d}$ where

$$X_s = \{1_j |\ j \equiv s\ (mod\ d)\}, \text{ and}$$
$$X_{d+s} = \{2_{(s-1)k+1}, 2_{(s-1)k+2}, \ldots, 2_{(s-1)k+k}\},$$

for $s = 1, 2, \ldots, d$. Observe that $|X_r| = k$ for all $r = 1, 2, \ldots, 2d$. First, we define an orientation F for $K(p,p,2k+2)$ as follows.

(I) $(V_2 - X_{d+s}) \to X_s \to X_{d+s} \to (V_1 - X_s)$, for $s = 1, 2, \ldots, d$.

(II) $V_1 \to 3_{2k+1} \to V_2 \to 3_{2k+2} \to V_1$.

(III) For $t = 1, 2, \ldots, k$,

$$(a)\ \{2_k, 2_{2k}, \ldots, 2_{dk}\} \cup (V_1 - \{1_{(t-1)d+1}, 1_{(t-1)d+2}, \ldots, 1_{(t-1)d+d}\}) \to 3_t \to$$
$$\{1_{(t-1)d+1}, 1_{(t-1)d+2}, \ldots, 1_{(t-1)d+d}\} \cup (V_2 - \{2_k, 2_{2k}, \ldots, 2_{dk}\}), \text{ and}$$
$$(b)\ \{1_1, 1_2, \ldots, 1_d\} \cup (V_2 - \{2_t, 2_{t+k}, \ldots, 2_{t+(d-1)k}\}) \to 3_{t+k} \to$$
$$\{2_t, 2_{t+k}, \ldots, 2_{t+(d-1)k}\} \cup (V_1 - \{1_1, 1_2, \ldots, 1_d\}).$$

Now, consider the case where $q > p+2$. Let $x_i = |O(3_j) \cap X_i|$ for some j, where $2k+2 < j \leq q$, and $i = 1, 2, \ldots, 2d$. So, for each solution $(x_1, x_2, \ldots, x_{2d})^*$ of (2.2), there are $\binom{k}{x_1}\binom{k}{x_2}\ldots\binom{k}{x_{2d}}$ ways to choose p vertices (as the outset of a vertex

3_j), where x_i vertices are selected from the set X_i, satisfying $1 \leq x_i \leq k-1$, for $i = 1, 2, \ldots, 2d$, and $x_1 + x_2 + \ldots + x_{2d} = p$. Summing over all possible solutions $(x_1, x_2, \ldots, x_{2d})^*$, there is a total of $\Phi^*(p, d) := \sum_{(x_1, x_2, \ldots, x_{2d})^*} \binom{k}{x_1}\binom{k}{x_2}\ldots\binom{k}{x_{2d}}$ of such combinations of p vertices of $V_1 \cup V_2$. Denote this set of combinations as Ψ.

Note that the $2k$ outsets of $3_1, 3_2, \ldots, 3_{2k}$ from (III) are elements of Ψ. That leaves $|\Psi| - 2k = \Phi^*(p, d) - 2k$ combinations of p vertices of $V_1 \cup V_2$. Note however that $O(3_{2k+1})$ and $O(3_{3k+2})$ from (II) are not elements of Ψ. Hence, for $2k + 2 < j \leq q \leq \max_d\{\Phi^*(p, d)\} + 2$, we extend the definition of the above orientation so that the outset of vertices $3_{2k+3}, 3_{2k+4}, \ldots, 3_q$ are these remaining elements of Ψ. (See Figure 2.3.1 for F when $d = 3$, and $k = 2$.)

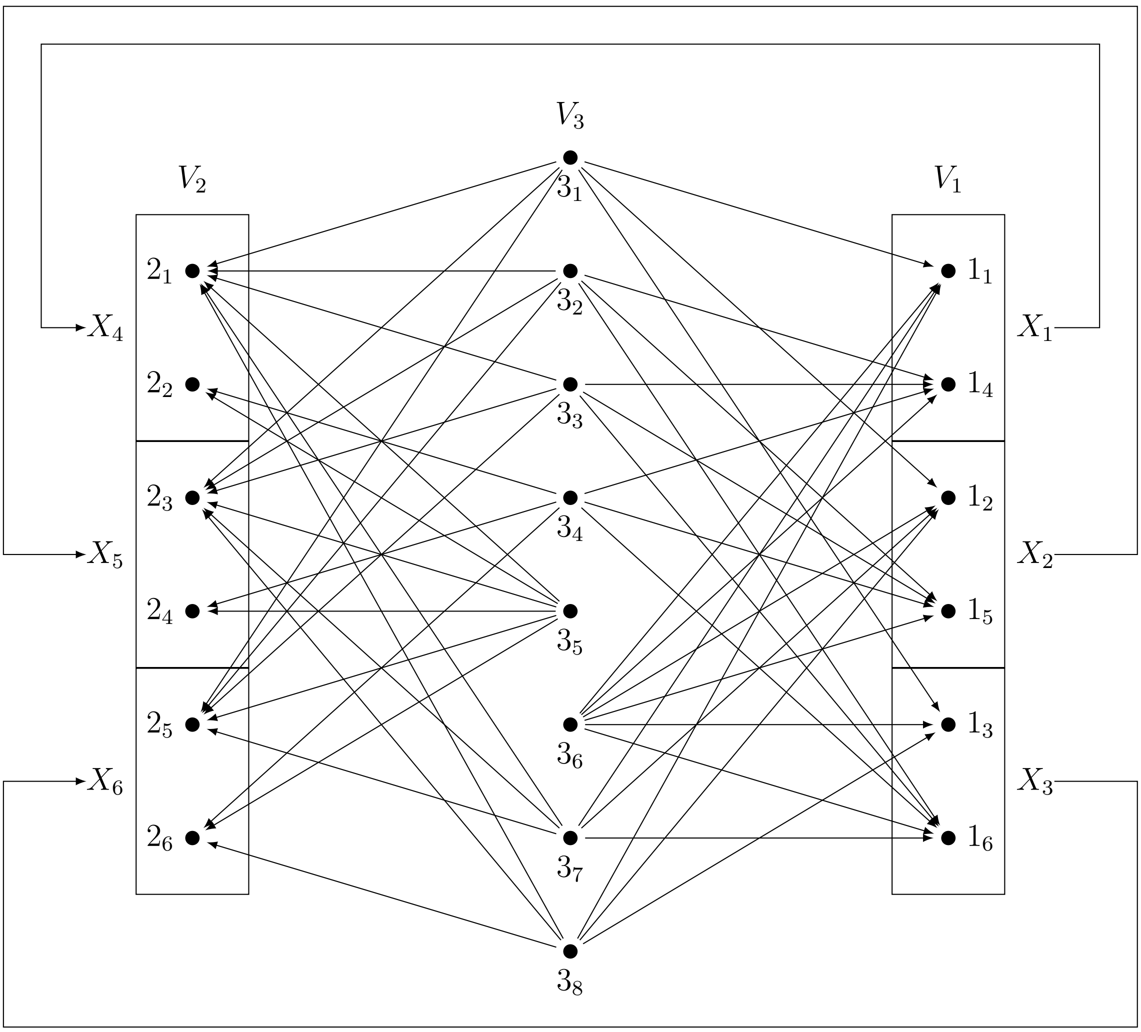

Figure 2.3.1: Orientation F for $d = 3$, and $k = 2$.
For clarity, only the arcs from (1) V_1 to V_2 and (2) V_3 to V_1 and V_2 are shown.

Claim: For all $u, v \in V(K(p,p,q))$, $d_F(u,v) \le 2$.

Case 1. $u = 1_a, v = 1_b$, $a \neq b$.

Since $1 \le a, b \le p = kd$, let $a = (\alpha_1 - 1)d + \alpha_2$ and $b = (\beta_1 - 1)d + \beta_2$ for some α_i, β_i, $i = 1, 2$, satisfying $1 \le \alpha_1, \beta_1 \le k$ and $1 \le \alpha_2, \beta_2 \le d$. If $\alpha_2 = \beta_2$, then $\alpha_1 \neq \beta_1$. Note that 1_a and 1_b are in the same X_i and by (III)(a) of orientation F, $1_a \to 3_{\beta_1} \to 1_b$. If $\alpha_2 \neq \beta_2$, then 1_a and 1_b are in different X_i's and by (I) of orientation F, $1_a \to X_{d+\alpha_2} \to 1_b$.

Case 2. $u = 2_a, v = 2_b$, $a \neq b$.

Since $1 \le a, b \le p = kd$, let $a = (\alpha_1 - 1)k + \alpha_2$ and $b = (\beta_1 - 1)k + \beta_2$ for some α_i, β_i, $i = 1, 2$, satisfying $1 \le \alpha_1, \beta_1 \le d$ and $1 \le \alpha_2, \beta_2 \le k$. If $\alpha_1 = \beta_1$, then $\alpha_2 \neq \beta_2$. Note that 2_a and 2_b are in the same X_i and by (III)(b) of orientation F, $2_a \to 3_{\beta_2+k} \to 2_b$. If $\alpha_1 \neq \beta_1$, then 2_a and 2_b are in different X_i's and by (I) of orientation F, $2_a \to X_{\beta_1} \to 2_b$.

Case 3. $u = 1_a, v = 2_b$.

By (II), $1_a \to 3_{2k+1} \to 2_b$.

Case 4. $u = 2_a, v = 1_b$.

By (II), $2_a \to 3_{2k+2} \to 1_b$.

Case 5. $u = 1_a, v = 3_b$.

Subcase 5a. $b = 2k + 1$.

By (II), $V_1 \to 3_{2k+1}$.

Subcase 5b. $b \neq 2k + 1$.

Suppose $1_a \in X_{i^*}$ for some $i^* = 1, 2, \ldots, d$. Then, $1_a \to X_{d+i^*}$ by (I). Since for each 3_b, $I(3_b) \cap X_{d+i} \neq \emptyset$ for each $i = 1, 2, \ldots, d$, by (II) and (III), let $w \in$

$I(3_b) \cap X_{d+i^*}$. It follows that $1_a \to w \to 3_b$.

Case 6. $u = 2_a, v = 3_b$.

Subcase 6a. $b = 2k + 2$.

By (II), $V_2 \to 3_{2k+2}$.

Subcase 6b. $b \neq 2k + 2$.

Suppose $2_a \in X_{d+i^*}$ for some $i^* = 1, 2, \ldots, d$. Then, $2_a \to V_1 - X_{i^*}$ by (I). Since for each 3_b, $I(3_b) \cap X_i \neq \emptyset$ for each $i = 1, 2, \ldots, d$, by (II) and (III), let $w \in I(3_b) \cap X_j$ for some $j = 1, 2, \ldots, d$ and $j \neq i^*$. It follows that $2_a \to w \to 3_b$.

Case 7. $u = 3_a, v = 1_b$.

Subcase 7a. $a = 2k + 2$.

By (II), $3_{2k+2} \to V_1$.

Subcase 7b. $a \neq 2k + 2$.

Suppose $1_b \in X_{i^*}$ for some $i^* = 1, 2, \ldots, d$. Then, $X_{d+j} \to 1_b$ for all $j = 1, 2, \ldots, d$ and $j \neq i^*$ by (I). Since for each 3_a, $O(3_a) \cap X_{d+i} \neq \emptyset$ for each $i = 1, 2, \ldots, d$, by (II) and (III), let $w \in O(3_a) \cap X_{d+j}$ for some $j = 1, 2, \ldots, d$, and $j \neq i^*$. It follows that $3_a \to w \to 1_b$.

Case 8. $u = 3_a, v = 2_b$.

Subcase 8a. $a = 2k + 1$.

By (II), $3_{2k+1} \to V_2$.

Subcase 8b. $a \neq 2k + 1$.

Suppose $2_b \in X_{d+i^*}$ for some $i^* = 1, 2, \ldots, d$. Then, $X_{i^*} \to 2_b$. Since for each 3_a, $O(3_a) \cap X_i \neq \emptyset$ for each $i = 1, 2, \ldots, d$, by (II) and (III), let $w \in O(3_a) \cap X_{i^*}$. It follows that $3_a \to w \to 2_b$.

Case 9. $u = 3_a, v = 3_b$.

Subcase 9a. $a \neq 2k+1, 2k+2$ and $b \neq 2k+1, 2k+2$.

Observe from (III) that $|O(3_x) \cap (V_1 \cup V_2)| = p$ for $x = a, b$. Furthermore, $O(3_a) \cap (V_1 \cup V_2) \nsubseteq O(3_b) \cap (V_1 \cup V_2)$ if $b \neq a$. Thus, there exists a vertex $w \in V_1 \cup V_2$ such that $3_a \to w \to 3_b$.

Subcase 9b. $a = 2k+1$ and $b \neq 2k+1, 2k+2$.

$3_{2k+1} \to V_2$ by (II), and $I(3_b) \cap X_{d+i} \neq \emptyset$ for every $i = 1, 2, \ldots, d$, imply the existence of $w \in I(3_b) \cap V_2$. Hence, $3_a \to w \to 3_b$.

Subcase 9c. $a = 2k+2$ and $b \neq 2k+1, 2k+2$.

$3_{2k+2} \to V_1$ by (II), and $I(3_b) \cap X_i \neq \emptyset$ for every $i = 1, 2, \ldots, d$, imply the existence of $w \in I(3_b) \cap V_1$. Hence, $3_a \to w \to 3_b$.

Subcase 9d. $a \neq 2k+1, 2k+2$ and $b = 2k+1$.

$V_1 \to 3_{2k+1}$ by (II), and $O(3_a) \cap X_i \neq \emptyset$ for every $i = 1, 2, \ldots, d$, imply the existence of $w \in O(3_a) \cap V_1$. Hence, $3_a \to w \to 3_b$.

Subcase 9e. $a \neq 2k+1, 2k+2$ and $b = 2k+2$.

$V_2 \to 3_{2k+2}$ by (II), and $O(3_a) \cap X_{d+i} \neq \emptyset$ for every $i = 1, 2, \ldots, d$, imply the existence of $w \in O(3_a) \cap V_2$. Hence, $3_a \to w \to 3_b$.

Subcase 9f. $a = 2k+1$ and $b = 2k+2$.

By (II), $3_{2k+1} \to V_2 \to 3_{2k+2}$.

Subcase 9g. $a = 2k+2$ and $b = 2k+1$.

By (II), $3_{2k+2} \to V_1 \to 3_{2k+1}$.

$\square$

Example 2.3.9 *If* $8 \leq q \leq 488$, *then* $\bar{d}(K(6,6,q)) = 2$.

Proof: Observe that the only non-trivial divisors of 6 are 2 and 3. By Corollary 2.3.7(i), $\Phi^*(6,2) = 486$ and $\Phi^*(6,3) = 64$. Hence, by Proposition 2.3.8, if $8 \leq q \leq max\{486, 64\} + 2$, then $\bar{d}(K(6,6,q)) = 2$.

Alternatively, we may verify the computation of $\Phi^*(6,2)$ and $\Phi^*(6,3)$ as follows.

Case 1. $d = 2, k = 3$.

Any solution (x_1, x_2, x_3, x_4) satisfying (2.2) is a permutation of $(1,1,2,2)$. Thus,
$\sum\limits_{(x_1,x_2,\ldots,x_{2d})} \binom{k}{x_1}\binom{k}{x_2}\cdots\binom{k}{x_{2d}} = \binom{4}{2}\binom{3}{1}^2\binom{3}{2}^2 = 486.$

Case 2. $d = 3, k = 2$.

The only solution of (2.2) is $(x_1, x_2, x_3, x_4, x_5, x_6) = (1,1,1,1,1,1)$. Thus,
$\sum\limits_{(x_1,x_2,\ldots,x_{2d})} \binom{k}{x_1}\binom{k}{x_2}\cdots\binom{k}{x_{2d}} = \binom{2}{1}^6 = 64.$

$\square$

Discussion 2.3.10

Since p *may have different factorisations, the natural question to ask is which non-trivial divisor(s)* d *of* p *gives the best bound. Verification, using Maple [55], for all non-trivial divisors* d *of each composite integer* $p \leq 100$ *shows that* $\max\limits_d\{\Phi^*(p,d)\} = \Phi^*(p, d_0)$ *with* d_0 *being the smallest non-trivial divisor of each* p. *Therefore, if* p *is even, we define*

$$\begin{aligned}
\Phi_{even}(p) &:= \Phi^*(p,2) \\
&= \sum_{s=0}^{2}\sum_{t=0}^{2}\left[(-1)^{(s+t)}\binom{4}{s,t,4-(s+t)}\binom{(4-(s+t))\frac{p}{2}}{(2-t)\frac{p}{2}}\right] \\
&= \binom{2p}{p} - 8\binom{\frac{3p}{2}}{p} + 12\binom{p}{\frac{p}{2}} - 6.
\end{aligned}$$

Furthermore, we wish to extend Definition 2.3.2 and Proposition 2.3.8 for prime numbers and $d = 2$ *seems to be the best candidate. Hence, we have the following* *generalisation, $\Phi_{odd}(p)$, for odd integers $p \geq 5$, which also provides a better bound than $\Phi(p, d_0)$ in cases where p is odd and composite.*

Definition 2.3.11 *Suppose $p \geq 5$ is an odd integer. Denote a solution $(x_1, x_2, x_3, x_4)^{**}$ if x_1, x_2, x_3, x_4 satisfies*

$$\begin{aligned} & x_1 + x_2 + x_3 + x_4 = p, \\ & 1 \leq x_i \leq \lfloor \frac{p}{2} \rfloor, \text{ for } i = 1, 2, \text{ and} \\ & 1 \leq x_i \leq \lfloor \frac{p}{2} \rfloor - 1, \text{ for } i = 3, 4. \end{aligned} \tag{2.3}$$

Define $\Phi_{odd}(p) := \sum_{(x_1,x_2,x_3,x_4)^{**}} \binom{\lfloor \frac{p}{2} \rfloor + 1}{x_1} \binom{\lfloor \frac{p}{2} \rfloor + 1}{x_2} \binom{\lfloor \frac{p}{2} \rfloor}{x_3} \binom{\lfloor \frac{p}{2} \rfloor}{x_4}$.

