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δ-primary Ideals of Commutative Rings

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In this paper we investigate δ-primary ideals which unify prime ideals and primary ideals. A number of main results about prime ideals and primary ideals are extended into this general framework.

Prime ideals and primary ideals are two of the most important structures in commutative algebra. Although different from each other in many aspects, they share quite a number of similar properties as well(see [1]). However, these two structures have been treated rather differently, and all of their properties were proved separately. It is therefore natural to examine whether it is possible to have a unified approach to studying these two structures. In this short paper we introduce the notion of δ-primary ideals where δ is a mapping that assigns to each ideal I an ideal $\delta(I)$ of the same ring. Such δ-primary ideals unify the prime and primary ideals under one frame. This approach clearly reveals how similar the two structures are and how they are related to each other. In the first section, we introduce ideal expansion and define primary ideals with respect to such an expansion. Besides the familiar expansions δ_0, δ_1 and $\mathbf{B}$, we also have a new expansion M defined by means of maximal ideals. In the second section, we investigate ideal expansions satisfying some additional conditions and prove more properties of the generalized primary ideals with respect to such expansions.

In this paper, all the rings used are commutative rings with an multiplication identity and all the ring homomorphisms preserve the identity. We shall use $Id(R)$ to denote the set of all ideals of the ring R.

1. Expansions of ideals and δ-primary ideals

Recall that an ideal I of a ring R is called a *prime ideal* if $I \neq R$ and $a \cdot b \in I$ implies $a \in I$ or $b \in I$ for any $a, b \in R$.

An ideal J of R is called *primary* if $a \cdot b \in I$ and $a \notin I$ implies $b \in \sqrt{I}$, where $\sqrt{I} = \{x \in R | \exists n, x^n \in I\}$ is the root of I. Prime ideals and primary ideals play roles which are (very roughly) similar to those played by prime numbers and by prime-power numbers in elementary arithmetic. One of the distinguished properties of commutative Noetherian rings is that every ideal of such rings can be decomposed as the product of finite number of primary ideals.

Every prime ideal is primary, and the converse is not true. For more properties

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of prime and primary ideals see [1] and [3].

Definition 1.1. An expansion of ideals, or briefly an ideal expansion, is a function δ which assigns to each ideal I of a ring R to another ideal $\delta(I)$ of the same ring, such that the following conditions are satisfied :

(E1) $I \subseteq \delta(I)$, and $P \subseteq Q$ implies $\delta(P) \subseteq \delta(Q)$.

Example 1.2. (1) The identity function δ_0, where $\delta_0(I) = I$ for every $I \in Id(R)$, is an expansion of ideals.

(2) For each ideal I define $\delta_1(I) = \sqrt{I}$, the root of I. Then 1 is an expansion of ideals.

(3) The function $\mathbf{B}$ that assigns the biggest idea R to each ideal is an expansion of ideals.

(4) For each ideal P, let $M(P)$ be the intersection of all maximal ideals containing P when P is a proper ideal of the ring R, and $M(R) = R$. Then M is an expansion of ideals.

Remark 1.3. Up to now all the expansions of ideals we know satisfy the property $\delta^2 = \delta$, that is $\delta(\delta(I)) = I$ for every ideal I. Thus δ restricts to a nucleus on each preframe $Id(R)$. See [5] for the latest discussion on nuclei on Z-frames. Since this condition would not be used in our discussion, so we do not include it as a requirement.

Definition 1.4. Given an expansion δ of ideals, an ideal I of R is called δ-primary if

$$ab \in I \text{ and } a \notin I \text{ imply } b \in \delta(I)$$

for all $a, b \in R$.

Obviously the definition of δ-primary ideals can be also stated as

$$ab \in I \text{ and } a \notin \delta(P) \text{ imply } b \in I.$$

Example 1.5. An ideal I is δ_0-primary if and only if it is prime. It is δ_1-primary if and only if it is primary. And every I is $\mathbf{B}$-primary for the $\mathbf{B}$ in example 1.2.(3).

Remark 1.6. (1) Apparently, if δ and γ are two ideal expansions, and $\delta(I) \subseteq \gamma(I)$ for each ideal I, then every δ–primary ideal is also γ primary. Thus, in particular, a prime ideal is δ-primary for every δ.

(2) Given two ideal expansions δ' and δ'', define $\delta(I) = \delta'(I) \cap \delta''(I)$. Then δ is also an ideal expansion. Generally, the intersection of any collection of ideal expansions is an ideal expansion.

