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A GENERAL TRANSFORMATION FOR THETA SERIES ASSOCIATED WITH THE QUADRATIC FORM $x^2 + ky^2$

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Abstract. Using elementary techniques, we prove a general transformation for theta series associated with the quadratic form $x^2 + ky^2$. The transformation is then applied to establish several infinite families of identities involving theta series whose Fourier coefficients are interlinked.

1. Introduction

Let $q = e^{2\pi i\tau}$, where τ is any complex number in the upper half plane. We define the Dedekind eta-function as

$$\eta(\tau) = q^{1/24}\prod_{j=1}^{\infty}(1-q^j).$$

Next we define $a(n)$ and $\tilde{a}(n)$ as the Fourier coefficients of the following pair of infinite products

$$\sum_{n\geq 0} a(n)q^n = \frac{\eta(16\tau)^2\eta(32\tau)^3}{\eta(8\tau)} = q^5 + q^{13} + q^{29} + \cdots, \tag{1.1a}$$

$$\sum_{n\geq 0} \tilde{a}(n)q^n = \frac{\eta(8\tau)^3\eta(16\tau)^2}{\eta(32\tau)} = q - 3q^9 - 2q^{17} + \cdots. \tag{1.1b}$$

Hirschhorn [3] used elementary techniques to prove that the coefficients of $a(n)$ and $\tilde{a}(n)$ are interlinked in the following manner.

$$\sum_{n\geq 0} a(5n)q^n + 5\sum_{n\geq 0} a(n)q^{5n} = \sum_{n\geq 0} \tilde{a}(n)q^n, \tag{1.2a}$$

$$\sum_{n\geq 0} \tilde{a}(5n)q^n + 5\sum_{n\geq 0} \tilde{a}(n)q^{5n} = 16\sum_{n\geq 0} a(n)q^n. \tag{1.2b}$$

He remarked that a question worth investigating was whether this was an isolated phenomenon or if there were more of such examples. One of the authors [5] showed that in fact Hirschhorn's identities were special cases of a more general phenomenon. He used the theory of modular forms to generalize (1.2) from the case $p = 5$ to all primes $p \equiv 1 \pmod 4$.

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Theorem 1.1 (Theorem 2.1 of [5]). *For any prime* $p \equiv 1 \pmod 4$, *we have*

$$\sum_{n\geq 0} a(pn)q^n + p\sum_{n\geq 0} a(n)q^{pn} = \begin{cases} \tilde{a}(p)\displaystyle\sum_{n\geq 0} a(n)q^n & \text{if } p \equiv 1 \pmod 8, \\ a(p)\displaystyle\sum_{n\geq 0} \tilde{a}(n)q^n & \text{if } p \equiv 5 \pmod 8; \end{cases} \tag{1.3a}$$

$$\sum_{n\geq 0} \tilde{a}(pn)q^n + p\sum_{n\geq 0} \tilde{a}(n)q^{pn} = \begin{cases} \tilde{a}(p)\displaystyle\sum_{n\geq 0} \tilde{a}(n)q^n & \text{if } p \equiv 1 \pmod 8, \\ 16a(p)\displaystyle\sum_{n\geq 0} a(n)q^n & \text{if } p \equiv 5 \pmod 8. \end{cases} \tag{1.3b}$$

In addition, he found another pair of infinite products whose Fourier coefficients satisfied analogous relations.

Theorem 1.2 (Theorem 2.3 of [5]). *If*

$$\sum_{n\geq 0} b(n)q^n = \frac{\eta(8\tau)^3\eta(64\tau)^2}{\eta(32\tau)} = q^5 - 3q^{13} + 5q^{29} + \cdots, \tag{1.4a}$$

$$\sum_{n\geq 0} \tilde{b}(n)q^n = \frac{\eta(8\tau)^3\eta(32\tau)^5}{\eta(16\tau)^2\eta(64\tau)^2} = q - 3q^9 + 2q^{17} + \cdots, \tag{1.4b}$$

then for any prime $p \equiv 1 \pmod 4$, *we have*

$$\sum_{n\geq 0} b(pn)q^n + p\sum_{n\geq 0} b(n)q^{pn} = \begin{cases} \tilde{b}(p)\displaystyle\sum_{n\geq 0} b(n)q^n & \text{if } p \equiv 1 \pmod 8, \\ b(p)\displaystyle\sum_{n\geq 0} \tilde{b}(n)q^n & \text{if } p \equiv 5 \pmod 8; \end{cases} \tag{1.5a}$$

$$\sum_{n\geq 0} \tilde{b}(pn)q^n + p\sum_{n\geq 0} \tilde{b}(n)q^{pn} = \begin{cases} \tilde{b}(p)\displaystyle\sum_{n\geq 0} \tilde{b}(n)q^n & \text{if } p \equiv 1 \pmod 8, \\ 4b(p)\displaystyle\sum_{n\geq 0} b(n)q^n & \text{if } p \equiv 5 \pmod 8. \end{cases} \tag{1.5b}$$

The Jacobi triple product identity [1, p. 10] allows us to write each of the four infinite products from Theorems 1.1 and 1.2 as a theta series of the following form

$$\sum_{\substack{x\equiv x_0 \pmod A \\ y\equiv y_0 \pmod B}} f(x,y)q^{x^2+4y^2},$$

for some function $f(x,y)$. By Fermat's theorem, every prime $p \equiv 1 \pmod 4$, can always be expressed as a sum of two squares. Since one of the two squares must be even, we can write $p = \alpha^2 + 4\beta^2$. Two questions arise naturally. Can Theorems 1.1 and 1.2 be proved using elementary methods? In other words, can they be proved without appealing to the theory of modular forms? The second question is whether there exist analogous identities associated to primes of the form $p = \alpha^2 + k\beta^2$ for other values of k. In this article, we answer both questions by deriving a general transformation associated with the quadratic form $x^2 + ky^2$. With the help of this transformation, we can provide elementary proofs of Theorems 1.1 and 1.2 and many other analogous infinite families of identities. In the following, we list some of the more striking examples.