Definition 2.3.12

For each $i = 1, 2, 3, 4$, denote the upper bound of x_i in (2.3) to be a_i, where $a_1 = a_2 = \lfloor \frac{p}{2} \rfloor$ and $a_3 = a_4 = \lfloor \frac{p}{2} \rfloor - 1$. Then, for any non-negative integers i, j, define $[i, j]$ to be the set of solutions (x_1, x_2, x_3, x_4) satisfying

$$\begin{aligned} & x_1 + x_2 + x_3 + x_4 = p, \\ & x_{s_m} = 0, \text{ for } m = 1, 2, \ldots, i, \text{ where } \{s_1, s_2, \ldots, s_i\} \subseteq \{1, 2, 3, 4\}, \\ & x_{t_n} = a_{t_n} + 1, \text{ for } n = 1, 2, \ldots, j, \text{ where } \{t_1, t_2, \ldots, t_j\} \subseteq \{1, 2, 3, 4\}, \text{ and} \\ & 1 \leq x_r \leq a_r, \text{ for } r \in \{1, 2, 3, 4\} - (\{s_1, s_2, \ldots, s_i\} \cup \{t_1, t_2, \ldots, t_j\}). \end{aligned}$$

Furthermore, we denote $\Phi_{odd}(p, [i, j]) := \sum_{(x_1,x_2,x_3,x_4) \in [i,j]} \binom{\lfloor \frac{p}{2} \rfloor + 1}{x_1} \binom{\lfloor \frac{p}{2} \rfloor + 1}{x_2} \binom{\lfloor \frac{p}{2} \rfloor}{x_3} \binom{\lfloor \frac{p}{2} \rfloor}{x_4}$.

We may find an expression of $\Phi_{odd}(p)$ by exhausting all possible values of i and j in $\Phi_{odd}(p, [i, j])$.

Lemma 2.3.13 *If $p \geq 5$ is an odd integer, then* $\Phi_{odd}(p) = \binom{2p}{p} - 4\binom{3x+2}{x+1} - 4\binom{3x+1}{x} + 2\binom{2x+2}{x+1} + 8\binom{2x+1}{x} + 2\binom{2x}{x} - 4$, *where* $x = \lfloor \frac{p}{2} \rfloor$.

Proof: By generalised Vandermonde's identity,

$$\binom{2p}{p} = \sum_{x_1+x_2+x_3+x_4=p} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} \tag{2.4}$$
$$= \sum_{i=0}^{2}\sum_{j=0}^{2} \Phi_{odd}(p,[i,j]).$$

By definition, $\Phi_{odd}(p) = \Phi_{odd}(p,[0,0])$. For $i \geq 1$ or $j \geq 1$, we shall count and subtract the contribution $\Phi_{odd}(p,[i,j])$ of each case from (2.4).

Claim 1: $\Phi_{odd}(p,[2,2]) = 4$.

Case 1.1. $\{x_1, x_2\} = \{x+1, 0\}$ and $\{x_3, x_4\} = \{x, 0\}$.

Then, $\sum_{\substack{x_1+x_2+x_3+x_4=p \\ Case\ 1.1}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{2}{1}\binom{2}{1}\binom{x+1}{x+1}\binom{x+1}{0}\binom{x}{x}\binom{x}{0} = 4.$

Case 1.2. $x_1 = x_2 = 0$ and $x_3 = x_4 = x$.

Then, $x_1 + x_2 + x_3 + x_4 = 2x < p$. Thus, this case is not possible.

Case 1.3. $x_1 = x_2 = x+1$ and $x_3 = x_4 = 0$.

Then, $x_1 + x_2 + x_3 + x_4 = 2x + 2 > p$. Thus, this case is not possible.

Claim 2: $\Phi_{odd}(p,[2,1]) = 2x + 2$.

Case 2.1. Exactly one of x_1, x_2 equals $x+1$ and $x_3 = x_4 = 0$.

Since $x_1+x_2+x_3+x_4 = p$, it follows that either $x_1 = x+1$, $x_2 = x$, $x_3 = x_4 = 0$ or $x_1 = x$, $x_2 = x+1$, $x_3 = x_4 = 0$. Thus, $\sum_{\substack{x_1+x_2+x_3+x_4=p \\ Case\ 2.1}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{2}{1}\binom{x+1}{x+1}\binom{x+1}{x}\binom{x}{0}\binom{x}{0} = 2x + 2.$

Case 2.2. $x_1 = x_2 = 0$ and exactly one of x_3, x_4 equals x.

Similar to Case 1.2, this case is not possible.

Case 2.3. Exactly one of x_1, x_2 is zero, exactly one of x_3, x_4 is zero and exactly one x_i equals $a_i + 1$.

W.l.o.g., suppose $x_1 = x_3 = 0$. Since $x_1 + x_2 + x_3 + x_4 = p$, it follows that $x_2 + x_4 = p$, which implies $x_2 = x + 1$ and $x_4 = x$. This contradicts the condition of having exactly one x_i equals $a_i + 1$. Thus, this case is not possible.

Claim 3: $\Phi_{odd}(p, [2, 0]) = 0$.

This means exactly two of x_i's, $i = 1, 2, 3, 4$, equals zero and none of the x_i's equals $a_i + 1$. This is not possible since $x_1 + x_2 + x_3 + x_4 = p$.

Claim 4: $\Phi_{odd}(p, [1, 2]) = 2x + 2$.

Since $x_1 + x_2 + x_3 + x_4 = p$, thus, we have the following.

Case 4. $x_3 = x_4 = x$ and either $x_1 = 1, x_2 = 0$ or $x_1 = 0, x_2 = 1$.

$$\sum_{\substack{x_1+x_2+x_3+x_4=p \\ \textit{Case } 4}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{2}{1}\binom{x+1}{1}\binom{x+1}{0}\binom{x}{x}\binom{x}{x} = 2x + 2.$$

Claim 5: $\Phi_{odd}(p, [1, 1]) = 2\binom{2x+2}{x+1} + 8\binom{2x+1}{x} + 2\binom{2x}{x} - 8x - 24$.

Case 5.1. $1 \le x_3, x_4 \le x - 1$ and either $x_1 = 0, x_2 = x + 1$ or $x_1 = x + 1, x_2 = 0$.

W.l.o.g., assume $x_1 = 0, x_2 = x + 1$. Then, $x_1 + x_2 + x_3 + x_4 = p$ implies that $x_3 + x_4 = x$. For $1 \le x_3, x_4 \le x - 1$, $\sum_{x_3+x_4=x} \binom{x}{x_3}\binom{x}{x_4} = \binom{2x}{x} - \binom{x}{0}\binom{x}{x} - \binom{x}{x}\binom{x}{0} = \binom{2x}{x} - 2$ by generalised Vandermonde's identity. A similar expression is obtained if $x_1 = x+1, x_2 = 0$. Thus, $\sum_{\substack{x_1+x_2+x_3+x_4=p \\ \textit{Case } 5.1}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{2}{1}[\binom{2x}{x} - 2] = 2\binom{2x}{x} - 4$.

Case 5.2. $1 \le x_1, x_2 \le x$ and either $x_3 = 0, x_4 = x$ or $x_3 = x, x_4 = 0$.

W.l.o.g., assume $x_3 = 0, x_4 = x$. Then, $x_1 + x_2 + x_3 + x_4 = p$ implies that $x_1 + x_2 = x + 1$. For $1 \le x_1, x_2 \le x$, $\sum_{x_1+x_2=x+1} \binom{x+1}{x_1}\binom{x+1}{x_2} = \binom{2x+2}{x+1} - \binom{x+1}{0}\binom{x+1}{x+1} -$

$\binom{x+1}{x+1}\binom{x+1}{0} = \binom{2x+2}{x+1} - 2$ by generalised Vandermonde's identity. A similar expression is obtained if $x_3 = x, x_4 = 0$. Thus, $\sum_{\substack{x_1+x_2+x_3+x_4=p \\ Case\ 5.2}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{2}{1}[\binom{2x+2}{x+1} - 2] = 2\binom{2x+2}{x+1} - 4$.

Case 5.3. Exactly one of x_1, x_2 equals zero and exactly one of x_3, x_4 equals x.

W.l.o.g., assume $x_1 = 0$ and $x_3 = x$. Then, $x_1 + x_2 + x_3 + x_4 = p$ implies that $x_2 + x_4 = x + 1$. For $1 \le x_2 \le x$ and $1 \le x_4 \le x - 1$, $\sum_{x_2+x_4=x+1} \binom{x+1}{x_2}\binom{x}{x_4} = \binom{2x+1}{x+1} - \binom{x+1}{x+1}\binom{x}{0} - \binom{x+1}{1}\binom{x}{x} = \binom{2x+1}{x+1} - x - 2$ by generalised Vandermonde's identity. A similar expression is obtained for the other three subcases. Thus, $\sum_{\substack{x_1+x_2+x_3+x_4=p \\ Case\ 5.3}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{2}{1}^2[\binom{2x+1}{x+1} - x - 2] = 4\binom{2x+1}{x+1} - 4x - 8$.

Case 5.4. Exactly one of x_1, x_2 equals $x + 1$ and exactly one of x_3, x_4 equals zero.

W.l.o.g., assume $x_1 = x + 1$ and $x_3 = 0$. Then, $x_1 + x_2 + x_3 + x_4 = p$ implies that $x_2 + x_4 = x$. For $1 \le x_2 \le x$ and $1 \le x_4 \le x - 1$, $\sum_{x_2+x_4=x} \binom{x+1}{x_2}\binom{x}{x_4} = \binom{2x+1}{x} - \binom{x+1}{0}\binom{x}{x} - \binom{x+1}{x}\binom{x}{0} = \binom{2x+1}{x} - x - 2$ by generalised Vandermonde's identity. A similar expression is obtained for the other three subcases. Thus, $\sum_{\substack{x_1+x_2+x_3+x_4=p \\ Case\ 5.4}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{2}{1}^2[\binom{2x+1}{x} - x - 2] = 4\binom{2x+1}{x} - 4x - 8$.

Summing all Cases 5.1-5.4, we have $\Phi_{odd}(p, [1, 1])$ as claimed.

Claim 6: $\Phi_{odd}(p, [1, 0]) = 2\binom{3x+2}{2x+1} + 2\binom{3x+1}{2x+1} - 2\binom{2x+2}{x+1} - 8\binom{2x+1}{x} - 2\binom{2x}{x} + 2x + 10$.

Case 6.1. Exactly one of x_1, x_2 equals 0.

Subcase 6.1.1. $x_1 = 0$ (and $1 \le x_2 \le x$ and $1 \le x_3, x_4 \le x - 1$).

Then, $x_2 + x_3 + x_4 = p$. Now, we want to count and exclude the following subcases.

Subcase 6.1.1.1. $x_2 = x+1$ and either $x_3 = x$, $x_4 = 0$ or $x_3 = 0$, $x_4 = x$.

$$\sum_{\substack{x_2+x_3+x_4=p \\ \textit{Subcase } 6.1.1.1}} \binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{x+1}{x+1}\binom{x}{x}\binom{x}{0} + \binom{x+1}{x+1}\binom{x}{0}\binom{x}{x} = 2.$$

Subcase 6.1.1.2. $x_2 = 1$ and $x_3 = x_4 = x$.

$$\sum_{\substack{x_2+x_3+x_4=p \\ \textit{Subcase } 6.1.1.2}} \binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{x+1}{1}\binom{x}{x}\binom{x}{x} = x+1.$$

Subcase 6.1.1.3. $x_2 = x+1$ and $1 \le x_3, x_4 \le x-1$.

Then, $x_3 + x_4 = x$. It follows from the generalised Vandermonde's identity that $\sum_{x_3+x_4=x} \binom{x}{x_3}\binom{x}{x_4} = \binom{2x}{x}$. So, $\sum_{\substack{x_2+x_3+x_4=p \\ \textit{Subcase } 6.1.1.3}} \binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \sum_{x_3+x_4=x} \binom{x}{x_3}\binom{x}{x_4} - \binom{x}{x}\binom{x}{0} - \binom{x}{0}\binom{x}{x} = \binom{2x}{x} - 2.$

Subcase 6.1.1.4. $x_3 = x$, $1 \le x_2 \le x$ and $1 \le x_4 \le x-1$.

Then, $x_2 + x_4 = x+1$. It follows from the generalised Vandermonde's identity that $\sum_{x_2+x_4=x+1} \binom{x+1}{x_2}\binom{x}{x_4} = \binom{2x+1}{x+1}$. So, $\sum_{\substack{x_2+x_3+x_4=p \\ \textit{Subcase } 6.1.1.4}} \binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \sum_{x_2+x_4=x+1} \binom{x+1}{x_2}\binom{x}{x_4} - \binom{x+1}{x+1}\binom{x}{0} - \binom{x+1}{1}\binom{x}{x} = \binom{2x+1}{x+1} - x - 2.$

Subcase 6.1.1.5. $x_4 = x$, $1 \le x_2 \le x$ and $1 \le x_3 \le x-1$.

By symmetry to Subcase 6.1.1.4, $\sum_{\substack{x_2+x_3+x_4=p \\ \textit{Subcase } 6.1.1.5}} \binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{2x+1}{x+1} - x - 2.$

Now, by generalised Vandermonde's identity, $\sum_{x_2+x_3+x_4=p} \binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{3x+1}{2x+1}$.

So, $\sum_{\substack{x_1+x_2+x_3+x_4=p \\ \textit{Subcase } 6.1.1}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \sum_{x_2+x_3+x_4=p} \binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} - \sum_{i=1}^{5} \sum_{\substack{x_2+x_3+x_4=p \\ \textit{Subcase } 6.1.1.i}} \binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{3x+1}{2x+1} - 2\binom{2x+1}{x+1} - \binom{2x}{x} + x + 3.$

Subcase 6.1.2. $x_2 = 0$ (and $1 \le x_1 \le x$ and $1 \le x_3, x_4 \le x-1$).

By symmetry to Subcase 6.1.1, $\sum_{\substack{x_1+x_2+x_3+x_4=p \\ \textit{Subcase } 6.1.2}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{3x+1}{2x+1} - 2\binom{2x+1}{x+1} - \binom{2x}{x} + x + 3.$

Hence, $\displaystyle\sum_{\substack{x_1+x_2+x_3+x_4=p\\ \textit{Case } 6.1}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = 2\binom{3x+1}{2x+1} - 4\binom{2x+1}{x+1} - 2\binom{2x}{x} + 2x + 6.$

Case 6.2. Exactly one of x_3, x_4 equals 0.

Subcase 6.2.1. $x_3 = 0$ (with $1 \leq x_1, x_2 \leq x$ and $1 \leq x_4 \leq x-1$).

Then, $x_1 + x_2 + x_4 = p$. Now, we want to count and exclude the following subcases.

Subcase 6.2.1.1. $x_4 = x$ and either $x_1 = x+1$, $x_2 = 0$ or $x_1 = 0$, $x_2 = x+1$.

$$\sum_{\substack{x_1+x_2+x_4=p\\ \textit{Subcase } 6.2.1.1}} \binom{x+1}{x_1}\binom{x}{x_2}\binom{x}{x_4} = \binom{x+1}{x+1}\binom{x+1}{0}\binom{x}{x} + \binom{x+1}{0}\binom{x+1}{x+1}\binom{x}{x} = 2.$$

Subcase 6.2.1.2. $x_4 = 0$ and either $x_1 = x+1$, $x_2 = x$ or $x_1 = x$, $x_2 = x+1$.

$$\sum_{\substack{x_1+x_2+x_4=p\\ \textit{Subcase } 6.2.1.2}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_4} = \binom{x+1}{x+1}\binom{x+1}{x}\binom{x}{0} + \binom{x+1}{x}\binom{x+1}{x+1}\binom{x}{0} = 2x+2.$$

Subcase 6.2.1.3. $x_1 = x+1$, $1 \leq x_2 \leq x$ and $1 \leq x_4 \leq x-1$.

Then, $x_2 + x_4 = x$. It follows from the generalised Vandermonde's identity that $\displaystyle\sum_{x_2+x_4=x} \binom{x+1}{x_2}\binom{x}{x_4} = \binom{2x+1}{x}$. So, $\displaystyle\sum_{\substack{x_1+x_2+x_4=p\\ \textit{Subcase } 6.2.1.3}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_4} = \sum_{x_2+x_4=x} \binom{x+1}{x_2}\binom{x}{x_4} - \binom{x+1}{x}\binom{x}{0} - \binom{x+1}{0}\binom{x}{x} = \binom{2x+1}{x} - x - 2.$

Subcase 6.2.1.4. $x_2 = x+1$, $1 \leq x_1 \leq x$ and $1 \leq x_4 \leq x-1$.