(3) Let δ be an ideal expansion. Define $E_\delta(P) = \bigcap\{J \in Id(R) | P \subseteq J, J \text{ is } \delta\text{-primary}\}$. Then E_δ is still an ideal expansion. For instance, $E_{\delta_0} = \delta_1$, $E_{\delta_1} = \delta_1$ and $E_M = M$.

Proposition 1.7. *Let $\{J_i : i \in D\}$ be a directed collection of δ-primary ideals of*

R, then the ideal $J = \cup_{i\in D} J_i$ is δ-primary.

Proof. Let $ab \in J$ and $a \notin J$. Then there is a J_i with $ab \in J_i$ but $a \notin J_i$. So $b \in \delta(J_i)$ and $\delta(J_i) \subseteq \delta(J)$ imply $b \in \delta(J)$. Hence J is δ-primary.

Thus the set of all δ-primary ideals is a direct complete poset with respect to the inclusion order. In general, the intersection of two δ-primary ideals is not δ-primary.

Lemma 1.8. *An ideal P is δ-primary if and only if for any two ideals I and J, if $IJ \subseteq P$ and $I \nsubseteq P$, then $J \subseteq \delta(P)$.*

Proof. Let P be δ-primary. Suppose $IJ \subseteq P$ and $I \nsubseteq P$, but $J \nsubseteq \delta(P)$, then we can chose $a \in I - P$ and $b \in J - \delta(P)$. Then $ab \in IJ \subseteq P$. But $a \notin P$ and $b \notin \delta(P)$. This contradicts the assumption that P is δ-primary.

Conversely if the condition is satisfied, for any two elements a and b, suppose $ab \in P$ and $a \notin P$. Then $(a)(b) \subseteq P$, and $(a) \nsubseteq P$. So $(b) \subseteq \delta(P)$. Hence $b \in (b) \subseteq \delta(P)$ implies $b \in \delta(P)$. Thus P is δ-primary.

Remark. As for the definition of δ-primary ideals, the statement "$IJ \subseteq P$ and $I \nsubseteq P$, then $J \subseteq \delta(P)$" in 1.8 Lemma can be replaced by "$IJ \subseteq P$ and $I \nsubseteq \delta(P)$, then $J \subseteq P$".

For any ideal P and any Q of a ring R, the residual division of P and Q is defined to be the ideal $P : Q = \{x \in R | xy \in P \text{ for all } y \in Q\}$.

Theorem 1.9. *Let δ be an ideal expansion. Then*

(1) *if P is a δ-primary ideal and I is an ideal with $I \nsubseteq \delta(P)$, then $P : I = P$;*

(2) *for any δ-primary ideal P and any subsete N of the ring R, $P : N$ is also δ-primary.*

Proof. (1) Clearly $P \subseteq P : I$. Also by the definition of $P : I$ we have $I(P : I) \subseteq P$. Since $I \nsubseteq \delta(P)$, by Lemma 1.4 we have $P : I \subseteq P$. Therefore $P = P : I$.

(2) Let $ab \in P : N$ and $a \notin P : N$. Then there is a $n \in N$ such that $an \notin P$. But $anb = abn \in P$. So $b \in \delta(P) \subseteq \delta(P : N)$. Hence $P : N$ is δ-primary.

Theorem 1.10. *If δ is an ideal expansion such that $\delta(I) \leq \delta_1(I)$ for every ideal I, then for any δ-primary ideal P, $\delta(P) = \delta_1(P)$.*

Proof. Let $a \in \delta_1(P)$. Then there exists k which is the least positive integer k with $a^k \in P$. If $k = 1$, then $a \in P \subseteq \delta(P)$. If $k > 1$, then $a^{k-1}a \in P$. But $a^{k-1} \notin P$, so $a \in \delta(P)$. Hence $\delta_1(P) \subseteq \delta(P)$ and $\delta(P) = \delta_1(P)$.

2. Expansions with extra properties

In this section we investigate δ-primary ideals where δ satisfy additional conditions, and prove more results with respect to such expansions.

Definition 2.1. An ideal expansion δ is intersection preserving if it satisfies

(E2) $\delta(I \cap J) = \delta(I) \cap \delta(J)$ for any $I, J \in Id(R)$.

An expansion is said to be global if for any ring homomorphism $f : R \longrightarrow S$,

(E3) $\delta(f^{-1}(I)) = f^{-1}(\delta(I))$ for all $I \in Id(S)$.