Theorem 1.3. *If*

$$\sum_{n\geq 0} c(n)q^n = \frac{\eta(4\tau)^2\eta(24\tau)^5}{\eta(8\tau)\eta(48\tau)^2} = q - 2q^5 + 2q^{17} + \cdots, \tag{1.6a}$$

$$\sum_{n\geq 0} \tilde{c}(n)q^n = \frac{\eta(16\tau)^2\eta(24\tau)^5}{\eta(8\tau)\eta(12\tau)^2} = q^5 + q^{13} + 2q^{17} + \cdots, \tag{1.6b}$$

then for any prime $p \equiv 1 \pmod{12}$, *we have*

$$\sum_{n\geq 0} c(pn)q^n + p\sum_{n\geq 0} c(n)q^{pn} = \begin{cases} c(p)\sum_{n\geq 0} c(n)q^n & \text{if } p = \alpha^2 + 36\beta^2, \\ 16\tilde{c}(p)\sum_{n\geq 0} \tilde{c}(n)q^n & \text{if } p = 9\alpha^2 + 4\beta^2; \end{cases} \tag{1.7a}$$

$$\sum_{n\geq 0} \tilde{c}(pn)q^n + p\sum_{n\geq 0} \tilde{c}(n)q^{pn} = \begin{cases} c(p)\sum_{n\geq 0} \tilde{c}(n)q^n & \text{if } p = \alpha^2 + 36\beta^2, \\ \tilde{c}(p)\sum_{n\geq 0} c(n)q^n & \text{if } p = 9\alpha^2 + 4\beta^2. \end{cases} \tag{1.7b}$$

Theorem 1.4. *If*

$$\sum_{n\geq 0} d_1(n)q^n = \frac{\eta(8\tau)\eta(12\tau)^2\eta(32\tau)\eta(48\tau)^2}{\eta(16\tau)\eta(24\tau)}, \qquad \sum_{n\geq 0} d_2(n)q^n = \frac{\eta(4\tau)^2\eta(48\tau)^{13}}{\eta(8\tau)\eta(24\tau)^5\eta(96\tau)^5},$$

$$\sum_{n\geq 0} d_3(n)q^n = \frac{\eta(8\tau)\eta(24\tau)^5\eta(32\tau)}{\eta(12\tau)^2\eta(16\tau)}, \qquad \sum_{n\geq 0} d_4(n)q^n = \frac{\eta(8\tau)^5\eta(48\tau)^{13}}{\eta(4\tau)^2\eta(16\tau)^2\eta(24\tau)^5\eta(96\tau)^5},$$

then for any prime $p \equiv 1 \pmod{12}$, *we have*

$$\sum_{n\geq 0} d_1(pn)q^n + (-1)^{\frac{p-1}{12}}p\sum_{n\geq 0} d_1(n)q^{pn} = \begin{cases} d_4(p)\sum_{n\geq 0} d_1(n)q^n & \text{if } p = \alpha^2 + 36\beta^2, \beta \text{ even}, \\ d_2(p)\sum_{n\geq 0} d_3(n)q^n & \text{if } p = \alpha^2 + 36\beta^2, \beta \text{ odd}, \\ d_1(p)\sum_{n\geq 0} d_4(n)q^n & \text{if } p = 9\alpha^2 + 4\beta^2, \beta \text{ even}, \\ d_3(p)\sum_{n\geq 0} d_2(n)q^n & \text{if } p = 9\alpha^2 + 4\beta^2, \beta \text{ odd}; \end{cases}$$

$$\sum_{n\geq 0} d_2(pn)q^n + (-1)^{\frac{p-1}{12}}p\sum_{n\geq 0} d_2(n)q^{pn} = \begin{cases} d_2(p)\sum_{n\geq 0} d_2(n)q^n & \text{if } p = \alpha^2 + 36\beta^2, \beta \text{ even}, \\ -d_4(p)\sum_{n\geq 0} d_4(n)q^n & \text{if } p = \alpha^2 + 36\beta^2, \beta \text{ odd}, \\ 16d_3(p)\sum_{n\geq 0} d_3(n)q^n & \text{if } p = 9\alpha^2 + 4\beta^2, \beta \text{ even}, \\ -16d_1(p)\sum_{n\geq 0} d_1(n)q^n & \text{if } p = 9\alpha^2 + 4\beta^2, \beta \text{ odd}; \end{cases}$$

$$\sum_{n\geq 0} d_3(pn)q^n + (-1)^{\frac{p-1}{12}} p \sum_{n\geq 0} d_3(n)q^{pn} = \begin{cases} d_2(p)\sum_{n\geq 0} d_3(n)q^n & \text{if } p=\alpha^2+36\beta^2, \beta \textit{ even},\\ d_4(p)\sum_{n\geq 0} d_1(n)q^n & \text{if } p=\alpha^2+36\beta^2, \beta \textit{ odd},\\ d_3(p)\sum_{n\geq 0} d_2(n)q^n & \text{if } p=9\alpha^2+4\beta^2, \beta \textit{ even},\\ d_1(p)\sum_{n\geq 0} d_4(n)q^n & \text{if } p=9\alpha^2+4\beta^2, \beta \textit{ odd}; \end{cases}$$