By symmetry to Subcase 6.2.1.3, $\displaystyle\sum_{\substack{x_1+x_2+x_4=p\\ \textit{Subcase } 6.2.1.4}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_4} = \binom{2x+1}{x} - x - 2.$

Subcase 6.2.1.5. $x_4 = x$ and $1 \leq x_1, x_2 \leq x$.

Then, $x_1 + x_2 = x+1$. It follows from the generalised Vandermonde's identity that $\displaystyle\sum_{x_1+x_2=x+1} \binom{x+1}{x_1}\binom{x+1}{x_2} = \binom{2x+2}{x+1}$. So, $\displaystyle\sum_{\substack{x_1+x_2+x_4=p\\ \textit{Subcase } 6.2.1.5}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_4} = \sum_{x_1+x_2=x+1} \binom{x+1}{x_1}\binom{x+1}{x_2} - \binom{x+1}{x+1}\binom{x+1}{0} - \binom{x+1}{0}\binom{x+1}{x+1} = \binom{2x+2}{x+1} - 2.$

Now, by generalised Vandermonde's identity, $\sum\limits_{x_1+x_2+x_4=p} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_4} = \binom{3x+2}{2x+1}$.

So, $\sum\limits_{\substack{x_1+x_2+x_3+x_4=p \\ Subcase\ 6.2.1}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \sum\limits_{x_1+x_2+x_4=p} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_4} - \sum\limits_{i=1}^{5} \sum\limits_{\substack{x_1+x_2+x_4=p \\ Subcase\ 6.2.1.i}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_4} = \binom{3x+2}{2x+1} - \binom{2x+2}{x+1} - 2\binom{2x+1}{x} + 2$.

Subcase 6.2.2. $x_4 = 0$ (with $1 \le x_1, x_2 \le x$ and $1 \le x_3 \le x-1$).

By symmetry to Subcase 6.2.1, $\sum\limits_{\substack{x_1+x_2+x_3+x_4=p \\ Subcase\ 6.2.2}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{3x+2}{2x+1} - \binom{2x+2}{x+1} - 2\binom{2x+1}{x} + 2$.

Hence, $\sum\limits_{\substack{x_1+x_2+x_3+x_4=p \\ Case\ 6.2}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = 2\binom{3x+2}{2x+1} - 2\binom{2x+2}{x+1} - 4\binom{2x+1}{x} + 4$.

So, $\Phi_{odd}(p, [1,0]) = \sum\limits_{i=1}^{2} \sum\limits_{\substack{x_1+x_2+x_3+x_4=p \\ Case\ 6.i}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = 2\binom{3x+2}{2x+1} + 2\binom{3x+1}{2x+1} - 2\binom{2x+2}{x+1} - 8\binom{2x+1}{x} - 2\binom{2x}{x} + 2x + 10$.

Claim 7: $\Phi_{odd}(p, [0,2]) = 0$.

If there are exactly two x_i's such that $x_i = a_i + 1$ and since $x_1 + x_2 + x_3 + x_4 = p$, it either results in Cases 1.1 or 4. i.e. Thus, this case is not possible.

Claim 8: $\Phi_{odd}(p, [0,1]) = 2\binom{3x+2}{x+1} + 2\binom{3x+1}{x} - 2\binom{2x+2}{x+1} - 8\binom{2x+1}{x} - 2\binom{2x}{x} + 2x + 10$.

Case 8.1. Exactly one of x_1, x_2 equals $x+1$.

Subcase 8.1.1. $x_1 = x + 1$ (with $1 \le x_2 \le x$ and $1 \le x_3, x_4 \le x - 1$).

Then, $x_2 + x_3 + x_4 = x$. Now, we want to count and exclude the following subcases.

Subcase 8.1.1.1. $x_2 = 0$ and either $x_3 = x$, $x_4 = 0$ or $x_3 = 0$, $x_4 = x$.

$$\sum_{\substack{x_2+x_3+x_4=x \\ Subcase\ 8.1.1.1}} \binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{x+1}{0}\binom{x}{x}\binom{x}{0} + \binom{x+1}{0}\binom{x}{0}\binom{x}{x} = 2.$$

Subcase 8.1.1.2. $x_2 = x$ and $x_3 = x_4 = 0$.

$$\sum_{\substack{x_2+x_3+x_4=x \\ Subcase\ 8.1.1.2}} \binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{x+1}{x}\binom{x}{0}\binom{x}{0} = x + 1.$$

Subcase 8.1.1.3. $x_2 = 0$ and $1 \le x_3, x_4 \le x-1$.

Then, $x_3 + x_4 = x$. It follows from the generalised Vandermonde's identity that $\sum\limits_{x_3+x_4=x} \binom{x}{x_3}\binom{x}{x_4} = \binom{2x}{x}$. So, $\sum\limits_{\substack{x_2+x_3+x_4=x \\ \textit{Subcase } 8.1.1.3}} \binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \sum\limits_{x_3+x_4=x} \binom{x}{x_3}\binom{x}{x_4} - \binom{x}{x}\binom{x}{0} - \binom{x}{0}\binom{x}{x} = \binom{2x}{x} - 2$.

Subcase 8.1.1.4. $x_3 = 0$, $1 \le x_2 \le x$ and $1 \le x_4 \le x-1$.

Then, $x_2 + x_4 = x$. It follows from the generalised Vandermonde's identity that $\sum\limits_{x_2+x_4=x} \binom{x+1}{x_2}\binom{x}{x_4} = \binom{2x+1}{x}$. So, $\sum\limits_{\substack{x_2+x_3+x_4=x \\ \textit{Subcase } 8.1.1.4}} \binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \sum\limits_{x_2+x_4=x} \binom{x+1}{x_2}\binom{x}{x_4} - \binom{x+1}{x}\binom{x}{0} - \binom{x+1}{0}\binom{x}{x} = \binom{2x+1}{x} - x - 2$.

Subcase 8.1.1.5. $x_4 = 0$, $1 \le x_2 \le x$ and $1 \le x_3 \le x-1$.

By symmetry to Subcase 8.1.1.4, $\sum\limits_{\substack{x_2+x_3+x_4=x \\ \textit{Subcase } 8.1.1.5}} \binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{2x+1}{x} - x - 2$.

Now, by generalised Vandermonde's identity, $\sum\limits_{x_2+x_3+x_4=x} \binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{3x+1}{x}$.

So, $\sum\limits_{\substack{x_1+x_2+x_3+x_4=p \\ \textit{Subcase } 8.1.1}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \sum\limits_{x_2+x_3+x_4=x} \binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} - \sum\limits_{i=1}^{5} \sum\limits_{\substack{x_2+x_3+x_4=x \\ \textit{Subcase } 8.1.1.i}} \binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{3x+1}{x} - 2\binom{2x+1}{x} - \binom{2x}{x} + x + 3$.

Subcase 8.1.2. $x_2 = x+1$ (and $1 \le x_1 \le x$ and $1 \le x_3, x_4 \le x-1$).

By symmetry to Subcase 8.1.1, $\sum\limits_{\substack{x_1+x_2+x_3+x_4=p \\ \textit{Subcase } 8.1.2}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{3x+1}{x} - 2\binom{2x+1}{x} - \binom{2x}{x} + x + 3$.

Hence, $\sum\limits_{\substack{x_1+x_2+x_3+x_4=p \\ \textit{Case } 8.1}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = 2\binom{3x+1}{x} - 4\binom{2x+1}{x} - 2\binom{2x}{x} + 2x + 6$.

Case 8.2. Exactly one of x_3, x_4 equals x.

Subcase 8.2.1. $x_3 = x$ (and $1 \le x_1, x_2 \le x$ and $1 \le x_4 \le x-1$).

Then, $x_1 + x_2 + x_4 = x + 1$. Now, we want to count and exclude the following subcases.

Subcase 8.2.1.1. $x_4 = 0$ and either $x_1 = x+1$, $x_2 = 0$ or $x_1 = 0$, $x_2 = x+1$.

$$\sum_{\substack{x_1+x_2+x_4=x+1\\ \textit{Subcase } 8.2.1.1}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_4} = \binom{x+1}{x+1}\binom{x+1}{0}\binom{x}{0} + \binom{x+1}{0}\binom{x+1}{x+1}\binom{x}{0} = 2.$$

Subcase 8.2.1.2. $x_4 = x$ and either $x_1 = 0$, $x_2 = 1$ or $x_1 = 1$, $x_2 = 0$.

$$\sum_{\substack{x_1+x_2+x_4=x+1\\ \textit{Subcase } 8.2.1.2}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_4} = \binom{x+1}{0}\binom{x+1}{1}\binom{x}{x} + \binom{x+1}{1}\binom{x+1}{0}\binom{x}{x} = 2x+2.$$

Subcase 8.2.1.3. $x_1 = 0$, $1 \leq x_2 \leq x$ and $1 \leq x_4 \leq x-1$.

Then, $x_2 + x_4 = x+1$. It follows from the generalised Vandermonde's identity that $\sum\limits_{x_2+x_4=x+1} \binom{x+1}{x_2}\binom{x}{x_4} = \binom{2x+1}{x+1}$. So, $\sum\limits_{\substack{x_1+x_2+x_4=x+1\\ \textit{Subcase } 8.2.1.3}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_4} = \sum\limits_{x_2+x_4=x+1} \binom{x+1}{x_2}\binom{x}{x_4} - \binom{x+1}{x+1}\binom{x}{0} - \binom{x+1}{1}\binom{x}{x} = \binom{2x+1}{x+1} - x - 2.$

Subcase 8.2.1.4. $x_2 = 0$, $1 \leq x_1 \leq x$ and $1 \leq x_4 \leq x-1$.

By symmetry to Subcase 8.2.1.3, $\sum\limits_{\substack{x_1+x_2+x_4=x+1\\ \textit{Subcase } 8.2.1.4}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_4} = \binom{2x+1}{x+1} - x - 2.$

Subcase 8.2.1.5. $x_4 = 0$ and $1 \leq x_1, x_2 \leq x$.

Then, $x_1 + x_2 = x+1$. It follows from the generalised Vandermonde's identity that $\sum\limits_{x_1+x_2=x+1} \binom{x+1}{x_1}\binom{x+1}{x_2} = \binom{2x+2}{x+1}$. So, $\sum\limits_{\substack{x_1+x_2+x_4=x+1\\ \textit{Subcase } 8.2.1.5}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_4} = \sum\limits_{x_1+x_2=x+1} \binom{x+1}{x_1}\binom{x+1}{x_2} - \binom{x+1}{x+1}\binom{x+1}{0} - \binom{x+1}{0}\binom{x+1}{x+1} = \binom{2x+2}{x+1} - 2.$

Now, by generalised Vandermonde's identity, $\sum\limits_{x_1+x_2+x_4=x+1} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_4} = \binom{3x+2}{x+1}$.

So, $\sum\limits_{\substack{x_1+x_2+x_3+x_4=p\\ \textit{Subcase } 8.2.1}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \sum\limits_{x_1+x_2+x_4=x+1} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_4} - \sum\limits_{i=1}^{5} \sum\limits_{\substack{x_1+x_2+x_4=x\\ \textit{Subcase } 8.2.1.i}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_4} = \binom{3x+2}{x+1} - \binom{2x+2}{x+1} - 2\binom{2x+1}{x+1} + 2.$

Subcase 8.2.2. $x_4 = x$ (and $1 \leq x_1, x_2 \leq x$ and $1 \leq x_3 \leq x-1$).

By symmetry to Subcase 8.2.1, $\sum\limits_{\substack{x_1+x_2+x_3+x_4=p\\ \textit{Subcase } 8.2.2}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = \binom{3x+2}{x+1} - \binom{2x+2}{x+1} - 2\binom{2x+1}{x+1} + 2.$

Hence, $\sum\limits_{\substack{x_1+x_2+x_3+x_4=p \\ Case\ 8.2}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = 2\binom{3x+2}{x+1} - 2\binom{2x+2}{x+1} - 4\binom{2x+1}{x+1} + 4.$

So, $\Phi_{odd}(p,[0,1]) = \sum\limits_{i=1}^{2} \sum\limits_{\substack{x_1+x_2+x_3+x_4=p \\ Case\ 8.i}} \binom{x+1}{x_1}\binom{x+1}{x_2}\binom{x}{x_3}\binom{x}{x_4} = 2\binom{3x+2}{x+1} + 2\binom{3x+1}{x} - 2\binom{2x+2}{x+1} - 8\binom{2x+1}{x} - 2\binom{2x}{x} + 2x + 10.$

Finally, by (2.4), $\Phi_{odd}(p) = \Phi_{odd}(p,[0,0]) = \binom{2p}{p} - \Phi_{odd}(p,[1,0]) - \Phi_{odd}(p,[0,1]) - \sum\limits_{i=1}^{2}\sum\limits_{j=1}^{2} \Phi_{odd}(p,[i,j]) = \binom{2p}{p} - 4\binom{3x+2}{x+1} - 4\binom{3x+1}{x} + 2\binom{2x+2}{x+1} + 8\binom{2x+1}{x} + 2\binom{2x}{x} - 4.$

$\square$

With an expression for $\Phi_{odd}(p)$ now, we will use its special property to construct an orientation F of $K(p,p,q)$ for odd $p \geq 5$. Similar to Proposition 2.3.8, we divide each of V_1 and V_2 into 2 groups, with sizes $\lfloor \frac{p}{2} \rfloor$ and $\lceil \frac{p}{2} \rceil$. Then, orientate F such that for all $1 \leq i \leq q$, $|O(3_i)| = p$ and $O(3_i)$ contains some but not all vertices of each group. As before, this design will aid in ensuring $d(F) = 2$.

Proposition 2.3.14 *Suppose $p \geq 5$ is an odd integer. If $p+3 \leq q \leq \Phi_{odd}(p)+2$, then $\bar{d}(K(p,p,q)) = 2$.*

Proof: Partition $V_1 \cup V_2$ into X_i, $i = 1,2,3,4$.

$$X_1 = \{1_j |\ j \equiv 1\ (mod\ 2)\}, X_3 = \{1_j |\ j \equiv 0\ (mod\ 2)\},\ \text{and}$$
$$X_4 = \{2_1, 2_2, \ldots, 2_{\lfloor p/2 \rfloor}\},\ \text{and}\ X_2 = \{2_{\lfloor p/2 \rfloor+1}, 2_{\lfloor p/2 \rfloor+2}, \ldots, 2_p\}.$$

Observe that $|X_1| = |X_2| = \lfloor \frac{p}{2} \rfloor + 1$ and $|X_3| = |X_4| = \lfloor \frac{p}{2} \rfloor$. First, we shall define an orientation F for $K(p,p,p+3)$ as follows.

(I) $X_1 \to X_4 \to X_3 \to X_2 \to X_1$.

(II) $V_1 \to 3_{p+2} \to V_2 \to 3_{p+3} \to V_1$.

(III) For $i = 1, \ldots, \lfloor \frac{p}{2} \rfloor$,

(a) $\{2_1, 2_{\lfloor p/2 \rfloor+1}\} \cup (V_1 - \{1_{2i-1}, 1_{2i}\}) \to 3_i \to \{1_{2i-1}, 1_{2i}\} \cup (V_2 - \{2_1, 2_{\lfloor p/2 \rfloor+1}\})$,

(b) $\{1_1, 1_2\} \cup (V_2 - \{2_i, 2_{i+\lfloor p/2 \rfloor}\}) \to 3_{i+\lfloor p/2 \rfloor+1} \to \{2_i, 2_{i+\lfloor p/2 \rfloor}\} \cup (V_1 - \{1_1, 1_2\})$;

and

(c) $\{2_1, 2_{\lfloor p/2 \rfloor+1}\} \cup (V_1 - \{1_{p-1}, 1_p\}) \to 3_{\lfloor p/2 \rfloor+1} \to \{1_{p-1}, 1_p\} \cup (V_2 - \{2_1, 2_{\lfloor p/2 \rfloor+1}\})$,

(d) $\{1_1, 1_2\} \cup (V_2 - \{2_{\lfloor p/2 \rfloor}, 2_p\}) \to 3_{p+1} \to \{2_{\lfloor p/2 \rfloor}, 2_p\} \cup (V_1 - \{1_1, 1_2\})$.

Now, consider the case where $q > p+3$. Let $x_i = |O(3_j) \cap X_i|$ for some j, where $p+3 < j \leq q$, and $i = 1,2,3,4$. So, for each solution $(x_1, x_2, x_3, x_4)^{**}$ of (2.3), there are $\binom{\lfloor p/2 \rfloor+1}{x_1}\binom{\lfloor p/2 \rfloor+1}{x_2}\binom{\lfloor p/2 \rfloor}{x_3}\binom{\lfloor p/2 \rfloor}{x_4}$ ways to choose p vertices (as the outset of a vertex 3_j), where x_i vertices are selected from the set X_i, satisfying $1 \leq x_1, x_2 \leq \lfloor \frac{p}{2} \rfloor$, $1 \leq x_3, x_4 \leq \lfloor \frac{p}{2} \rfloor - 1$ and $x_1 + x_2 + x_3 + x_4 = p$. Summing over all possible solutions $(x_1, x_2, x_3, x_4)^{**}$ of (2.3), there is a total of $\Phi_{odd}(p) := \sum\limits_{(x_1,x_2,x_3,x_4)} \binom{\lfloor \frac{p}{2} \rfloor+1}{x_1}\binom{\lfloor \frac{p}{2} \rfloor+1}{x_2}\binom{\lfloor \frac{p}{2} \rfloor}{x_3}\binom{\lfloor \frac{p}{2} \rfloor}{x_4}$ of such combinations of p vertices of $V_1 \cup V_2$. Denote this set of combinations as Ψ_{odd}.