The expansions δ_0, δ_1 and $\mathbf{B}$ are both intersection preserving and global.

Lemma 2.2. *For each ideal I, let $M(I) = \cap\{H|I \subseteq H,\ H$ is a maximal ideal $\}$. Then $M(I \cap J) = M(I) \cap M(J)$ holds for any two ideals I and J.*

Proof. Let $\mathcal{H}_1 = \{H|I \cap J \subseteq H,\ H$ is a maximal ideal $\}$ and $\mathcal{H}_2 = \{H|I \subseteq H$ or $J \subseteq H,\ H$ is a maximal ideal $\}$. Then $\cap\mathcal{H}_1 = M(I \cap J)$ and $\cap\mathcal{H}_2 = M(I) \cap M(J)$. Obviously $\mathcal{H}_2 \subseteq \mathcal{H}_1$. Now if $H \in \mathcal{H}_1$, then $I \cap J \subseteq H$. But H is maximal, so is prime, hence $I \subseteq H$ or $J \subseteq H$, that is $H \in \mathcal{H}_2$. So $\mathcal{H}_1 = \mathcal{H}_2$ and $M(I \cap J) = M(I) \cap M(J)$.

It then follows that the ideal expansion M defined in Example 1.2.(3) is intersection preserving. However it is not global as is shown by the following example.

Example 2.3. Let $P = (X)$ be the principal ideal of $C[X]$ generated by X, where C is the ring of all complex numbers. Then P is a prime ideal of $C[X]$. It is easy(for instance, use Lemma 3.17 of [1]) to show that $Jac(C[X])$, the Jacboson radical of $C[X]$, is $\{0\}$. Let $F = (C[X])_P$ be the localization of $C[X]$ at P and let $f : C[X] \longrightarrow F$ be the natural ring homomorphism. Then $H = \{\frac{p}{s}|p \in P, s \in P^c\}$ is the unique maximal ideal of F(see 5.2 Lemma of [1]). Let $I = \{0\}$ be the trivial proper ideal of F. Then $f^{-1}(I) = \{0\}$. But $M(f^{-1}(I)) = M(\{0\}) = \mathrm{Jac}(C[X]) = \{0\} \neq P = f^{-1}(H) = f^{-1}(M(I))$.

Lemma 2.4. *Let δ be an intersection preserving ideal expansion. If $Q_1, \cdots, Q_n$ are δ-primary ideals of R, and $P = \delta(Q_i)$ for all i, then*

$$Q = \cap_{i=1}^{i=n} Q_i$$

is δ-primary.

Proof. If $xy \in Q$ and $x \notin Q$, then $x \notin Q_k$ for some k. But $xy \in Q \subseteq Q_k$ and Q_k is δ-primary, so $y \in \delta(Q_i)$. But $\delta(Q) = \delta(\cap_{i=1}^{i=n} Q_i) = \cap_{i=1}^{i=n} \delta(Q_i) = P = \delta(Q_k)$. Thus $y \in \delta(Q)$. So Q is δ-primary.

An element a of a ring R is called δ-nilpotent if $a \in \delta(\{0\})$. So δ_0-nilpotent element is the zero element. Also δ_1-nilpotent elements are exactly the ordinary nilpotent elements.

Theorem 2.5. *Let δ be a global expansion. An ideal I of R is δ-primary if and only if every zero-divisor of the quotient ring R/I is δ-nilpotent.*

Proof. Let I be δ-primary. If $\tilde{r} = r + I$ is a zero-divisor of R/I, then there is a $\tilde{s} = s + I \neq I$ with $\tilde{r}\tilde{s} = rs + I = I$. This means that $rs \in I$ and $s \notin I$. By the assumption, I is δ-primary, so $r \in \delta(I)$, that is $\tilde{r} \in \delta(I)/I$. Let $q : R \longrightarrow R/I$ be the natural quotient homomorphism. As δ is global, we have

$$\delta(I) = \delta(q^{-1}(\{0_{R/I}\})) = q^{-1}(\delta(\{0_{R/I}\})).$$

Since q is onto, so $\delta(I)/I = q(\delta(I)) = \delta(\{0_{R/I}\})$. Hence we get $\tilde{r} \in \delta(\{0_{R/I}\})$, i.e. $\tilde{r}$ is δ-nilpotent.