$$\sum_{n\geq 0} d_4(pn)q^n + (-1)^{\frac{p-1}{12}} p \sum_{n\geq 0} d_4(n)q^{pn} = \begin{cases} d_4(p)\sum_{n\geq 0} d_4(n)q^n & \text{if } p=\alpha^2+36\beta^2, \beta \textit{ even},\\ -d_2(p)\sum_{n\geq 0} d_2(n)q^n & \text{if } p=\alpha^2+36\beta^2, \beta \textit{ odd},\\ 16d_1(p)\sum_{n\geq 0} d_1(n)q^n & \text{if } p=9\alpha^2+4\beta^2, \beta \textit{ even},\\ -16d_3(p)\sum_{n\geq 0} d_3(n)q^n & \text{if } p=9\alpha^2+4\beta^2, \beta \textit{ odd}. \end{cases}$$

Theorem 1.5. *If*

$$\sum_{n\geq 0} f(n)q^n = \frac{\eta(2\tau)^2\eta(24\tau)^5}{\eta(4\tau)\eta(48\tau)^2} = q - 2q^3 + 2q^9 - 2q^{19} - 5q^{25} + \cdots, \tag{1.8a}$$

$$\sum_{n\geq 0} \tilde{f}(n)q^n = \frac{\eta(12\tau)^5\eta(16\tau)^2}{\eta(6\tau)^2\eta(8\tau)} = q^3 + 2q^9 + q^{11} + 2q^{17} + q^{27} + \cdots, \tag{1.8b}$$

then for any prime $p \equiv 1$ *or* $3 \pmod 8$ *and* $p \neq 3$, *we have*

(1.9a)

$$\sum_{n\geq 0} f(pn)q^n + \left(\frac{p}{3}\right) p \sum_{n\geq 0} f(n)q^{pn} = \begin{cases} f(p)\sum_{n\geq 0} f(n)q^n & \text{if } p=\alpha^2+18\beta^2,\\ -8\tilde{f}(p)\sum_{n\geq 0} \tilde{f}(n)q^n & \text{if } p=9\alpha^2+2\beta^2; \end{cases}$$

(1.9b)

$$\sum_{n\geq 0} \tilde{f}(pn)q^n + \left(\frac{p}{3}\right) p \sum_{n\geq 0} \tilde{f}(n)q^{pn} = \begin{cases} f(p)\sum_{n\geq 0} \tilde{f}(n)q^n & \text{if } p=\alpha^2+18\beta^2,\\ \tilde{f}(p)\sum_{n\geq 0} f(n)q^n & \text{if } p=9\alpha^2+2\beta^2. \end{cases}$$

Theorem 1.6. *If*

$$\sum_{n\geq 0} g(n)q^n = \frac{\eta(3\tau)^2\eta(24\tau)^5}{\eta(6\tau)\eta(48\tau)^2} = q - 2q^4 + 2q^{13} + \cdots, \tag{1.10a}$$

$$\sum_{n\geq 0} \tilde{g}(n)q^n = \frac{\eta(6\tau)^5\eta(48\tau)^2}{\eta(3\tau)^2\eta(24\tau)} = q^4 + 2q^7 - 4q^{19} + \cdots, \tag{1.10b}$$

then for any prime $p \equiv 1 \pmod 3$*, we have*

$$\sum_{n\geq 0} g(pn)q^n + p\sum_{n\geq 0} g(n)q^{pn} = \begin{cases} g(p)\sum_{n\geq 0} g(n)q^n & \text{if } p=\alpha^2+12\beta^2, \\ 4\tilde{g}(p)\sum_{n\geq 0}\tilde{g}(n)q^n & \text{if } p=4\alpha^2+3\beta^2; \end{cases} \tag{1.11a}$$

$$\sum_{n\geq 0} \tilde{g}(pn)q^n + p\sum_{n\geq 0} \tilde{g}(n)q^{pn} = \begin{cases} g(p)\sum_{n\geq 0} \tilde{g}(n)q^n & \text{if } p=\alpha^2+12\beta^2, \\ \tilde{g}(p)\sum_{n\geq 0} g(n)q^n & \text{if } p=4\alpha^2+3\beta^2. \end{cases} \tag{1.11b}$$

Theorem 1.7. *For a prime* $p \equiv 1$ *or* $9 \pmod{20}$*, define*

$$\delta(p) = \begin{cases} 0 & \text{if } p=\alpha^2+45\beta^2, \\ 1 & \text{if } p=9\alpha^2+5\beta^2. \end{cases}$$