Note from (III) that the $p+1$ outsets of $3_1, 3_2, \ldots, 3_{p+1}$ are elements of Ψ_{odd}. That leaves $|\Psi_{odd}| - (p+1) = \Phi_{odd}(p) - (p+1)$ combinations of p vertices of $V_1 \cup V_2$. Note however that $O(3_{p+2})$ and $O(3_{p+3})$ from (II) are not elements of Ψ_{odd}. Hence, for $p+1+2 < j \leq q \leq \max\limits_d\{\Phi_{odd}(p)\} + 2$, we extend the definition of the above orientation so that the outsets of vertices $3_{p+3}, 3_{p+4}, \ldots, 3_q$ are these remaining elements of Ψ_{odd}. (See Figure 2.3.2 for F for $p = 5$ and $q = 9$.)

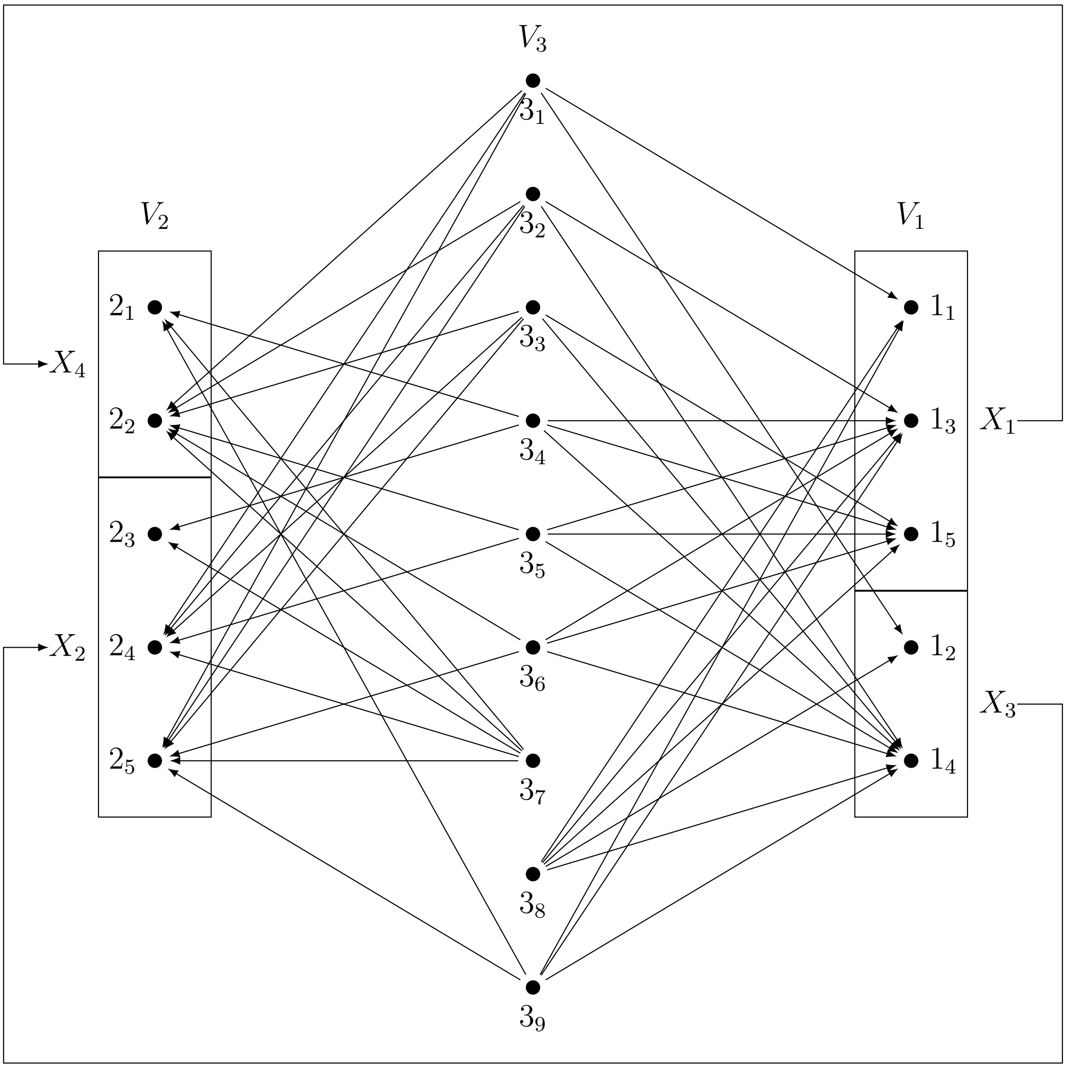

Figure 2.3.2: Orientation F for $p = 5$ and $q = 9$.
For clarity, only the arcs from (1) V_1 to V_2 and (2) V_3 to V_1 and V_2 are shown.

Claim: For all $u, v \in V(K(p, p, q))$, $d_F(u, v) \leq 2$.

Case 1. $u = 1_a, v = 1_b$, $a \neq b$.

If $1_a \in X_1$ and $1_b \in X_3$, then we have $1_a \to 2_1 \to 1_b$ by (I). If $1_a \in X_3$ and $1_b \in X_1$, then we have $1_a \to 2_p \to 1_b$ by (I). If $1_a, 1_b \in X_1$ (X_3, respectively), and $b = 2i - 1$ ($2i$, respectively) for some $1 \leq i \leq \lfloor \frac{p}{2} \rfloor + 1$ ($1 \leq i \leq \lfloor \frac{p}{2} \rfloor$, respectively), then $1_a \to 3_i \to 1_b$ by (III)(a) and (c).

Case 2. $u = 2_a, v = 2_b$, $a \neq b$.

If $2_a \in X_4$ and $2_b \in X_2$, then we have $2_a \to 1_2 \to 2_b$ by (I). If $2_a \in X_2$ and $2_b \in X_4$, then we have $2_a \to 1_1 \to 2_b$ by (I). If $2_a, 2_b \in X_4$, and $1 \leq b \leq \lfloor \frac{p}{2} \rfloor$, then $2_a \to 3_{b+\lfloor \frac{p}{2} \rfloor+1} \to 2_b$ by (III)(b). If $2_a, 2_b \in X_2$, and $b = i + \lfloor \frac{p}{2} \rfloor$ for some i, $1 \leq i \leq \lfloor \frac{p}{2} \rfloor + 1$, then $2_a \to 3_{i+\lfloor \frac{p}{2} \rfloor+1} \to 2_b$ by (III)(b) and (d).

Case 3. $u = 1_a, v = 2_b$.

By (II), $1_a \to 3_{p+2} \to 2_b$.

Case 4. $u = 2_a, v = 1_b$.

By (II), $2_a \to 3_{p+3} \to 1_b$.

Case 5. $u = 1_a, v = 3_b$.

Subcase 5a. $b = p + 2$.

By (II), $V_1 \to 3_{p+2}$.

Subcase 5b. $b \neq p + 2$.

Suppose $1_a \in X_1$. Then, $1_a \to X_4$ by (I). Since for each $3_b \in V_3$, $I(3_b) \cap X_4 \neq \emptyset$ by (II) and (III), let $w \in I(3_b) \cap X_4$. It follows that $1_a \to w \to 3_b$. A similar argument follows if $1_a \in X_3$.

Case 6. $u = 2_a, v = 3_b$.

Subcase 6a. $b = p + 3$.

By (II), $V_2 \to 3_{p+3}$.

Subcase 6b. $b \neq p + 3$.

Suppose $2_a \in X_4$. Then, $2_a \to X_3$ by (I). Since for each $3_b \in V_3$, $I(3_b) \cap X_3 \neq \emptyset$ by (II) and (III), let $w \in I(3_b) \cap X_3$. It follows that $2_a \to w \to 3_b$. A similar argument follows if $2_a \in X_2$.

Case 7. $u = 3_a, v = 1_b$.

Subcase 7a. $a = p + 3$.

By (II), $3_{p+3} \to V_1$.

Subcase 7b. $a \neq p + 3$.

Suppose $1_b \in X_1$. Recall that $X_2 \to X_1$ by (I). Since for each $3_a \in V_3$, $O(3_a) \cap X_2 \neq \emptyset$ by (II) and (III), let $w \in O(3_a) \cap X_2$. It follows that $3_a \to w \to 1_b$. A similar argument follows if $1_b \in X_3$.

Case 8. $u = 3_a, v = 2_b$.

Subcase 8a. $a = p + 2$.

By (II), $3_{p+2} \to V_2$.

Subcase 8b. $a \neq p + 2$.

Suppose $2_b \in X_4$. Then, $X_1 \to 2_b$ by (I). Since for each $3_a \in V_3$, $O(3_a) \cap X_1 \neq \emptyset$ by (II) and (III), let $w \in O(3_a) \cap X_1$. It follows that $3_a \to w \to 2_b$. A similar argument follows if $2_b \in X_2$.

Case 9. $u = 3_a, v = 3_b$.

Subcase 9a. $a \neq p + 2, p + 3$ and $b \neq p + 2, p + 3$.

Observe from (III) that $|O(3_x) \cap (V_1 \cup V_2)| = p$ for $x = a, b$. Furthermore, $O(3_a) \cap (V_1 \cup V_2) \nsubseteq O(3_b) \cap (V_1 \cup V_2)$ if $b \neq a$. Thus, there exists a vertex $w \in V_1 \cup V_2$ such that $3_a \to w \to 3_b$.

Subcase 9b. $a = p + 2$ and $b \neq p + 2, p + 3$.

$3_{p+2} \to V_2$ by (II) and $I(3_b) \cap X_i \neq \emptyset$ for all $i = 2, 4$, imply the existence of $w \in I(3_b) \cap V_2$. Hence, $3_a \to w \to 3_b$.

Subcase 9c. $a = p+3$ and $b \neq p+2, p+3$.

$3_{p+3} \to V_1$ by (II) and $I(3_b) \cap X_i \neq \emptyset$ for all $i = 1, 3$, imply the existence of $w \in I(3_b) \cap V_1$. Hence, $3_a \to w \to 3_b$.

Subcase 9d. $a \neq p+2, p+3$ and $b = p+2$.

$V_1 \to 3_{p+2}$ and $O(3_a) \cap X_i \neq \emptyset$ for all $i = 1, 3$, imply the existence of $w \in O(3_a) \cap V_1$. Hence, $3_a \to w \to 3_b$.

Subcase 9e. $a \neq p+2, p+3$ and $b = p+3$.

$V_2 \to 3_{p+3}$ by (II) and $O(3_a) \cap X_i \neq \emptyset$ for all $i = 2, 4$, imply the existence of $w \in O(3_a) \cap V_2$. Hence, $3_a \to w \to 3_b$.

Subcase 9f. $a = p+2$ and $b = p+3$.

By (II), $3_{p+2} \to V_2 \to 3_{p+3}$.

Subcase 9g. $a = p+3$ and $b = p+2$.

By (II), $3_{p+3} \to V_1 \to 3_{p+2}$.

$\square$

Example 2.3.15 *If* $10 \leq q \leq 2090$, *then* $\bar{d}(K(7,7,q)) = 2$.

Proof: By Lemma 2.3.13 and Proposition 2.3.14 , if $10 \leq q \leq \Phi_{odd}(7) + 2 = 2090$, then $\bar{d}(K(7,7,q)) = 2$.

Alternatively, we may verify the computation of $\Phi_{odd}(7)$ as follows. A solution $(x_1, x_2, x_3, x_4)^{**}$ of (2.3) is either a permutation of $(1,1,2,3)$ or a permutation of $(1,2,2,2)$. Hence, by enumerating all solutions, we have $\Phi_{odd}(7) = 4\binom{3}{1}\binom{3}{2}\binom{4}{3}\binom{4}{1} + 2\binom{3}{1}\binom{3}{1}\binom{4}{3}\binom{4}{2} + 2\binom{3}{1}\binom{3}{2}\binom{4}{2}\binom{4}{2} + 2\binom{3}{2}\binom{3}{2}\binom{4}{2}\binom{4}{1} = 2088$.

$\square$

In Discussion 2.3.10, we claimed that $\Phi_{even}(p)$ is the best possible bound of $\Phi^*(p,d)$ amidst all non-trivial divisors d of p when $p \geq 4$ is even, and $\Phi_{odd}(p)$ is a better bound than $\Phi^*(p,d)$ when $p \geq 5$ is odd and composite. Now, we shall prove it, i.e. $\Phi_{even}(p)$ ($\Phi_{odd}(p)$, respectively) is greater than $\max\limits_{3 \leq d < p}\{\Phi^*(p,d)\}$ for each even (odd and composite, respectively) $p \geq 4$.

Proposition 2.3.16 *Suppose $p \geq 4$ is a composite integer and d is a divisor of p, where $3 \leq d < p$.*

$$\max_{3 \leq d < p}\{\Phi^*(p,d)\} < \begin{cases} \Phi_{even}(p), & \text{if } p \text{ is even,} \\ \Phi_{odd}(p), & \text{if } p \text{ is odd.} \end{cases}$$

Proof:

Case 1. p is even.

Claim 1: For any even integer $p \geq 14$ and any divisor $3 \leq d < p$ of p, $\binom{2p-\frac{p}{d}}{p} - 8\binom{\frac{3p}{2}}{p} + 12\binom{p}{\frac{p}{2}} - 6 > 0$.

$$\begin{aligned} \binom{2p-\frac{p}{d}}{p} - 8\binom{\frac{3p}{2}}{p} + 12\binom{p}{\frac{p}{2}} - 6 &\geq \binom{2p-\frac{p}{d}}{p} - 8\binom{\frac{3p}{2}}{p} \\ &\geq \binom{\frac{5p}{3}}{p} - 8\binom{\frac{3p}{2}}{p} \\ &> 0. \end{aligned}$$

The first inequality is due to $12\binom{p}{\frac{p}{2}} \geq 6$, while the second inequality follows as $d \geq 3$ and $f(z) := \binom{z}{p}$ is an increasing function for $z \geq p$. Since $f(z)$ is also strictly convex for $z \geq p$ and $\binom{\frac{5(13)}{3}}{13} - 8\binom{\frac{3(13)}{2}}{13} > 0$, the last inequality follows for all $p \geq 13$. So, Claim 1 follows.

Now, for each even integer $p \leq 12$, we verified, using Maple, $\Phi^*(p,d) < \Phi_{even}(p)$ for all divisors $3 \leq d < p$ of p. (See Table 2.1.) Let $p \geq 14$ be an even integer.

Note that $\sum_{i=1}^{d}\sum_{j=0}^{d}\Phi(p,d,[i,j]) \geq \binom{k}{0}\binom{(2d-1)k}{p} = \binom{2p-\frac{p}{d}}{p}$ as the expression $\binom{k}{0}\binom{(2d-1)k}{p}$ counts the number of ways such that none is selected from a (fixed) group of k elements and p elements are selected from the remaining $2d-1$ groups of k elements. Also, recall that $\binom{2p}{p} = \sum_{i=0}^{d}\sum_{j=0}^{d}\Phi(p,d,[i,j]) = \Phi(p,d,[0,0]) + \sum_{i=1}^{d}\sum_{j=0}^{d}\Phi(p,d,[i,j]) + \sum_{j=1}^{d}\Phi(p,d,[0,j])$ by generalised Vandermonde's identity. It follows for each even integer $p \geq 14$ and each divisor $3 \leq d < p$ of p that,

$$
\begin{aligned}
\binom{2p}{p} - \Phi^*(p,d) &= \binom{2p}{p} - \Phi(p,d,[0,0]) \\
&= \sum_{i=1}^{d}\sum_{j=0}^{d}\Phi(p,d,[i,j]) + \sum_{j=1}^{d}\Phi(p,d,[0,j]) \\
&\geq \binom{2p-\frac{p}{d}}{p} \\
&> 8\binom{\frac{3p}{2}}{p} - 12\binom{p}{\frac{p}{2}} + 6 \\
&= \binom{2p}{p} - \Phi_{even}(p),
\end{aligned}
$$

where the last inequality is due to Claim 1.

Case 2. p is odd and composite.

Set $x := \lfloor \frac{p}{2} \rfloor$.

Claim 2: For any composite and odd integer $p \geq 17$ and any divisor $3 \leq d < p$ of p, $\binom{2p-\frac{p}{d}}{p} - 4\binom{3x+2}{x+1} - 4\binom{3x+1}{x} > 0$.

$$
\begin{aligned}
&\binom{2p-\frac{p}{d}}{p} - 4\binom{3x+2}{x+1} - 4\binom{3x+1}{x} \\
&= \binom{2p-\frac{p}{d}}{p} - 4\binom{3x+2}{2x+1} - 4\binom{3x+1}{2x+1} \\
&\geq \binom{2p-\frac{p}{3}}{p} - 8\binom{3x+2}{2x+1} \\
&\geq \binom{\frac{10x+5}{3}}{2x+1} - 8\binom{3x+2}{2x+1} \\
&> 0.
\end{aligned}
$$

The first inequality is due to $d \geq 3$ and $f(z)$ is an increasing function for $z \geq p$. Since $f(z)$ is also strictly convex for $z \geq p$ and $\binom{\frac{10(8)+5}{3}}{2(8)+1} - 8\binom{3(8)+2}{2(8)+1} > 0$, the last inequality follows for all $x \geq 8$. Hence, Claim 2 follows.