Conversely suppose every zero-divisor of R/I is δ-nipotent. Let $r, s \in R$ with $rs \in I$ and $r \notin I$. Then $\tilde{r}\tilde{s} = 0_{R/I}$ and $\tilde{r} \neq 0_{R/I}$. So $\tilde{s}$ is a zero-divisor of R/I. By the assumption, $\tilde{s} \in \delta(\{0_{R/I}\}) = \delta(I)/I$. Then there is an $s' \in \delta(I)$ such that $s - s' \in I$. So $s - s'$ is in $\delta(I)$ also. It follows that $s = (s - s') + s' = \delta(I)$. Hence I is δ-primary.

Corollary. *An ideal I of R is prime if and only if R/I has only $0_{R/I}$ as its zero divisor, equivalently, if R/I is an integral domain.*
Also, I is a primary ideal if and only if every zero divisor of R/I is nilpotent.

Lemma 2.6. *If δ is global and $f : R \longrightarrow S$ is a ring homomorphism, then for any δ-primary ideal I of S, $f^{-1}(I)$ is a δ-primary ideal of R.*

Proof. Let $a, b \in R$ with $ab \in f^{-1}(I)$. If $a \notin f^{-1}(I)$ then $f(a)f(b) \in I$ but $f(a) \notin I$. So, as I is δ-primary, $f(b) \in \delta(I)$. So $b \in f^{-1}(\delta(I)) = \delta(f^{-1}(I))$. Hence $f^{-1}(I)$ is δ-primary.

It can easily be proved that if $f : R \longrightarrow S$ is a ring homomorphism then for any ideal I that contains $ker(f)$ we have $f^{-1}(f(I)) = I$.

Proposition 2.7. *Let $f : R \longrightarrow S$ be a surjective ring homomorphism. Then an ideal I of R that contains $ker(f)$ is δ-primary iff $f(I)$ is a δ-primary ideal of S.*

Proof. If $f(I)$ is δ-primary, then by $I = f^{-1}(f(I))$ and the lemma 2.6, I is δ-primary. Now suppose I is δ-primary. If $a, b \in S$ and $ab \in f(I)$ and $a \notin f(I)$, then there are $x, y \in R$ with $f(x) = a, f(y) = b$. Then $f(xy) = f(x)f(y) = ab \in f(I)$ implies $xy \in f^{-1}(f(I)) = I$, and $f(x) = a \notin f(I)$ implies $x \notin I$. So $y \in \delta(I)$, and hence $b = f(y) \in f(\delta(I))$. Now it only needs to prove $f(\delta(I)) = \delta(f(I))$. But this follows directly from $\delta(I) = \delta(f^{-1}(f(I))) = f^{-1}(\delta(f(I)))$ and that f is surjective.

Remark. From the proof of the above lemma it follows that for a global ideal expansion δ, if $f : R \longrightarrow S$ is a surjective homomorphism and $ker(f) \subseteq I$ then $f(\delta(I)) = \delta(f(I))$. In particular, if f is an isomorphism, then $f(\delta(I)) = \delta(f(I))$ holds for every ideal I of R.

Corollary 2.8. *Let I be any ideal of R, and let J be an ideal of R containing I. Then J/I is a δ-primary ideal of R/I if and only if J is a δ-primary ideal of R.*

Remarks 2.9. Lemma 1.2 says that for every commutative ring R, the set $Spec_\delta(R)$ of all primary ideals of R is a direct complete poset. As it was pointed out in [4], for any commutative ring R, the set $Spec(R)$ of all prime ideals of R is a profinite poset, or equivalently, there is a topology τ on $Spec(R)$ so that $(Spec(R), \tau)$ is a spectral space. One question about generalized primary ideals is under which condition the poset $Spec_\delta(R)$ of all δ-primary ideals of R is profinite. In particular, we ask: for which ring R the poset of all primary ideals of a ring is profinite?

References

[1] R. Y. Sharp, Steps in Commutative Algebra, LMSST 19, 1990.

[2] P. T. Johnstone, Stone spaces, volume 3 of Cambridge studies in Advanced Mathematics. Cambridge University Press, 1982.

[3] D. G. Northecott, Ideal Theory, Cambridge University Press, Cambridge, 1953.

[4] H. A. Priestley, *Spectral Sets*, J. Pure and Applied Algebra, **94**(1994), 101-114.

[5] D. Zhao, *Nuclei on Z-frames*, Sochoo Journal of Mathematics, **22**(1996), 59-74.