If

$$\sum_{n\geq 0} h_1(n)q^n = \frac{\eta(5\tau)^2\eta(24\tau)^5}{\eta(10\tau)\eta(48\tau)^2}, \qquad \sum_{n\geq 0} h_2(n)q^n = \frac{\eta(6\tau)^5\eta(80\tau)^2}{\eta(3\tau)^2\eta(40\tau)},$$

$$\sum_{n\geq 0} h_3(n)q^n = \frac{\eta(\tau)^2\eta(120\tau)^5}{\eta(2\tau)\eta(240\tau)^2}, \qquad \sum_{n\geq 0} h_4(n)q^n = \frac{\eta(16\tau)^2\eta(30\tau)^5}{\eta(8\tau)\eta(15\tau)^2},$$

then for any prime $p \equiv 1$ *or* $9 \pmod{20}$*, we have*

$$\sum_{n\geq 0} h_1(pn)q^n + (-1)^{\delta(p)}p\sum_{n\geq 0} h_1(n)q^{pn} = \begin{cases} h_1(p)\sum_{n\geq 0} h_1(n)q^n & \text{if } p=\alpha^2+45\beta^2, \beta \text{ even}, \\ 4h_2(p)\sum_{n\geq 0} h_2(n)q^n & \text{if } p=\alpha^2+45\beta^2, \beta \text{ odd}, \\ -20h_4(p)\sum_{n\geq 0} h_4(n)q^n & \text{if } p=9\alpha^2+5\beta^2, \beta \text{ even}, \\ -5h_3(p)\sum_{n\geq 0} h_3(n)q^n & \text{if } p=9\alpha^2+5\beta^2, \beta \text{ odd}; \end{cases}$$

$$\sum_{n\geq 0} h_2(pn)q^n + (-1)^{\delta(p)}p\sum_{n\geq 0} h_2(n)q^{pn} = \begin{cases} h_1(p)\sum_{n\geq 0} h_2(n)q^n & \text{if } p=\alpha^2+45\beta^2, \beta \text{ even}, \\ h_2(p)\sum_{n\geq 0} h_1(n)q^n & \text{if } p=\alpha^2+45\beta^2, \beta \text{ odd}, \\ -5h_4(p)\sum_{n\geq 0} h_3(n)q^n & \text{if } p=9\alpha^2+5\beta^2, \beta \text{ even}, \\ -5h_3(p)\sum_{n\geq 0} h_4(n)q^n & \text{if } p=9\alpha^2+5\beta^2, \beta \text{ odd}; \end{cases}$$

$$\sum_{n\geq 0} h_3(pn)q^n + (-1)^{\delta(p)} p \sum_{n\geq 0} h_3(n)q^{pn} = \begin{cases} h_1(p)\displaystyle\sum_{n\geq 0} h_3(n)q^n & \text{if } p=\alpha^2+45\beta^2, \beta \textit{ even},\\ 4h_2(p)\displaystyle\sum_{n\geq 0} h_4(n)q^n & \text{if } p=\alpha^2+45\beta^2, \beta \textit{ odd},\\ 4h_4(p)\displaystyle\sum_{n\geq 0} h_2(n)q^n & \text{if } p=9\alpha^2+5\beta^2, \beta \textit{ even},\\ h_3(p)\displaystyle\sum_{n\geq 0} h_1(n)q^n & \text{if } p=9\alpha^2+5\beta^2, \beta \textit{ odd}; \end{cases}$$

$$\sum_{n\geq 0} h_4(pn)q^n + (-1)^{\delta(p)} p \sum_{n\geq 0} h_4(n)q^{pn} = \begin{cases} h_1(p)\displaystyle\sum_{n\geq 0} h_4(n)q^n & \text{if } p=\alpha^2+45\beta^2, \beta \textit{ even},\\ h_2(p)\displaystyle\sum_{n\geq 0} h_3(n)q^n & \text{if } p=\alpha^2+45\beta^2, \beta \textit{ odd},\\ h_4(p)\displaystyle\sum_{n\geq 0} h_1(n)q^n & \text{if } p=9\alpha^2+5\beta^2, \beta \textit{ even},\\ h_3(p)\displaystyle\sum_{n\geq 0} h_2(n)q^n & \text{if } p=9\alpha^2+5\beta^2, \beta \textit{ odd}. \end{cases}$$

Theorems 1.3 and 1.4 are both associated with the quadratic form x^2+4y^2 while Theorems 1.5, 1.6 and 1.7 are related to the forms x^2+2y^2, x^2+3y^2 and x^2+5y^2 respectively. In the next section, we will derive the aforementioned transformation. Details of the proofs of Theorems 1.1 and 1.6 are described in the subsequent sections. The proofs of the remaining theorems can be obtained in an analogous way and are hence omitted. In our final section, we state several generalizations of recent results by Mahadeva Naika and Gireesh [4] which can be proved with our transformation formula.

2. General Transformation

Theorem 2.1. *Let k be a positive integer and p an odd prime (distinct from k) that can be expressed as $\mu^2+k\nu^2$. Let A and B be positive integers such that $A \mid p^2-1$ and $B \mid p^2-1$. Suppose there exist integers $a>0, b>0, \alpha, \beta$ such that* $\gcd(a,b)=\gcd(\alpha,\beta)=1$,

$$p = a^2\alpha^2 + kb^2\beta^2, \tag{2.1}$$

and

$$EF = AB, \qquad E \mid p^2-1, \qquad F \mid p^2-1, \tag{2.2}$$

where

$$E = \gcd(Aa, kBb) \qquad \textit{and} \qquad F = \gcd(Ab, Ba). \tag{2.3}$$

Then the following transformation holds.