For each composite and odd integer $p \leq 15$, we verified, using Maple, $\Phi^*(p,d) \leq \Phi_{odd}(p)$ for all divisors $3 \leq d < p$ of p. (See Table 2.1.) Now, consider any composite and odd integer $p \geq 17$. As in Case 1, $\sum\limits_{i=1}^{d}\sum\limits_{j=0}^{d} \Phi(p,d,[i,j]) \geq \binom{2p-\frac{p}{d}}{p}$. It follows for each composite and odd integer $p \geq 17$ and each divisor $3 \leq d < p$ of p that,

$$
\begin{aligned}
&\binom{2p}{p} - \Phi^*(p,d) \\
=&\binom{2p}{p} - \Phi(p,d,[0,0]) \\
=&\sum_{i=1}^{d}\sum_{j=0}^{d} \Phi(p,d,[i,j]) + \sum_{j=1}^{d} \Phi(p,d,[0,j]) \\
\geq&\binom{2p-\frac{p}{d}}{p} \\
>&4\binom{3x+2}{x+1} + 4\binom{3x+1}{x} \\
\geq&4\binom{3x+2}{x+1} + 4\binom{3x+1}{x} - 2\binom{2x+2}{x+1} - 8\binom{2x+1}{x} - 2\binom{2x}{x} + 4 \\
=&\binom{2p}{p} - \Phi_{odd}(p),
\end{aligned}
$$

where the second last inequality follows from Claim 2.

$\square$

p	d	$\begin{cases} \Phi_{even}(p) - \Phi^*(p,d), & \text{if } p \text{ is even,} \\ \Phi_{odd}(p) - \Phi^*(p,d), & \text{if } p \text{ is odd.} \end{cases}$
4	2	16-16=0
6	2	486-486=0
6	3	486-64=422
8	2	9,744-9,744=0
8	4	9,744-256=9,488
9	3	39,400-14,580=24,820
10	2	163,750-163,750=0
10	5	163,750-1,024=162,726
12	2	2,566,726-2,566,726=0
12	3	2,566,726-1,580,096=986,630
12	4	2,566,726-459,270=2,107,456
12	6	2,566,726-4,096=2,562,630
14	2	39,227,538-39,227,538=0
14	7	39,227,538-16,384=39,211,154
15	3	152,558,168-121,562,500=30,995,668
15	5	152,558,168-14,880,348=137,677,820
16	2	595,351,056-595,351,056=0
16	4	595,351,056-269,992,192=325,358,864
16	8	595,351,056-65,536=595,285,520
18	2	9,038,224,134-9,038,224,134=0
18	3	9,038,224,134-8,120,234,620=917,989,514
18	6	9,038,224,134-491,051,484=8,547,172,650
18	9	9,038,224,134-262,144=9,037,961,990
20	2	137,608,385,766-137,608,385,766=0
20	4	137,608,385,766-95,227,343,750=42,381,042,016
20	5	137,608,385,766-47,519,843,328=90,088,542,438
20	10	137,608,385,766-1,048,576=137,607,337,190

Table 2.1: Comparison of $\Phi^*(p,d)$ with $\Phi_{even}(p)$ and $\Phi_{odd}(p)$ for $4 \leq p \leq 20$.

For clarity, we summarise Propositions 2.3.8 and 2.3.14 as follows.

Theorem 2.3.17 *Suppose $p \geq 4$ is an integer. Then,*

$$\bar{d}(K(p,p,q)) = 2 \text{ if } \begin{cases} p+2 \leq q \leq \Phi_{even}(p)+2, & \text{if } p \text{ is even,} \\ p+3 \leq q \leq \Phi_{odd}(p)+2, & \text{if } p \text{ is odd,} \end{cases}$$

where $\Phi_{even}(p) = \binom{2p}{p} - 8\binom{\frac{3p}{2}}{p} + 12\binom{p}{\frac{p}{2}} - 6$ *and* $\Phi_{odd}(p) = \binom{2p}{p} - 4\binom{3x+2}{x+1} - 4\binom{3x+1}{x} + 2\binom{2x+2}{x+1} + 8\binom{2x+1}{x} + 2\binom{2x}{x} - 4$, $x = \lfloor \frac{p}{2} \rfloor$.

For a clearer picture of the 'gap' between Theorems 2.3.17 and 1.2.9, we compare some values of $\Phi_{even}(p)+2$ and $\Phi_{odd}(p)+2$ with the bound (1.2) in the following table.

p	$\binom{2p}{p} - (\Phi_{even}(p)+2)$, **if p is even,** $\binom{2p}{p} - (\Phi_{odd}(p)+2)$, **if p is odd.**
4	70-18=52
5	252-74=178
6	924-488=436
7	3,432-2,090=1,342
8	12,870-9,746=3,124
9	48,620-39,402=9,218
10	184,756-163,752=21,004
11	705,432-644,502=60,930
12	2,704,156-2,566,728=137,428
13	10,400,600-10,004,430=396,170
14	40,116,600-39,227,540=889,060
15	155,117,520-152,558,170=2,559,350
16	601,080,390-595,351,058=5,729,332
17	2,333,606,220-2,317,099,178=16,507,042
18	9,075,135,300-9,038,224,136=36,911,164
19	35,345,263,800-35,238,721,934=106,541,866
20	137,846,528,820-137,608,385,768=238,143,052

Table 2.2: Comparison of $\binom{2p}{p}$ with $\Phi_{even}(p)$ and $\Phi_{odd}(p)$ for $4 \leq p \leq 20$.

Since complete tripartite graphs can be spanning subgraphs of the complete multipartite graphs, we have the following corollary.

Corollary 2.3.18 *Suppose* $n \geq 2$ *and* p_i *are positive integers for* $i = 1, 2, \ldots, n$ *such that* $p_1 + p_2 + \ldots + p_r = p_{r+1} + p_{r+2} + \ldots + p_n = p \geq 4$ *for some integers* r *and* p*. Let* $G = K(p_1, p_2, \ldots, p_n, q)$*. Then,* $\bar{d}(G) = 2$ *if*

$$\begin{cases} p+2 \leq q \leq \Phi_{even}(p)+2, & \textit{if } p \textit{ is even,} \\ p+3 \leq q \leq \Phi_{odd}(p)+2, & \textit{if } p \textit{ is odd.} \end{cases}$$

Proof: Note that G is a supergraph of $K(p,p,q)$ and $\bar{d}(K(p,p,q)) = 2$ by Theorem 2.3.17. So, there exists an orientation D for $K(p,p,q)$, where $d(D) = 2$. Partition $V(G)$ into three parts $\bigcup_{i=1}^{r} V_i$, $\bigcup_{i=r+1}^{n} V_i$ and V_{n+1}, and define an orientation F for G such that D is a subdigraph of F, and edges not in D are oriented arbitrarily. It follows that $d(F) = 2$.

□

3. Tree Vertex-multiplications

3.1. Existing Results

In [30], Koh and Tay studied vertex-multiplications of trees. Since trees of diameter at most 2 are parent graphs to complete bipartite graphs which have a characterisation (see Theorem 1.2.5), they only considered trees of diameter at least 3. Specifically, they proved the following results for trees of diameter 3 or 4. Recall the assumption, $s_i \geq 2$ for $i = 1, 2, \ldots, n$, holds unless otherwise stated.

Theorem 3.1.1 *(Koh and Tay [30])*

If T is a tree of order n and $d(T) = 3$ or 4, then $T(s_1, s_2, ..., s_n) \in \mathscr{C}_0 \cup \mathscr{C}_1$.

Theorem 3.1.2 *(Koh and Tay [30])*

Let T be a tree with diameter 4 and its only central vertex be u.

(i) If $deg_T(u) = 2$, then $T(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$.

(ii) If $deg_T(u) \geq 3$, then $T^{(2)} \in \mathscr{C}_1$.

3.2. New Results On Trees With Diameter 4

In light of the above theorems, we are interested in determining conditions in which $T(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$ or $\mathscr{C}_1$. Particularly, Theorem 3.2.8 generalises Theorem 3.1.2.

For convenience, we shall introduce some notations. Let D be an orientation of $G(s_1, s_2, \ldots, s_n)$ with $s_i \geq 2$ for $1 \leq i \leq n$. If v_p and v_q, $1 \leq p, q \leq n$ and $p \neq q$, are adjacent vertices in G, then for each i, $1 \leq i \leq s_p$, we denote $O_D^{v_q}((i,p)) := \{(j,q)|\ (i,p) \rightarrow (j,q), 1 \leq j \leq s_q\}$ and $I_D^{v_q}((i,p)) := \{(j,q)|\ (j,q) \rightarrow (i,p), 1 \leq j \leq$

$s_q\}$. If there is no danger of confusion, we shall omit the subscript D for the above notations.

Let T be a tree of diameter 4 with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$. We further denote by u, the unique central vertex of T, i.e. $e_T(u) = 2$, and the neighbours of u by $[i]$. i.e. $N_T(u) = \{[i] | \ i = 1, 2 \ldots, deg_T(u)\}$. For each $i = 1, 2, \ldots, deg_T(u)$, we further denote the neighbours of $[i]$, excluding u, by $[\alpha, i]$. i.e. $N_T([i]) - \{u\} = \{[\alpha, i] | \ \alpha = 1, 2, \ldots, deg_T([i]) - 1\}$. In the vertex-multiplication graph $G := T(s_1, s_2, \ldots, s_n)$ of T, the integer s_i corresponds to the vertex v_i, $i \neq n$, while $s_n := s$ corresponds to u. We will loosely use the two denotations of a vertex, for example, if $v_i = [j]$, then $s_i = s_{[j]}$. Also, we set $(\mathbb{N}_s, u) := \{(1, u), (2, u) \ldots, (s, u)\}$. Since each shortest $v - w$ path is unique for all $v, w \in V(T)$, the above notation is well-defined. Example 3.2.1 illustrates the use of this notation.

Example 3.2.1 *Let u be the unique central vertex in T, a tree of diameter 4.*

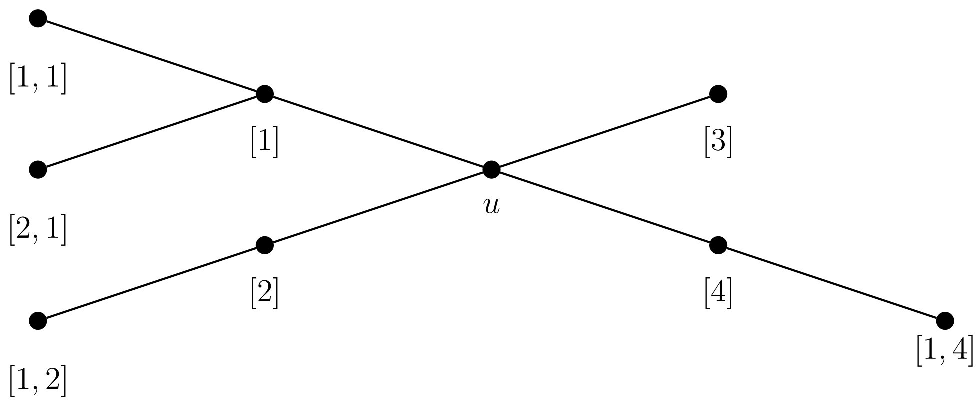

Figure 3.2.1: Labelling vertices in T

Also, we set $G(A_j) := \{i | \ s_{[i]} = j, \ 1 \leq i \leq deg_T(u)$ and $[i]$ is not an end-vertex in $T\}$, where j is a positive integer. If there is no ambiguity, we will use A_j instead of $G(A_j)$. Similarly, $A_{\leq j}$ and $A_{\geq j}$ denote the corresponding sets, when the condition $s_{[i]} = j$ is replaced by $s_{[i]} \leq j$ and $s_{[i]} \geq j$ respectively. If T is as given in Example 3.2.1 and $G := T^{(2)}$, then $G(A_2) = \{1, 2, 4\}$ since none of the vertices $[1], [2], [4]$ is an end-vertex in T.

Now, we will see that tree vertex-multiplications of a tree T with diameter 4 lies in $\mathscr{C}_0$ whenever the vertex-multiplication of $[i]$ is not too small, i.e. $s_{[i]} \geq 4$, for all $i = 1, 2, \ldots, deg_T(u)$.

Theorem 3.2.2 *Let T be a tree of diameter 4 and u be its unique central vertex. If $A_{\geq 4} \neq \emptyset$ and $A_2 = A_3 = \emptyset$, then $G := T(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$.*

Proof:

Let $H := T(t_1, t_2, \ldots, t_n)$ be the subgraph of G, where $t_n = 2$, $t_{[i]} = 4$ and $t_{[\alpha,i]} = 2$ for all $i \in G(A_{\geq 4})$ and $\alpha = 1, 2, \ldots, deg_T([i]) - 1$, while $t_{[j]} = 2$ for all $j \in N_T(u) - G(A_{\geq 4})$ (These $[j]$'s are end-vertices in T). Note that we will use A_j for $H(A_j)$ from here onwards. So, $H(A_j) \neq \emptyset$ if and only if $j = 4$. Define an orientation D for H as follows.

(I) $\{(2,[i]),(3,[i])\} \rightarrow (1,[\alpha,i]) \rightarrow \{(1,[i]),(4,[i])\} \rightarrow (2,[\alpha,i]) \rightarrow \{(2,[i]),(3,[i])\}$,

and

(II) $\{(1,[i]),(2,[i])\} \rightarrow (1,u) \rightarrow \{(3,[i]),(4,[i])\} \rightarrow (2,u) \rightarrow \{(1,[i]),(2,[i])\}$,

for all $i \in A_4$ and $\alpha = 1, 2, \ldots, deg_T([i]) - 1$.

(III) $(2,u) \rightarrow \{(1,[j]),(2,[j])\} \rightarrow (1,u)$

for all $j \in N_T(u) - A_4$.

(See Figure 3.2.2.)

Claim: For all $v, w \in V(D)$, $d_D(v,w) \leq 4$.

Case 1.1. $v, w \in \{(1,[\alpha,i]),(2,[\alpha,i]) |\ \alpha = 1, 2, \ldots, deg_T([i]) - 1\}$ for each $i \in A_4$.

This is clear since, by (I), $(2,[i]) \rightarrow (1,[\alpha_1,i]) \rightarrow (1,[i]) \rightarrow (2,[\alpha_2,i]) \rightarrow (2,[i])$ is a directed C_4 for all $1 \leq \alpha_1 \leq \alpha_2 \leq deg_T([i]) - 1$.

Case 1.2. $v \in \{(1, [\alpha, i]), (2, [\alpha, i])\}$ and $w \in \{(1, [i]), (2, [i]), (3, [i]), (4, [i])\}$ for all $i \in A_4$ and $\alpha = 1, 2, \ldots, deg_T([i]) - 1$.

By symmetry, it suffices to show for the case $v = (1, [\alpha, i])$. $(1, [\alpha, i]) \to \{(1, [i]), (4, [i])\} \to (2, [\alpha, i]) \to \{(2, [i]), (3, [i])\}$ by (I).

Case 1.3. $v \in \{(1, [\alpha, i]), (2, [\alpha, i])\}$ and $w \in \{(1, u), (2, u)\}$ for all $i \in A_4$ and $\alpha = 1, 2, \ldots, deg_T([i]) - 1$.

By symmetry, it suffices to show for the case $v = (1, [\alpha, i])$. Note that $(1, [\alpha, i]) \to (1, [i]) \to (1, u)$ and $(1, [\alpha, i]) \to (4, [i]) \to (2, u)$ by (I) and (II).

Case 1.4. $v \in \{(1, [\alpha, i]), (2, [\alpha, i])\}$ and $w \in \{(1, [j]), (2, [j]), (3, [j]), (4, [j])\}$ for all $i, j \in A_4$, $i \neq j$, and all $\alpha = 1, 2, \ldots, deg_T([i]) - 1$.

By symmetry, it suffices to show for the case $v = (1, [\alpha, i])$. Note that $(1, [\alpha, i]) \to (1, [i]) \to (1, u) \to \{(3, [j]), (4, [j])\}$ and $(1, [\alpha, i]) \to (4, [i]) \to (2, u) \to \{(1, [j]), (2, [j])\}$ by (I) and (II).

Case 1.5. $v \in \{(1, [\alpha, i]), (2, [\alpha, i])\}$ and $w \in \{(1, [\beta, j]), (2, [\beta, j])\}$ for all $i, j \in A_4$, $i \neq j$, all $\alpha = 1, 2, \ldots, deg_T([i]) - 1$ and $\beta = 1, 2, \ldots, deg_T([j]) - 1$.

By symmetry, it suffices to show for the case $v = (1, [\alpha, i])$. Note that $(1, [\alpha, i]) \to (1, [i]) \to (1, u) \to \{(3, [j]), (4, [j])\}$ and while $(3, [j]) \to (1, [\beta, j])$ and $(4, [j]) \to (2, [\beta, j])$ by (I) and (II).

Case 2.1. $v \in \{(1, [i]), (2, [i]), (3, [i]), (4, [i])\}$ and $w \in \{(1, [\alpha, i]), (2, [\alpha, i])\}$ for all $i \in A_4$ and $\alpha = 1, 2, \ldots, deg_T([i]) - 1$.

By symmetry, it suffices to show for the case $v = (1, [i])$. Note that $(1, [i]) \to (2, [\alpha, i]) \to (2, [i]) \to (1, [\alpha, i])$ by (I).

Case 2.2. $v, w \in \{(1,[i]),(2,[i]),(3,[i]),(4,[i])\}$ for all $i \in A_4$ and $\alpha = 1, 2, \ldots, deg_T([i]) - 1$.

By symmetry, it suffices to show for the case $v = (1,[i])$. Note that $(1,[i]) \to (1,u) \to \{(3,[i]),(4,[i])\} \to (2,u) \to \{(1,[i]),(2,[i])\}$ by (II).

Case 2.3. $v \in \{(1,[i]),(2,[i]),(3,[i]),(4,[i])\}$ and $w \in \{(1,u),(2,u)\}$ for all $i \in A_4$.

By symmetry, it suffices to show for the case $v \in \{(1,[i]),(2,[i])\}$. Note that $\{(1,[i]),(2,[i])\} \to (1,u) \to \{(3,[i]),(4,[i])\} \to (2,u)$ by (II).

Case 2.4. $v \in \{(1,[i]),(2,[i]),(3,[i]),(4,[i])\}$ and $w \in \{(1,[j]),(2,[j]),(3,[j]),(4,[j])\}$ for $i, j \in A_4$ and $i \neq j$.

By symmetry, it suffices to show for the case $v \in \{(1,[i]),(2,[i])\}$. Note that $\{(1,[i]),(2,[i])\} \to (1,u) \to \{(3,[j]),(4,[j])\} \to (2,u) \to \{(1,[j]),(2,[j])\}$ by (II).

Case 2.5. $v \in \{(1,[i]),(2,[i]),(3,[i]),(4,[i])\}$ and $w \in \{(1,[\beta,j]),(2,[\beta,j])\}$ for $i, j \in A_4$, $i \neq j$, and $\beta = 1, 2, \ldots, deg_T([j]) - 1$.

By symmetry, it suffices to show for the case $v \in \{(1,[i]),(2,[i])\}$. Note that $\{(1,[i]),(2,[i])\} \to (1,u) \to \{(3,[j]),(4,[j])\}$, while $(3,[j]) \to (1,[\beta,j])$ and $(4,[j]) \to (2,[\beta,j])$ by (I) and (II).

Case 3.1. $v, w \in \{(1,u),(2,u)\}$.

Note that $(1,u) \to \{(3,[i]),(4,[i])\} \to (2,u) \to \{(1,[i]),(2,[i])\} \to (1,u)$ for all i by (II).

Case 3.2. $v \in \{(1,u),(2,u)\}, w \in \{(1,[i]),(2,[i]),(3,[i]),(4,[i])\}$ for all $i \in A_4$.

Note that $(1,u) \to \{(3,[i]),(4,[i])\} \to (2,u) \to \{(1,[i]),(2,[i])\} \to (1,u)$ by (II).

Case 3.3. $v \in \{(1,u),(2,u)\}, w \in \{(1,[\alpha,i]),(2,[\alpha,i])\}$ for all $i \in A_4$ and $\alpha = 1,2,\ldots,deg_T([i])-1$.

By symmetry, it suffices to show for the case $v=(1,u)$. Note that $(1,u) \to \{(3,[i]),(4,[i])\}$, while $(3,[i]) \to (1,[\alpha,i])$ and $(4,[i]) \to (2,[\alpha,i])$ by (I) and (II).

Case 4.1. $v \in \{(1,[\alpha,i]),(2,[\alpha,i])\}$ and $w \in \{(1,[j]),(2,[j])\}$ for all $i \in A_4$, $\alpha = 1,2,\ldots,deg_T([i])-1$ and $j \in N_T(u)-A_4$.

Note that $(1,[\alpha,i]) \to (4,[i]) \to (2,u)$, $(2,[\alpha,i]) \to (3,[i]) \to (2,u)$, and $(2,u) \to \{(1,[j]),(2,[j])\}$ by (I)-(III).

Case 4.2. $v \in \{(1,[j]),(2,[j])\}$ and $w \in \{(1,[\alpha,i]),(2,[\alpha,i])\}$ for all $i \in A_4$, $\alpha = 1,2,\ldots,deg_T([i])-1$ and $j \in N_T(u)-A_4$.

Note that $\{(1,[j]),(2,[j])\} \to (1,u) \to \{(3,[i]),(4,[i])\}$, $(3,[i]) \to (1,[\alpha,i])$, and $(4,[i]) \to (2,[\alpha,i])$ by (I)-(III).

Case 4.3. $v \in \{(1,[i]),(2,[i]),(3,[i]),(4,[i])\}$ and $w \in \{(1,[j]),(2,[j])\}$ for all $i \in A_4$, $\alpha = 1,2,\ldots,deg_T([i])-1$ and $j \in N_T(u)-A_4$.

Note that $\{(1,[i]),(2,[i])\} \to (1,u) \to \{(3,[i]),(4,[i])\} \to (2,u) \to \{(1,[j]),(2,[j])\}$ by (II) and (III).

Case 4.4. $v \in \{(1,[j]),(2,[j])\}$ and $w \in \{(1,[i]),(2,[i]),(3,[i]),(4,[i])\}$ for all $i \in A_4$, $\alpha = 1,2,\ldots,deg_T([i])-1$ and $j \in N_T(u)-A_4$.

$\{(1,[j]),(2,[j])\} \to (1,u) \to \{(3,[i]),(4,[i])\} \to (2,u) \to \{(1,[i]),(2,[i])\}$ by (II) and (III).

Case 4.5.

(i) $v \in \{(1,u),(2,u)\}$ and $w \in \{(1,[j]),(2,[j])\}$ for all $j \in N_T(u)-A_4$, or

(ii) $v \in \{(1,[j]),(2,[j])\}$ and $w \in \{(1,u),(2,u)\}$ for all $j \in N_T(u)-A_4$.

Note that $\{(1,[j]),(2,[j])\} \to (1,u) \to (3,[k]) \to (2,u) \to \{(1,[j]),(2,[j])\}$ is a directed C_4 for any $k \in A_4$.

Case 4.6. $v \in \{(1,[i]),(2,[i])\}$ and $w \in \{(1,[j]),(2,[j])\}$ for all $i,j \in N_T(u) - A_4$.

Note that $\{(1,[i]),(2,[i])\} \to (1,u) \to (3,[k]) \to (2,u) \to \{(1,[j]),(2,[j])\}$ is a directed C_4 for any $k \in A_4$ by (II) and (III).

Hence, $d(D) \leq 4$. Notice that every vertex in D lies in a directed C_4. So, by Lemma 1.3.2, $4 = d(T) \leq \bar{d}(G) \leq max\{4, d(D)\} = 4$. Therefore, $\bar{d}(G) = 4$.

$\square$

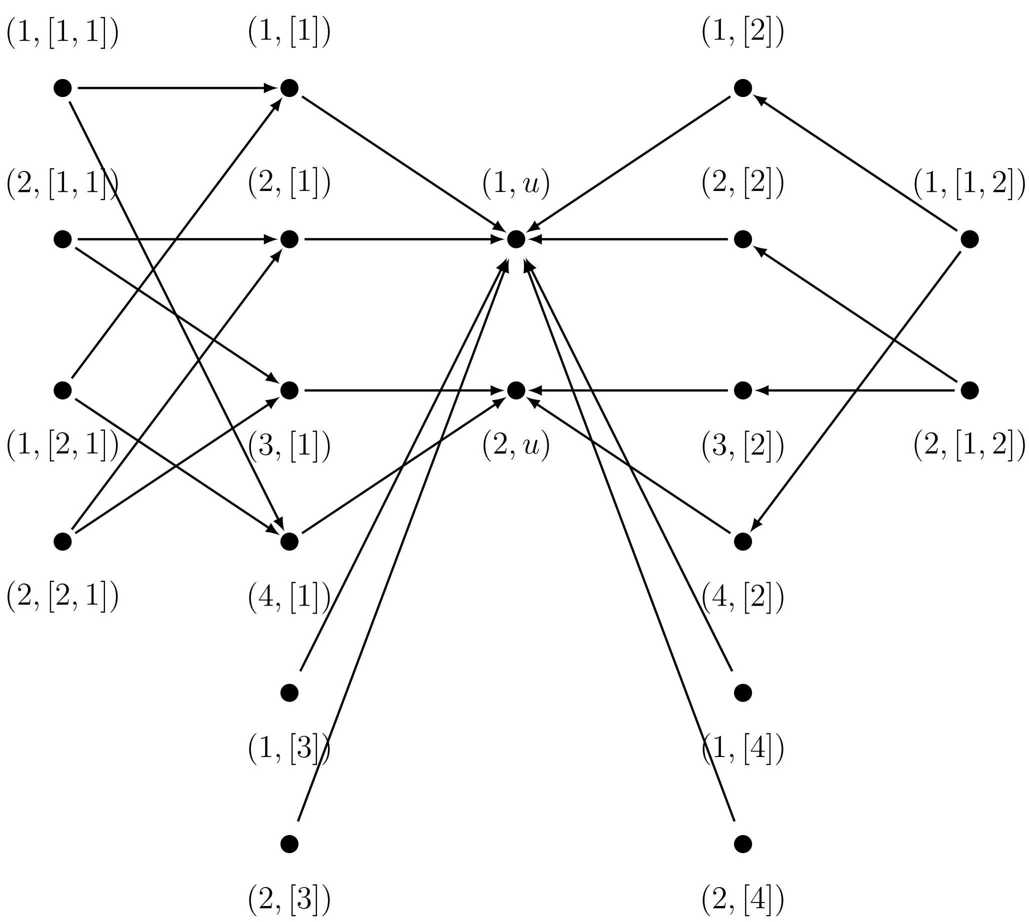

Figure 3.2.2: Orientation D, where $A_4 = \{1, 2\}$ and $N_T(u) - A_4 = \{3, 4\}$. For clarity, the arcs directed from (p, u) to $(q, [i])$ and $(q, [i])$ to $(r, [\alpha, i])$ are omitted.

Next, we consider the smallest possible size $s_{[i]}$ of our concern. Given $s_{[i]} = 2$ for all $i = 1, 2, \ldots, deg_T(u)$, we seek a necessary and sufficient condition for $T(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$, where T is a tree of diameter 4. We start by introducing some lemmas.

Remark 3.2.3 *Note that it is not necessarily true that* $A_2 = N_T(u)$ *in a* $T^{(2)}$. *Consider* $T^{(2)}$, *where* T *is as given in Example 3.2.1. Then,* $A_2 = \{1, 2, 4\} \neq N_T(u) = \{1, 2, 3, 4\}$ *since* $[3]$ *is an end-vertex in* T.

Lemma 3.2.4 *Let* T *be a tree of diameter 4,* $G := T(s_1, s_2, \ldots, s_n)$ *and* D *be an orientation of* G *where* $d(D) = 4$. *If* $s_{[i]} = 2$ *for some* $1 \leq i \leq deg_T(u)$, *then for all* $1 \leq j \leq s_{[\alpha,i]}$ *and* $1 \leq \alpha \leq deg_T([i]) - 1$, *either* $(2, [i]) \to (j, [\alpha, i]) \to (1, [i])$ *or* $(1, [i]) \to (j, [\alpha, i]) \to (2, [i])$.

Proof: This follow from the fact that $deg^+((j, [\alpha, i])) > 0$ and $deg^-((j, [\alpha, i])) > 0$ for all $j = 1, 2, \ldots, s_{[\alpha,i]}$ so that D is a strong orientation.

$\square$

Lemma 3.2.5 *Let* T *be a tree of diameter 4,* $G := T(s_1, s_2, \ldots, s_n)$ *and* D *be an orientation of* G *where* $d(D) = 4$. *Then,* $d_D((p, [\alpha, i]), (q, [j])) = d_D((q, [j]), (p, [\alpha, i])) = 3$ *for all* $1 \leq i, j \leq deg_T(u)$, $i \neq j$, $1 \leq \alpha \leq deg_T([i]) - 1$, $1 \leq p \leq s_{[\alpha,i]}$ *and* $1 \leq q \leq s_{[j]}$.

Proof: Note that $3 = d_T([\alpha, i], [j]) \leq d_D((p, [\alpha, i]), (q, [j])) \leq d(D) = 4$. Since there is no $[\alpha, i] - [j]$ path of even length in T, there is no $(p, [\alpha, i]) - (q, [j])$ path of even length in G, in particular, no path of length 4. Hence, $d_D((p, [\alpha, i]), (q, [j])) = 3$. Similarly, $d_D((q, [j]), (p, [\alpha, i])) = 3$ may be proved.

$\square$

Lemma 3.2.6 *Let* T *be a tree of diameter 4,* $G := T(s_1, s_2, \ldots, s_n)$ *and* D *be an orientation of* G *where* $d(D) = 4$. *For some* $1 \leq i < j \leq deg_T(u)$, $1 \leq \alpha \leq deg_T([i]) - 1$, *and* $1 \leq \beta \leq deg_T([j]) - 1$, *(i) if $O((1,[\alpha,i])) = \{(1,[i])\}$ and $O((1,[\beta,j])) = \{(1,[j])\}$, then $O^u((1,[i]))$ and $O^u((1,[j]))$ are independent.*

(ii) if $I((1,[\alpha,i])) = \{(1,[j])\}$ and $I((1,[\beta,j])) = \{(1,[j])\}$, then $I^u((1,[i]))$ and $I^u((1,[j]))$ are independent.

Proof:

(i) By Lemma 3.2.5, $d_D((1,[\alpha,i]),(1,[j])) = 3$. Now, $d_D((1,[\alpha,i]),(1,[j])) = 3$ implies $d_D((1,[i]),(1,[j])) = 2$. Hence, it follows that $O^u((1,[i])) \not\subseteq O^u((1,[j]))$. A similar argument shows $O^u((1,[j])) \not\subseteq O^u((1,[i]))$.

(ii) This part follows from (i) and the Duality Lemma.

$\square$

The next theorem by Lih will be useful in shortening our proof, as we will explain in a moment.

Theorem 3.2.7 *(Lih [34])*

Let $n \in \mathbb{Z}^+$ and $Y \subseteq \mathbb{N}_n$. If $\mathscr{A}$ is an antichain of $\mathbb{N}_n$ such that $X \cap Y \neq \emptyset$ for all $X \in \mathscr{A}$, then

$$|\mathscr{A}| \leq \binom{n}{\lceil n/2 \rceil} - \binom{n-|Y|}{\lceil n/2 \rceil}.$$

We return to our aim of seeking a necessary and sufficient condition for $T(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0$, where T is a tree with diameter 4. The condition $d_D((1,[1,i]),(1,[j])) = 3 = d_D((1,[1,j]),(1,[i]))$ for all $i \neq j$ is fundamental to our proof for the "necessary" direction. As in Lemma 3.2.6, it is consequent that $\{O^u((1,[i]))|\ i \in A_2\}$ is an antichain, assuming $O((1,[1,i])) = \{(1,[i])\}$ for each $i \in A_2$. Hence, $|A_2| \leq \binom{s}{\lceil s/2 \rceil}$ by Sperner's Lemma. This bound is tight if $A_2 = N_T(u)$. And, if $A_2 \subset N_T(u)$, we invoke Lih's Theorem to obtain a tighter bound $|A_2| \leq \binom{s}{\lceil s/2 \rceil} - 1$.

Theorem 3.2.8 *Let T be a tree of diameter 4 and u be its unique central vertex. Suppose $A_2 \neq \emptyset$, and $A_{\geq 3} = \emptyset$. Then,*

$$G := T(s_1, s_2, \ldots, s_n) \in \mathscr{C}_0 \iff \begin{cases} |A_2| \leq \binom{s}{\lceil s/2 \rceil}, & \text{if } |A_2| = deg_T(u), \\ |A_2| \leq \binom{s}{\lceil s/2 \rceil} - 1, & \text{if } |A_2| < deg_T(u). \end{cases}$$

Proof:

($\Rightarrow$)

Let D be an orientation of G such that $d(D) = 4$. By Lemma 3.2.4, we may assume w.l.o.g. for each $i \in A_2$ that, $(2, [i]) \to (1, [1, i]) \to (1, [i])$. Then, $\{O^u((1, [i]))|\ i \in A_2\}$ is an antichain of $(\mathbb{N}_s, u)$ by Lemma 3.2.6. It follows from Sperner's Lemma that $|A_2| = |\{O^u((1, [i]))|\ i \in A_2\}| \leq \binom{s}{\lceil s/2 \rceil}$. Now, we are done if $A_2 = N_T(u)$.

Assume $A_2 \subset N_T(u)$ and let $i^* \in N_T(u) - A_2$. By a similar argument as in the previous paragraph, $\{I^u((2, [i]))|\ i \in A_2\}$ is also an antichain of $(\mathbb{N}_s, u)$. If $|O^u((1, [i^*]))| \geq \lceil \frac{s}{2} \rceil$, then $d_D((1, [1, i]), (1, [i^*])) = 3$ implies $O^u((1, [i])) \cap I^u((1, [i^*])) \neq \emptyset$ for all $i \in A_2$. It follows from Theorem 3.2.7 that $|A_2| = |\{O^u((1, [i]))|\ i \in A_2\}| \leq \binom{s}{\lceil s/2 \rceil} - \binom{s - |I^u((1,[i^*]))|}{\lceil s/2 \rceil} \leq \binom{s}{\lceil s/2 \rceil} - \binom{\lceil s/2 \rceil}{\lceil s/2 \rceil} = \binom{s}{\lceil s/2 \rceil} - 1$. If $|O^u((1, [i^*]))| \leq \lfloor \frac{s}{2} \rfloor$, then $d_D((1, [i^*]), (1, [1, i])) = 3$ implies $I^u((2, [i])) \cap O^u((1, [i^*])) \neq \emptyset$ for all $i \in A_2$. It follows from Theorem 3.2.7 that $|A_2| = |\{I^u((2, [i]))|\ i \in A_2\}| \leq \binom{s}{\lceil s/2 \rceil} - \binom{s - |O^u((1,[i^*]))|}{\lceil s/2 \rceil} \leq \binom{s}{\lceil s/2 \rceil} - \binom{\lceil s/2 \rceil}{\lceil s/2 \rceil} = \binom{s}{\lceil s/2 \rceil} - 1$.