$$\sum_{\substack{x\equiv x_0 \pmod{A}\\ y\equiv y_0 \pmod{B}\\ x^2+ky^2\equiv 0 \pmod{p}}} f(x,y)q^{x^2+ky^2} = T_1 + T_2 - T_3, \tag{2.4}$$

where

$$T_1 = \sum_{\substack{m\equiv p(a\alpha x_0 - kb\beta y_0) \pmod{E} \\ n\equiv p(b\beta x_0 + a\alpha y_0) \pmod{F}}} f(a\alpha m + kb\beta n, -b\beta m + a\alpha n) q^{p(m^2+kn^2)}; \tag{2.5}$$

$$T_2 = \sum_{\substack{m\equiv p(a\alpha x_0 + kb\beta y_0) \pmod{E} \\ n\equiv p(b\beta x_0 - a\alpha y_0) \pmod{F}}} f(a\alpha m + kb\beta n, b\beta m - a\alpha n) q^{p(m^2+kn^2)}; \tag{2.6}$$

$$T_3 = \sum_{\substack{x\equiv x_0 \pmod{A} \\ y\equiv y_0 \pmod{B} \\ x\equiv y\equiv 0 \pmod{p}}} f(x,y) q^{x^2+ky^2}. \tag{2.7}$$

Proof. Let p be an odd prime that can be expressed as $\mu^2 + k\nu^2$ for some positive integer k. We rewrite $\mu = a\alpha$ and $\nu = b\beta$, so that

$$p = a^2\alpha^2 + kb^2\beta^2, \tag{2.8}$$

for some integers a, b, α, β such that $\gcd(a,b) = \gcd(\alpha,\beta) = 1$.

Since $b\beta$ is relatively prime to p, it has a multiplicative inverse modulo p. Hence there exists s such that

$$s^2 \equiv -k \pmod{p}. \tag{2.9}$$

We now fix a choice of $a > 0, b > 0, \alpha, \beta$ and s to satisfy

$$a\alpha \equiv -sb\beta \pmod{p}. \tag{2.10}$$

It is then straightforward to see that

$$sa\alpha \equiv kb\beta \pmod{p}. \tag{2.11}$$

Now consider a binary quadratic form $x^2 + ky^2$, where x and y are of the form

$$x = Ax_1 + x_0 \qquad \text{and} \qquad y = By_1 + y_0, \tag{2.12}$$

with the additional requirement that

$$A \mid p^2 - 1 \qquad \text{and} \qquad B \mid p^2 - 1. \tag{2.13}$$

We wish to make a change of variables that holds whenever p divides any quadratic form $x^2 + ky^2$ satisfying (2.12) and (2.13).

$$\sum_{\substack{x\equiv x_0 \pmod{A} \\ y\equiv y_0 \pmod{B} \\ x^2+ky^2\equiv 0 \pmod{p}}} f(x,y) q^{x^2+ky^2} = \sum_{\substack{x\equiv x_0 \pmod{A} \\ y\equiv y_0 \pmod{B} \\ x\equiv sy \pmod{p}}} f(x,y) q^{x^2+ky^2} + \sum_{\substack{x\equiv x_0 \pmod{A} \\ y\equiv y_0 \pmod{B} \\ x\equiv -sy \pmod{p}}} f(x,y) q^{x^2+ky^2} - \sum_{\substack{x\equiv x_0 \pmod{A} \\ y\equiv y_0 \pmod{B} \\ x\equiv y\equiv 0 \pmod{p}}} f(x,y) q^{x^2+ky^2}, \tag{2.14}$$

with the three sums corresponding to T_1, T_2 and T_3 respectively in the statement of the theorem. We next obtain the stated form for T_1. From

$$x \equiv sy \pmod{p}, \tag{2.15}$$

we use (2.12) to obtain

$$Ax_1 - sBy_1 \equiv -x_0 + sy_0 \pmod{p}. \tag{2.16}$$

Multiplying (2.16) by $a\alpha$ and $b\beta$ respectively and using (2.10), (2.11), we obtain

$$\begin{aligned} Aa\alpha x_1 - kBb\beta y_1 &\equiv a\alpha(-x_0 + sy_0) \\ &\equiv -a\alpha x_0 + kb\beta y_0 \pmod p \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} Ab\beta x_1 + Ba\alpha y_1 &\equiv b\beta(-x_0 + sy_0) \\ &\equiv -b\beta x_0 - a\alpha y_0 \pmod p. \end{aligned} \tag{2.18}$$

Dividing by E in (2.17) and F in (2.18), we have

$$\frac{Aa\alpha}{E}x_1 - \frac{kBb\beta}{E}y_1 = pu + \frac{1-p^2}{E}(-a\alpha x_0 + kb\beta y_0), \tag{2.19a}$$

$$\frac{Ab\beta}{F}x_1 + \frac{Ba\alpha}{F}y_1 = pv + \frac{1-p^2}{F}(-b\beta x_0 - a\alpha y_0), \tag{2.19b}$$

for some integers u, v. Solving this system, we obtain x_1 and y_1 which lead to the following

$$x = Ax_1 + x_0 = Ea\alpha u + kFb\beta v + p^2 x_0, \tag{2.20a}$$

$$y = By_1 + y_0 = -Eb\beta u + Fa\alpha v + p^2 y_0. \tag{2.20b}$$

Using the expressions above and (2.8), we can show that

$$x^2 + ky^2 = p\left(Eu + p(a\alpha x_0 - kb\beta y_0)\right)^2 + kp\left(Fv + p(b\beta x_0 + a\alpha y_0)\right)^2. \tag{2.21}$$

To reiterate, what we have done is to transform the quadratic form x^2+ky^2 (which are multiples of p) into an expression involving the variables,

$$m = Eu + p(a\alpha x_0 - kb\beta y_0) \text{ and } n = Fv + p(b\beta x_0 + a\alpha y_0). \tag{2.22}$$

The expressions for x and y from (2.20) can also be rewritten in terms of these two variables.

$$\begin{aligned} x &= a\alpha m + kb\beta n \\ &= a\alpha\left(Eu + p(a\alpha x_0 - kb\beta y_0)\right) + kb\beta\left(Fv + p(b\beta x_0 + a\alpha y_0)\right), \end{aligned} \tag{2.23a}$$

$$\begin{aligned} y &= -b\beta m - a\alpha n \\ &= -b\beta\left(Eu + p(a\alpha x_0 - kb\beta y_0)\right) + a\alpha\left(Fv + p(b\beta x_0 + a\alpha y_0)\right). \end{aligned} \tag{2.23b}$$

Replacing the variables x and y in the first sum of (2.14) completes the expression for T_1.