On account of the last proof, it is intuitive to assign $\lfloor \frac{s}{2} \rfloor$-element subsets of $(\mathbb{N}_s, u)$ as $O^u((1, [i])) = O^u((2, [i]))$ in constructing an orientation D of G. Indeed, this is our plan if $|A_2|$ is big enough (i.e. $|A_2| \geq s$). However, there are some potential drawbacks of this approach if $|A_2|$ is small (i.e. $|A_2| < s$). For instance, consider $s = 5$ and $deg_T(u) = |A_2| = 2$. Should we have assigned $O^u((p, [1])) = \{(1, u), (2, u)\}$ and $O^u((p, [2])) = \{(1, u), (3, u)\}$ for $p = 1, 2$, then $deg^+((1, u)) = 0$ and $deg^-((j, u)) = 0$ for $j = 4, 5$. Consequently, D will not be a strong orientation. Hence, we consider cases dependent on $|A_2|$ to circumvent this problem; namely, they are Cases 1 and 3 if $A_2 = N_T(u)$, and Cases 2 and 4 if

$A_2 \subset N_T(u)$.

($\Leftarrow$)

W.l.o.g., assume $A_2 = \mathbb{N}_{|A_2|}$. Thus, it is taken that $N_T(u) - A_2 = \{|A_2| + 1, |A_2| + 2, \ldots, deg_T(u)\}$, if $A_2 \subset N_T(u)$.

Case 1. $A_2 = N_T(u)$ and $|A_2| < s$.

Define an orientation D_1 for G as follows.

(I) $(2, [i]) \to (1, [\alpha, i]) \to (1, [i]) \to (2, [\alpha, i]) \to (2, [i])$ for $i \in A_2$ and $\alpha = 1, 2, \ldots, deg_T([i]) - 1$.

(II) $(\mathbb{N}_s, u) - \{(i, u)\} \to \{(1, [i]), (2, [i])\} \to (i, u)$ for $i = 1, 2, \ldots, |A_2| - 1$.

(III) $(\mathbb{N}_s, u) - \{(k, u)|\ |A_2| \le k \le s\} \to \{(1, [|A_2|]), (2, [|A_2|])\} \to \{(k, u)|\ |A_2| \le k \le s\}$. (See Figure 3.2.3 for D_1 when $s = 5$.)

Claim 1: For all $v, w \in V(D_1)$, $d_{D_1}(v, w) \le 4$.

Subcase 1.1. $v, w \in \{(1, [\alpha, i]), (2, [\alpha, i]), (1, [i]), (2, [i])\}$ for each $i \in A_2$ and $\alpha = 1, 2, \ldots, deg_T([i]) - 1$.

This is clear since, by (I), $(2, [i]) \to (1, [\alpha, i]) \to (1, [i]) \to (2, [\alpha, i]) \to (2, [i])$ is a directed C_4.

Subcase 1.2. For each $i, j \in A_2$, $i \ne j$, each $\alpha = 1, 2, \ldots, deg_T([i]) - 1$, and each $\beta = 1, 2, \ldots, deg_T([j]) - 1$,

(i) $v = (p, [\alpha, i]), w = (q, [\beta, j])$ for $p, q = 1, 2$, or

(ii) $v = (p, [\alpha, i]), w = (q, [i])$ for $p, q = 1, 2$, or

(iii) $v = (p, [i]), w = (q, [\beta, j])$ for $p, q = 1, 2$.

If $i \ne j$, then, by (I)-(III), $(p, [\alpha, i]) \to (p, [i]) \to (i, u) \to (1, [j]) \to (2, [\beta, j])$ and $(p, [\alpha, i]) \to (p, [i]) \to (i, u) \to (2, [j]) \to (1, [\beta, j])$ for all α, β and $p = 1, 2$.

Subcase 1.3. $v = (x_1, u)$ and $w = (x_2, u)$ for $x_1 \neq x_2$ and $1 \leq x_1, x_2 \leq s$.

If $x_2 < |A_2|$, then $(x_1, u) \to (1, [x_2]) \to (x_2, u)$ by (II). If $x_1 < |A_2| \leq x_2 \leq s$, then $(x_1, u) \to (1, [|A_2|]) \to (x_2, u)$ by (III). If $|A_2| \leq x_1, x_2 \leq s$, then $(x_1, u) \to (1, [1]) \to (1, u) \to (1, [|A_2|]) \to (x_2, u)$ by (II) and (III).

Subcase 1.4. $v \in \{(1, [i]), (2, [i]), (1, [\alpha, i]), (2, [\alpha, i])\}$ for each $i \in A_2$, $\alpha = 1, 2, \ldots, deg_T([i]) - 1$, and $w = (j, u)$ for $j = 1, 2, \ldots, s$.

If $j = i$, then $(p, [\alpha, i]) \to (p, [i]) \to (j, u)$ for $p = 1, 2$, by (I) and (II). If $j \neq i$ and $j < |A_2|$, then $(p, [\alpha, i]) \to (p, [i]) \to (i, u) \to (1, [j]) \to (j, u)$ for $p = 1, 2$, by (I) and (II). If $j \neq i$ and $|A_2| \leq j \leq s$, then $(p, [\alpha, i]) \to (p, [i]) \to (i, u) \to (1, [|A_2|]) \to (j, u)$ for $p = 1, 2$, by (I)-(III).

Subcase 1.5.

$v = (j, u)$ for each $j = 1, 2, \ldots, s$, and $w \in \{(1, [i]), (2, [i]), (1, [\alpha, i]), (2, [\alpha, i])\}$ for each $i \in A_2$ and $\alpha = 1, 2, \ldots, deg_T([i]) - 1$.

If $j < |A_2|$ and $j \neq i$, or $i < |A_2| \leq j \leq s$, then $(j, u) \to (p, [i]) \to (3 - p, [\alpha, i])$ for $p = 1, 2$, by (I)-(III). If $i = j < |A_2|$, then $(j, u) \to (1, [|A_2|]) \to (|A_2|, u) \to (p, [i]) \to (3 - p, [\alpha, i])$ for $p = 1, 2$ by (I) and (III). If $i = |A_2| \leq j \leq s$, then $(j, u) \to (1, [1]) \to (1, u) \to (p, [|A_2|]) \to (3 - p, [\alpha, |A_2|])$, for $p = 1, 2$ by (I)-(III).

Subcase 1.6. $v = (p, [i])$ and $w = (q, [j])$, where $1 \leq p, q \leq 2$, $i \neq j$, and $i, j \in A_2$.

This follows from the fact that $|O^u((p, [i]))| > 0$, $|I^u((q, [j]))| > 0$, and $d_{D_1}((r_1, u), (r_2, u)) = 2$ for any $r_1 \neq r_2$ and $1 \leq r_1, r_2 \leq |A_2|$ by Subcase 1.3.

Case 2. $A_2 \subset N_T(u)$ and $|A_2| < s$.

We define an orientation D_2 for G such that $\langle V(D_2) - \{(1, [|A_2|]), (2, [|A_2|])\}\rangle_{D_2} \cong \langle V(D_2) - \{(1, [|A_2|]), (2, [|A_2|])\}\rangle_{D_1}$.

Furthermore, in D_2, we have

(IV) $(2,[|A_2|) \to (1,[\alpha,|A_2|]) \to (1,[|A_2|]) \to (2,[\alpha,|A_2|]) \to (2,[|A_2|])$ for $\alpha = 1,2,\ldots, deg_T([|A_2|]) - 1$,

(V) $(\mathbb{N}_s,u) - \{(k,u)|\ |A_2| \le k \le s-1\} \to \{(1,[|A_2|]),(2,[|A_2|])\} \to \{(k,u)|\ |A_2| \le k \le s-1\}$, and

(VI) $(\mathbb{N}_s,u) - \{(s,u)\} \to \{(1,[i]),(2,[i])\} \to (s,u)$ for all $i \in N_T(u) - A_2$.

(See Figure 3.2.4 for D_2 when $s = 5$.)

Claim 2: For all $v,w \in V(D_2)$, $d_{D_2}(v,w) \le 4$.

In view of the similarity between D_1 and D_2, it suffices to check the following subcases.

Subcase 2.1. For each $i \in A_2$, each $j \in N_T(u) - A_2$, and each $\alpha = 1,2,\ldots, deg_T([i]) - 1$,

(i) $v = (p,[\alpha,i]), w = (q,[j])$ for $p,q = 1,2$, or

(ii) $v = (p,[i]), w = (q,[j])$ for $p,q = 1,2$, or

(iii) $v = (q,[j]), w = (p,[\alpha,i])$ for $p,q = 1,2$.

(i) and (ii) follow from $(p,[\alpha,i]) \to (p,[i]) \to (i,u) \to \{(1,[j]),(2,[j])\}$ for all $p = 1,2$, by (I), (II) and (IV)-(VI). Similarly, for (iii), $\{(1,[j]),(2,[j])\} \to (s,u) \to (3-p,[i]) \to (p,[\alpha,i])$ for $p = 1,2$ by (I), (II) and (IV)-(VI).

Subcase 2.2. $v = (x_1,u)$ and $w = (x_2,u)$ for $x_1 \ne x_2$ and $1 \le x_1,x_2 \le s$.

If $x_2 \in A_2$, then $(x_1,u) \to (1,[x_2]) \to (x_2,u)$ by (II) and (V). If $x_1 \in A_2 \cup \{s\} - \{|A_2|\}$ and $|A_2| \le x_2 \le s-1$, then $(x_1,u) \to (1,[|A_2|]) \to (x_2,u)$ by (II) and (V). If $|A_2| \le x_1,x_2 \le s-1$, then $(x_1,u) \to (1,[1]) \to (1,u) \to (1,[|A_2|]) \to (x_2,u)$ by (II) and (V). If $x_2 = s$, then $(x_1,u) \to (1,[w]) \to (x_2,u)$ by (II), (V) and (VI), where w can be any element of $N_T(u) - A_2$.

Subcase 2.3. $v \in \{(1,[i]),(2,[i])\}$ for each $i \in N_T(u) - A_2$, and $w = (j,u)$ for $j = 1,2,\ldots,s$.

For $1 \leq j < |A_2|$, $\{(1,[i]),(2,[i])\} \to (s,u) \to (1,[j]) \to (j,u)$ by (II) and (VI). For $|A_2| \leq j \leq s-1$, $\{(1,[i]),(2,[i])\} \to (s,u) \to (1,[|A_2|]) \to (j,u)$ by (V) and (VI). And, of course, $\{(1,[i]),(2,[i])\} \to (j,u)$ if $j = s$ by (VI).

Subcase 2.4.

$v = (j,u)$ for each $j = 1,2,\ldots,s$, and $w \in \{(1,[i]),(2,[i])\}$ for each $i \in N_T(u) - A_2$.

By (VI), for any $1 \leq j \leq s-1$, $(j,u) \to \{(1,[i]),(2,[i])\}$. Furthermore, $(s,u) \to (1,[1]) \to (1,u) \to \{(1,[j]),(1,[j])\}$ by (II) and (VI).

Subcase 2.5. $v = (p,[i])$ and $w = (q,[j])$, where $1 \leq p,q \leq 2$, and $i,j \in N_T(u) - A_2$.

Here, it is possible that $i = j$. Note that $\{(1,[i]),(2,[i])\} \to (s,u) \to (1,[1]) \to (1,u) \to \{(1,[j]),(1,[j])\}$ by (II) and (VI).

Case 3. $A_2 = N_T(u)$ and $s \leq |A_2| \leq \binom{s}{\lceil s/2 \rceil}$.

Let Ψ_s be the set containing all $\lfloor \frac{s}{2} \rfloor$-element subsets of $(\mathbb{N}_s, u)$. In particular, denote $\psi_j \in \Psi_s$ where $\psi_j := \{(j,u),\ldots,(j+\lfloor \frac{s}{2} \rfloor - 1,u)\}$ for $j = 1,2,\ldots,s$, where the addition is taken modulo s. For $j > s$, the denotation of ψ_j is arbitrary. Define an orientation D_3 for G as follows.

(VII) $(2,[i]) \to (1,[\alpha,i]) \to (1,[i]) \to (2,[\alpha,i]) \to (2,[i])$, and

(VIII) $\bar{\psi}_i \to \{(1,[i]),(2,[i])\} \to \psi_i$,

for $i \in A_2$ and $\alpha = 1,2,\ldots,deg_T([i]) - 1$.

(See Figure 3.2.5 for D_3 when $s = 5$.)

Claim 3: For all $v, w \in V(D_3)$, $d_{D_3}(v, w) \leq 4$.

Subcase 3.1. $v, w \in \{(1, [\alpha, i]), (2, [\alpha, i]), (1, [i]), (2, [i])\}$ for each $i \in A_2$ and $\alpha = 1, 2, \ldots, deg_T([i]) - 1$.

This is clear since, by (VII), $(2, [i]) \to (1, [\alpha, i]) \to (1, [i]) \to (2, [\alpha, i]) \to (2, [i])$ is a directed C_4.

Subcase 3.2. For each $i, j \in A_2$, $i \neq j$, each $\alpha = 1, 2, \ldots, deg_T([i]) - 1$, and each $\beta = 1, 2, \ldots, deg_T([j]) - 1$,

(i) $v = (p, [\alpha, i]), w = (q, [\beta, j])$ for $p, q = 1, 2$, or

(ii) $v = (p, [\alpha, i]), w = (q, [j])$ for $p, q = 1, 2$, or

(iii) $v = (p, [i]), w = (q, [\beta, j])$ for $p, q = 1, 2$.

By (VII) and (VIII), since $O^u((p, [i])) = \psi_i \not\subseteq \psi_j = O^u((q, [j]))$, there exists a vertex $(x, u) \in \psi_i \cap \bar{\psi}_j$ such that $(p, [\alpha, i]) \to (p, [i]) \to (x, u) \to (q, [j]) \to (3 - q, [\beta, j])$ for $p, q = 1, 2$.

Subcase 3.3. $v = (r_1, u)$ and $w = (r_2, u)$ for $r_1 \neq r_2$ and $1 \leq r_1, r_2 \leq s$.

Let $t = r_1 - \lfloor \frac{s}{2} \rfloor (mod\ s)$. Since $(r_1, u) \notin \psi_{r_1+1} \cup \psi_t$, it follows that $(r_1, u) \to \{(1, [r_1 + 1]), (1, [t])\}$. Taking addition modulo s, $(1, [r_1 + 1]) \to \psi_{r_1+1} = \{(r_1 + 1, u), (r_1+2, u), \ldots, (r_1+\lfloor \frac{s}{2} \rfloor, u)\}$ and $(1, [t]) \to \psi_t = \{(t, u), (t+1, u), \ldots, (r_1-1, u)\}$ since $t + \lfloor \frac{s}{2} \rfloor - 1 = r_1 - 1 (mod\ s)$. Noting that $\psi_{r_1+1} \cup \psi_t = (\mathbb{N}_n, u) - \{(r_1, u)\}$, it follows w.l.o.g. that $d_{D_3}(v, w) = 2$.

Subcase 3.4. $v \in \{(1, [i]), (2, [i]), (1, [\alpha, i]), (2, [\alpha, i])\}$ for each $i \in A_2$ and $\alpha = 1, 2, \ldots, deg_T([i]) - 1$, and $w = (r, u)$ for $r = 1, 2, \ldots, s$.

Note that there exists some $1 \leq k \leq s$ such that $d_{D_3}(v, (k, u)) \leq 2$ by (VII) and (VIII). If $k = r$, we are done. If $k \neq r$, then $d_{D_3}((k, u), (r, u)) = 2$ by Subcase 3.3. Hence, it follows that $d_{D_3}(v, w) \leq d_{D_3}(v, (k, u)) + d_{D_3}((k, u), w) = 4$.

Subcase 3.5.

$v = (r, u)$ for $r = 1, 2, \ldots, s$ and $w \in \{(1, [i]), (2, [i]), (1, [\alpha, i]), (2, [\alpha, i])\}$ for each $i \in A_2$ and $\alpha = 1, 2, \ldots, deg_T([i]) - 1$.

Note that there exists some $1 \leq k \leq s$ such that $d_{D_3}((k, u), v) \leq 2$ by (VII) and (VIII). If $k = r$, we are done. If $k \neq r$, then $d_{D_3}((r, u), (k, u)) = 2$ by Subcase 3.3. Hence, it follows that $d_{D_3}(v, w) \leq d_{D_3}(v, (k, u)) + d_{D_3}((k, u), w) = 4$.

Subcase 3.6. $v = (p, [i])$ and $w = (q, [j])$, where $1 \leq p, q \leq 2$ and $i, j \in A_2$.

This follows from the fact that $|O^u((p, [i]))| > 0$, $|I^u((q, [j]))| > 0$, and $d_{D_3}((r_1, u), (r_2, u)) = 2$ for any $r_1 \neq r_2$ and $1 \leq r_1, r_2 \leq s$ by Subcase 3.3.

Case 4. $A_2 \subset N_T(u)$ and $s \leq |A_2| \leq \binom{s}{\lceil s/2 \rceil} - 1$. (If $s = 3$, this case does not apply, and we refer to Case 2 instead.)