Likewise for T_2, we begin with

$$x \equiv -sy \pmod p \tag{2.24}$$

and multiply appropriate factors to get

$$\frac{Aa\alpha}{E}x_1 - \frac{kBb\beta}{E}y_1 = pu + \frac{1-p^2}{E}(-a\alpha x_0 - kb\beta y_0), \tag{2.25a}$$

$$\frac{Ab\beta}{F}x_1 + \frac{Ba\alpha}{F}y_1 = pv + \frac{1-p^2}{F}(-b\beta x_0 + a\alpha y_0), \tag{2.25b}$$

for some integers u, v. Solving this system for x_1 and y_1 allows us to compute

$$x = Ax_1 + x_0 = Ea\alpha u + kFb\beta v + p^2 x_0, \tag{2.26a}$$

$$y = By_1 + y_0 = Eb\beta u - Fa\alpha v + p^2 y_0. \tag{2.26b}$$

and

$$x^2 + ky^2 = p\left(Eu + p(a\alpha x_0 + kb\beta y_0)\right)^2 + kp\left(Fv + p(b\beta x_0 - a\alpha y_0)\right)^2. \tag{2.27}$$

As before, we can rewrite

$$\begin{aligned} x &= a\alpha m + kb\beta n \\ &= a\alpha\left(Eu + p(a\alpha x_0 + kb\beta y_0)\right) + kb\beta\left(Fv + p(b\beta x_0 - a\alpha y_0)\right), \end{aligned} \tag{2.28a}$$

$$\begin{aligned} y &= b\beta m - a\alpha n \\ &= b\beta\left(Eu + p(a\alpha x_0 + kb\beta y_0)\right) - a\alpha\left(Fv + p(b\beta x_0 - a\alpha y_0)\right). \end{aligned} \tag{2.28b}$$

The transformation in (2.4) is thus verified. □

3. Proof of Theorem 1.1

Having established the transformation in the previous section, we are now ready to prove Theorem 1.1. We first note that the Jacobi triple product identity [1, p. 10] allows us to prove the following results.

Lemma 3.1. *The following identities hold:*

$$\eta(8\tau)^3 = \sum_{n\in\mathbb{Z}}(4n+1)q^{(4n+1)^2} = -\sum_{n\in\mathbb{Z}}(4n+3)q^{(4n+3)^2}; \tag{3.1a}$$

$$0 = \sum_{n\in\mathbb{Z}}(4n)q^{(4n)^2} = \sum_{n\in\mathbb{Z}}(4n+2)q^{(4n+2)^2}; \tag{3.1b}$$

$$\frac{\eta(16\tau)^2}{\eta(8\tau)} = \sum_{n\in\mathbb{Z}} q^{(4n+1)^2} = \sum_{n\in\mathbb{Z}} q^{(4n+3)^2}; \tag{3.1c}$$

$$\frac{\eta(4\tau)^2}{\eta(8\tau)} = \sum_{n\in\mathbb{Z}}(-1)^n q^{4n^2} = \sum_{n\in\mathbb{Z}} q^{(4n)^2} - \sum_{n\in\mathbb{Z}} q^{(4n+2)^2}. \tag{3.1d}$$

Moreover, these identities are invariant under the transformation $n \mapsto n + k$ for any integer k.

Using (3.1a), (3.1c) and (3.1d), we can rewrite the infinite products in (1.1a) and (1.1b) respectively as

$$\sum_{n\geq 0} a(n)q^n = \sum_{\substack{x\equiv 1 \pmod 4\\ y\equiv 1 \pmod 4}} yq^{x^2+4y^2} \tag{3.2a}$$

and

$$\sum_{n\geq 0} \tilde{a}(n)q^n = \sum_{\substack{x\equiv 1 \pmod 4\\ y\in\mathbb{Z}}} (-1)^y xq^{x^2+16y^2} = \sum_{\substack{x\equiv 1 \pmod 4\\ y\equiv 0 \pmod 2}} (-1)^{y/2} xq^{x^2+4y^2}. \tag{3.2b}$$

Let p be an odd prime satisfying $p \equiv 1 \pmod 4$, thus

$$p = \alpha^2 + 4\beta^2$$

for some unique $\alpha \equiv 1 \pmod 4$. As for β, if $p \equiv 5 \pmod 8$ then β is odd and we can pick $\beta \equiv 1 \pmod 4$. With these choices of α and β, we can see from (3.2a) that

$$a(p) = \beta. \tag{3.3}$$

However, if $p \equiv 1 \pmod 8$, one can show that β is even. In this case, we choose $\beta > 0$. Since there is no constraint on the sign of the variable y in the sum (3.2b), we conclude that

$$\tilde{a}(p) = 2(-1)^{\beta/2}\alpha. \tag{3.4}$$

With these choices for α, β, we shall use Theorem 2.1 with $k = 4$, $a = b = 1$, and $A = B = 4$ to obtain

$$\sum_{\substack{x \equiv y \equiv 1 \pmod 4 \\ x^2+4y^2 \equiv 0 \pmod p}} y q^{x^2+4y^2} = T_1 + T_2 - T_3. \tag{3.5}$$