Using the notations in Case 3, we define an orientation D_4 for G by making a slight modification to D_3. Noting that $|A_2| \leq \binom{s}{\lceil s/2 \rceil} - 1$, define D_4 as follows.

(VII)′ $(2, [i]) \rightarrow (1, [\alpha, i]) \rightarrow (1, [i]) \rightarrow (2, [\alpha, i]) \rightarrow (2, [i])$.

(VIII)′ $\bar{\psi}_i \rightarrow \{(1, [i]), (2, [i])\} \rightarrow \psi_i$ for $i \in A_2$ and $\alpha = 1, 2, \ldots, deg_T([i]) - 1$.

(IX)′ $\bar{\psi}_{\binom{s}{\lceil s/2 \rceil}} \rightarrow \{(1, [j]), (2, [j])\} \rightarrow \psi_{\binom{s}{\lceil s/2 \rceil}}$ for $j \in N_T(u) - A_2$.

(See Figure 3.2.6 for D_4 when $s = 5$.)

Claim 4: For all $v, w \in V(D_4)$, $d_{D_4}(v, w) \leq 4$.

In view of the similarity between D_3 and D_4, it suffices to check the following subcases.

Subcase 4.1. $v \in \{(1, [i]), (2, [i])\}$ for each $i \in N_T(u) - A_2$ and $\alpha = 1, 2, \ldots, deg_T([i]) - 1$, and $w = (r, u)$ for $r = 1, 2, \ldots, s$.

This follows from the fact that $|O^u((p, [i]))| > 0$ for $p = 1, 2$, and $d_{D_4}((r_1, u), (r_2, u)) = 2$ for any $r_1 \neq r_2$ and $1 \leq r_1, r_2 \leq s$ by Subcase 3.3.

Subcase 4.2.

$v = (r, u)$ for $r = 1, 2, \ldots, s$ and $w \in \{(1, [i]), (2, [i])\}$ for each $i \in N_T(u) - A_2$ and $\alpha = 1, 2, \ldots, deg_T([i]) - 1$.

This follows from the fact that $|I^u((p, [i]))| > 0$ for $p = 1, 2$, and $d_{D_4}((r_1, u), (r_2, u)) = 2$ for any $r_1 \neq r_2$ and $1 \leq r_1, r_2 \leq s$ by Subcase 3.3.

Subcase 4.3. $v = (p, [i])$ and $w = (q, [j])$, where $1 \leq p, q \leq 2$ and $i, j \in N_T(u) - A_2$.

This follows from the fact that $|O^u((p, [i]))| > 0$, $|I^u((q, [j]))| > 0$, and $d_{D_4}((r_1, u), (r_2, u)) = 2$ for any $r_1 \neq r_2$ and $1 \leq r_1, r_2 \leq s$ by Subcase 3.3.

Hence, $d(D_i) \leq 4$ for $i = 1, 2, 3, 4$. Notice that every vertex in D lies in a directed C_4. So, by Lemma 1.3.2, $4 = d(T) \leq \bar{d}(G) \leq max\{4, d(D)\} = 4$. Therefore, $\bar{d}(G) = 4$.

$\square$

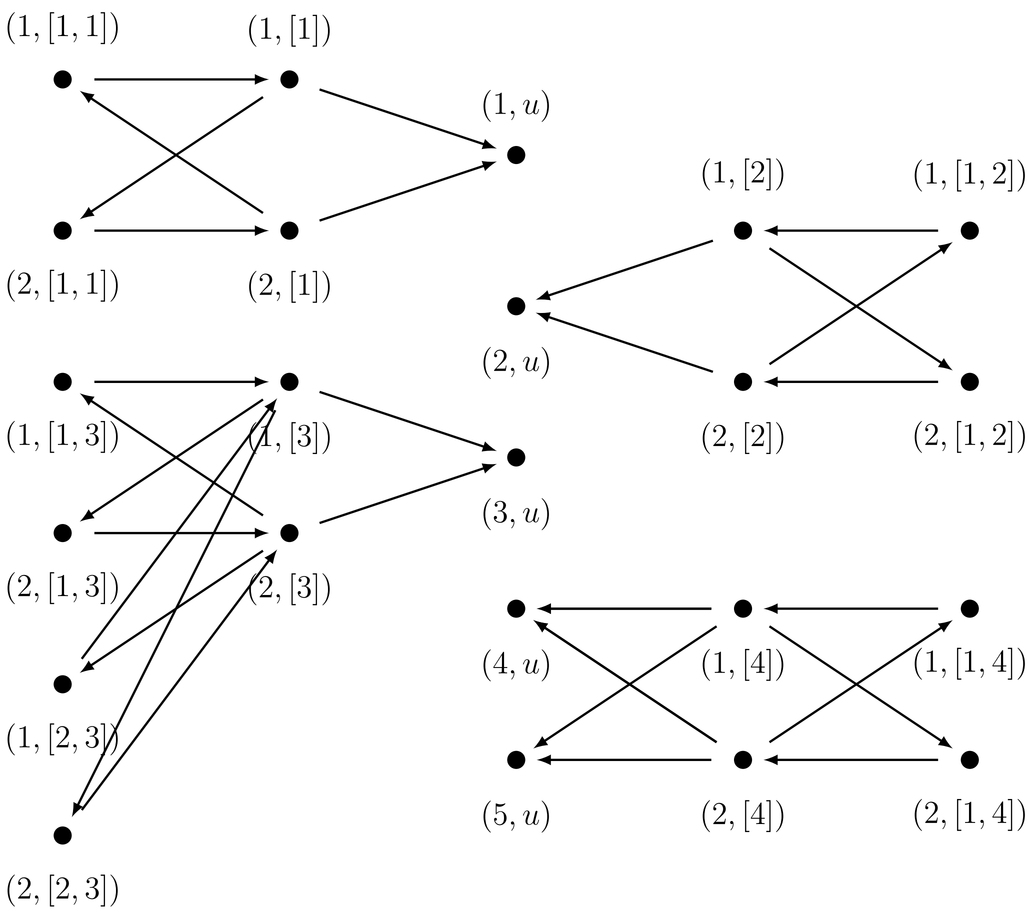

Figure 3.2.3: D_1 for Case 1. $s = 5$ and $A_2 = \{1, 2, 3, 4\}$.
For clarity, the arcs directed from (p, u) to $(q, [i])$ are omitted.

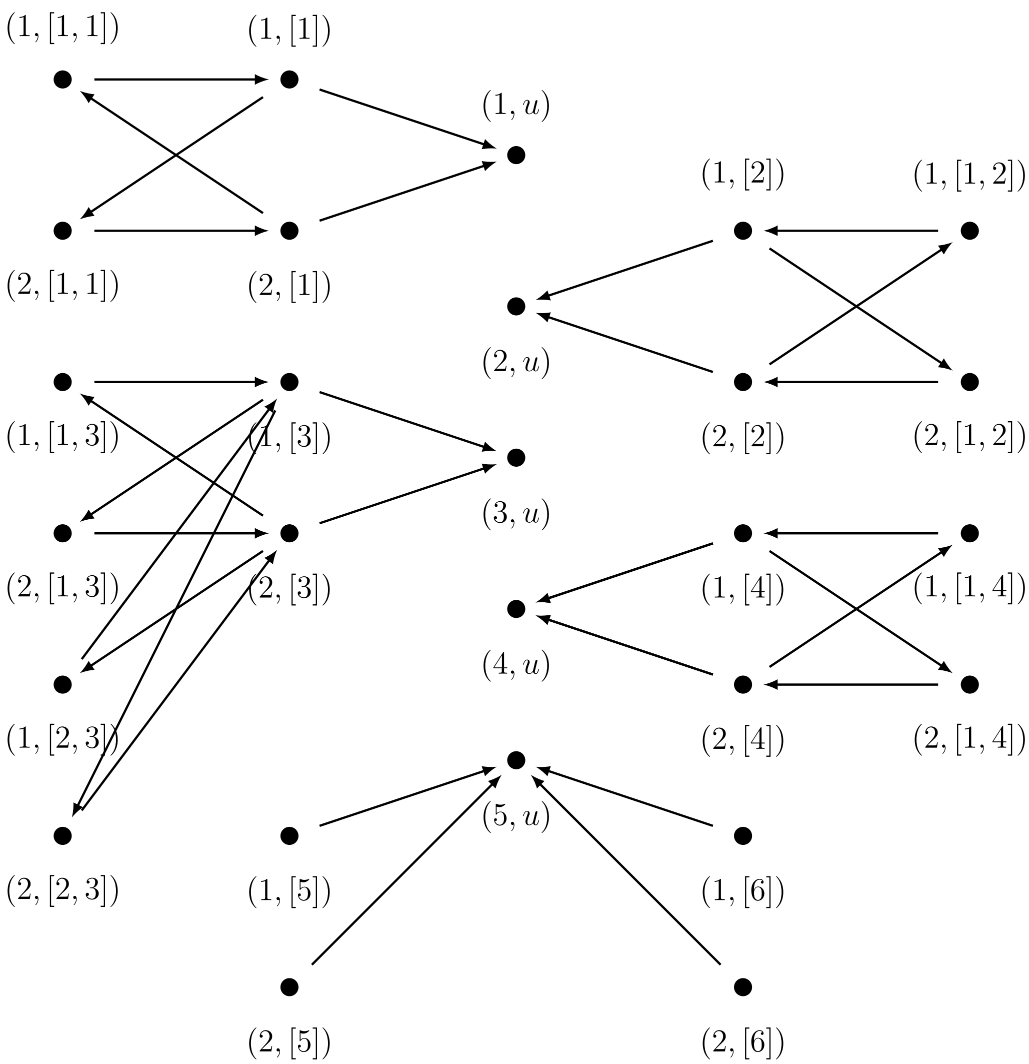

Figure 3.2.4: D_2 for Case 2. $s = 5$, $deg_T(u) = 6$, $A_2 = \{1, 2, 3, 4\}$ and $N_T(u) - A_2 = \{5, 6\}$.
For clarity, the arcs directed from (p, u) to $(q, [i])$ are omitted.

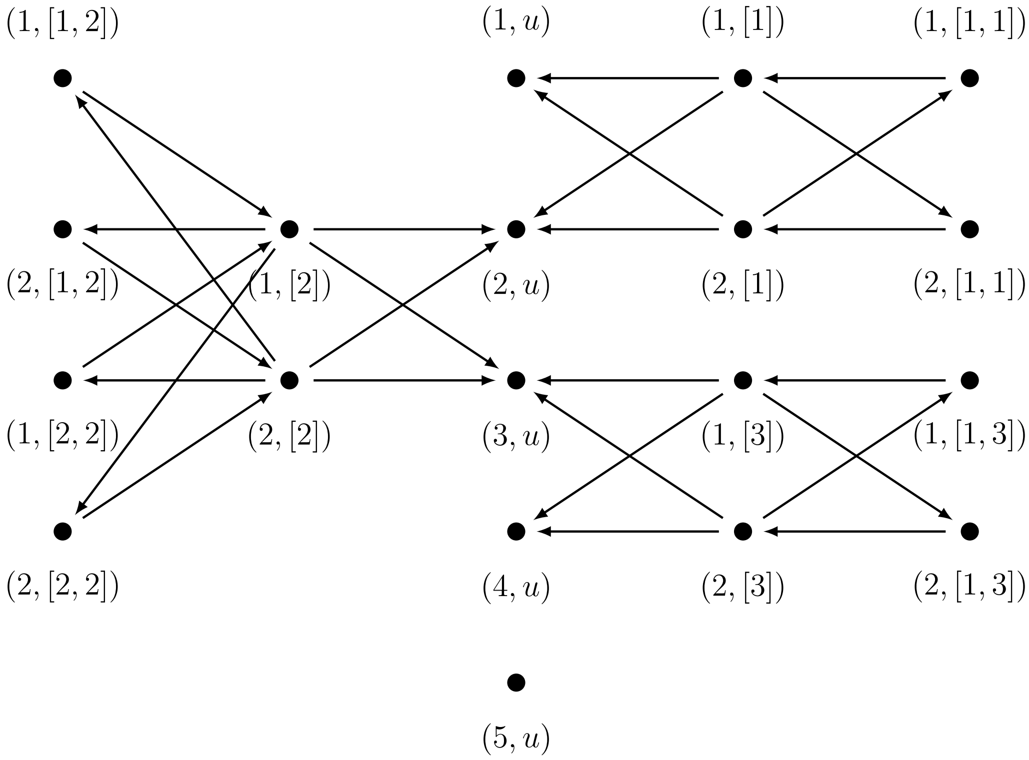

Figure 3.2.5: D_3 for Case 3. $s = 5$, and $A_2 = \{1, 2, 3, 4, 5\}$.
For clarity, we only show the vertices $[\alpha, i]$ and $[i]$ for $i = 1, 2, 3$,
and the arcs directed from (p, u) to $(q, [i])$ are omitted.

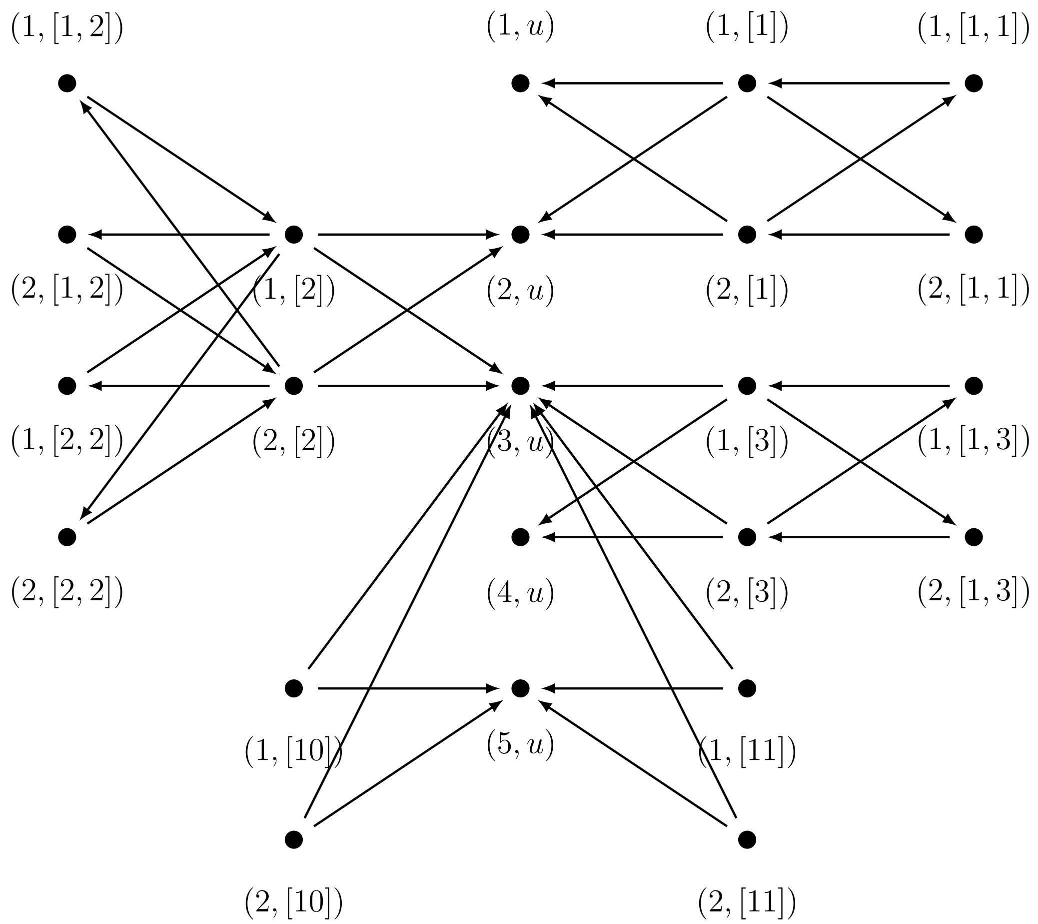

Figure 3.2.6: D_4 for Case 4. $s = 5$, $deg_T(u) = 11$, $A_2 = \{1, 2, \ldots, 9\}$ and $N_T(u) - A_2 = \{10, 11\}$.
Here, assume we define $\psi_{\binom{s}{\lceil s/2 \rceil}} = \{(3, u), (5, u)\}$.
For clarity, we only show the vertices $[\alpha, i]$ and $[i]$ for $i = 1, 2, 3, 10, 11$, and the arcs directed from (p, u) to $(q, [i])$ are omitted.

Corollary 3.2.9 *Let T be a tree of diameter 4 and u be its unique central vertex. If*

(i) $|A_{\geq 2}| \leq \binom{s}{\lceil s/2 \rceil}$ *and* $|A_2| = deg_T(u)$, *or*

(ii) $|A_{\geq 2}| \leq \binom{s}{\lceil s/2 \rceil} - 1$,

then $G := T(s_1, s_2, \ldots, s) \in \mathscr{C}_0$.

Proof:

Note that every vertex lies in a directed C_4 for each orientation D_i and $d(D_i) \leq 4$, for $i = 1, 2, 3, 4$. Thus, $\bar{d}(G) \leq max\{4, d(D_i)\}$ for $i = 1, 2, 3, 4$, by Lemma 1.3.2. Since $\bar{d}(G) \geq d(T) = 4$, it follows that $\bar{d}(G) = 4$.

□

Further research may be done to characterise the tree vertex-multiplications $T_4(s_1, s_2, \ldots, s_n)$ which belong to $\mathscr{C}_0$.

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