To simplify T_3, we can write $x = pm$ and $y = pn$, which means

$$T_3 = \sum_{m \equiv n \equiv 1 \pmod 4} pn q^{(pm)^2+4(pn)^2} = p \sum_{n \geq 0} a(n) q^{p^2 n}. \tag{3.6}$$

For the sum T_1 in (3.5), we have $x_0 = y_0 = 1$, $E = F = 4$ and thus

$$\begin{aligned} T_1 &= \sum_{u,v \in \mathbb{Z}} -\beta(4u + p(\alpha - 4\beta)) q^{p(4u+p(\alpha-4\beta))^2 + 4p(4v+p(\alpha+\beta))^2} \\ &\quad + \sum_{u,v \in \mathbb{Z}} \alpha(4v + p(\alpha+\beta)) q^{p(4u+p(\alpha-4\beta))^2 + 4p(4v+p(\alpha+\beta))^2} \\ &= -\beta \sum_{v \in \mathbb{Z}} q^{4p(4v+p(\alpha+\beta))^2} \left(\sum_{u \in \mathbb{Z}} (4u + p(\alpha - 4\beta)) q^{p(4u+p(\alpha-4\beta))^2} \right) \\ &\quad + \alpha \sum_{u \in \mathbb{Z}} q^{p(4u+p(\alpha-4\beta))^2} \left(\sum_{v \in \mathbb{Z}} (4v + p(\alpha+\beta)) q^{4p(4v+p(\alpha+\beta))^2} \right). \end{aligned} \tag{3.7}$$

Similarly,

$$\begin{aligned} T_2 &= \sum_{u,v \in \mathbb{Z}} \beta(4u + p(\alpha + 4\beta)) q^{p(4u+p(\alpha+4\beta))^2 + 4p(4v+p(\beta-\alpha))^2} \\ &\quad - \sum_{u,v \in \mathbb{Z}} \alpha(4v + p(\beta-\alpha)) q^{p(4u+p(\alpha+4\beta))^2 + 4p(4v+p(\beta-\alpha))^2} \\ &= \beta \sum_{v \in \mathbb{Z}} q^{4p(4v+p(\beta-\alpha))^2} \left(\sum_{u \in \mathbb{Z}} (4u + p(\alpha + 4\beta)) q^{p(4u+p(\alpha+4\beta))^2} \right) \\ &\quad - \alpha \sum_{u \in \mathbb{Z}} q^{p(4u+p(\alpha+4\beta))^2} \left(\sum_{v \in \mathbb{Z}} (4v + p(\beta-\alpha)) q^{4p(4v+p(\beta-\alpha))^2} \right). \end{aligned} \tag{3.8}$$

At this point, we need to consider separately the cases $p \equiv 1 \pmod 8$ and $p \equiv 5 \pmod 8$. In the latter case, recall that $\alpha \equiv \beta \equiv 1 \pmod 4$. We can thus use both (3.1a) and (3.1b) to simplify (3.7) and (3.8) to

$$T_1 = -\beta \left(\sum_{v \in \mathbb{Z}} q^{4p(4v+2)^2} \right) \eta(8p\tau)^3 + 0, \tag{3.9}$$

$$T_2 = \beta \left(\sum_{v \in \mathbb{Z}} q^{4p(4v)^2} \right) \eta(8p\tau)^3 + 0. \tag{3.10}$$

Consequently, the results in (3.9), (3.10), (3.6), (3.1d) and (3.3) imply

$$\begin{aligned}
\sum_{n\geq 0} a(pn)q^{pn} &= T_1 + T_2 - T_3 \\
&= -\beta \left(\sum_{v\in\mathbb{Z}} q^{4p(4v+2)^2}\right)\eta(8p\tau)^3 + \beta\left(\sum_{v\in\mathbb{Z}} q^{4p(4v)^2}\right)\eta(8p\tau)^3 \\
&\quad - p\sum_{n\geq 0} a(n)q^{p^2 n} \\
&= \beta\left(\frac{\eta(16p\tau)^2}{\eta(32p\tau)}\right)\eta(8p\tau)^3 - p\sum_{n\geq 0} a(n)q^{p^2 n} \\
&= a(p)\sum_{n\geq 0}\tilde{a}(n)q^{pn} - p\sum_{n\geq 0} a(n)q^{p^2 n}. \qquad (3.11)
\end{aligned}$$

On the other hand, when $p \equiv 1 \pmod 8$, we have $\alpha \equiv 1 \pmod 4$ and β is even. Therefore, using (3.1a) again, we have

$$T_1 = -\beta\left(\sum_{v\in\mathbb{Z}} q^{4p(4v+1)^2}\right)\eta(8p\tau)^3 + (-1)^{\beta/2}\alpha\left(\sum_{u\in\mathbb{Z}} q^{p(4u+1)^2}\right)\eta(32p\tau)^3, \qquad (3.12)$$

$$T_2 = \beta\left(\sum_{v\in\mathbb{Z}} q^{4p(4v+1)^2}\right)\eta(8p\tau)^3 + (-1)^{\beta/2}\alpha\left(\sum_{u\in\mathbb{Z}} q^{p(4u+1)^2}\right)\eta(32p\tau)^3. \qquad (3.13)$$

By virtue of (3.12), (3.13), (3.6), (3.1c) and (3.4), we obtain

$$\begin{aligned}
\sum_{n\geq 0} a(pn)q^{pn} &= -\beta\left(\sum_{v\in\mathbb{Z}} q^{4p(4v+1)^2}\right)\eta(8p\tau)^3 + (-1)^{\beta/2}\alpha\left(\sum_{u\in\mathbb{Z}} q^{p(4u+1)^2}\right)\eta(32p\tau)^3 \\
&\quad + \beta\left(\sum_{v\in\mathbb{Z}} q^{4p(4v+1)^2}\right)\eta(8p\tau)^3 + (-1)^{\beta/2}\alpha\left(\sum_{u\in\mathbb{Z}} q^{p(4u+1)^2}\right)\eta(32p\tau)^3 \\
&\quad - p\sum_{n\geq 0} a(n)q^{p^2 n} \\
&= 2(-1)^{\beta/2}\alpha\left(\frac{\eta(16p\tau)^2}{\eta(8p\tau)}\right)\eta(32p\tau)^3 - p\sum_{n\geq 0} a(n)q^{p^2 n} \\
&= \tilde{a}(p)\sum_{n\geq 0} a(n)q^{pn} - p\sum_{n\geq 0} a(n)q^{p^2 n}. \qquad (3.14)
\end{aligned}$$

Finally, replacing q^{pn} by q^n in both (3.11) and (3.14) proves (1.3a).

We now devote our attention to (1.3b). As before, we extract only the terms in (3.2b) where the power of q is divisible by p

(3.15)
$$\begin{aligned}\sum_{\substack{x\equiv 1 \pmod 4\\ y\equiv 0 \pmod 2\\ x^2+4y^2\equiv 0 \pmod p}} (-1)^{y/2}xq^{x^2+4y^2} &= \sum_{\substack{x\equiv 1 \pmod 4\\ y\equiv 0 \pmod 2\\ x\equiv sy \pmod p}} (-1)^{y/2}xq^{x^2+4y^2} + \sum_{\substack{x\equiv 1 \pmod 4\\ y\equiv 0 \pmod 2\\ x\equiv -sy \pmod p}} (-1)^{y/2}xq^{x^2+4y^2}\\ &\quad - \sum_{\substack{x\equiv 1 \pmod 4\\ y\equiv 0 \pmod 2\\ x\equiv y\equiv 0 \pmod p}} (-1)^{y/2}xq^{x^2+4y^2}\\ &= 2T_1 - T_3.\end{aligned}$$

The transformation $y \mapsto -y$ shows that the first and second sums on the right-hand side of (3.15) are equal.

Similar to (3.6), one can show that

$$T_3 = p\sum_{n\geq 0} \tilde{a}(n)q^{p^2n}. \tag{3.16}$$

For the sum T_1, we use the parameters $A = 4$, $B = 2$, $x_0 = 1$, $y_0 = 0$, $E = 4$ and $F = 2$ in Theorem 2.1

$$\begin{aligned}T_1 = &\sum_{u,v\in\mathbb{Z}} (-1)^v\alpha(4u+p\alpha)q^{p(4u+p\alpha)^2+4p(2v+p\beta)^2}\\ &+ \sum_{u,v\in\mathbb{Z}} 4(-1)^v\beta(2v+p\beta)q^{p(4u+p\alpha)^2+4p(2v+p\beta)^2}.\end{aligned} \tag{3.17}$$

As before, when $p \equiv 5 \pmod 8$, we have $\alpha \equiv \beta \equiv 1 \pmod 4$. Using (3.1a) and (3.1c), we simplify (3.17) to

(3.18)
$$\begin{aligned}T_1 &= \alpha\eta(8p\tau)^3\left(\sum_{v\in\mathbb{Z}}(-1)^vq^{4p(2v+p\beta)^2}\right)\\ &\qquad + 4\beta\left(\frac{\eta(16p\tau)^2}{\eta(8p\tau)}\right)\left(\sum_{v\in\mathbb{Z}}(-1)^v(2v+p\beta)q^{4p(2v+p\beta)^2}\right)\\ &= 0 + 4\beta\left(\frac{\eta(16p\tau)^2}{\eta(8p\tau)}\right)\left(\sum_{v\in\mathbb{Z}}(4v+p\beta)q^{4p(4v+p\beta)^2} - \sum_{v\in\mathbb{Z}}(4v+2+p\beta)q^{4p(4v+2+p\beta)^2}\right)\\ &= 8\beta\left(\frac{\eta(16p\tau)^2\eta(32p\tau)^3}{\eta(8p\tau)}\right).\end{aligned}$$

The results in (3.18), (3.16) and (3.3) give us

$$\begin{aligned}\sum_{n\geq 0}\tilde{a}(pn)q^{pn} &= 2T_1 - T_3\\ &= 16\beta\left(\frac{\eta(16p\tau)^2\eta(32p\tau)^3}{\eta(8p\tau)}\right) - p\sum_{n\geq 0}\tilde{a}(n)q^{p^2n}\\ &= 16a(p)\sum_{n\geq 0}a(n)q^{pn} - p\sum_{n\geq 0}\tilde{a}(n)q^{p^2n}.\end{aligned} \tag{3.19}$$

In the case $p \equiv 1 \pmod 8$, we have $\alpha \equiv 1 \pmod 4$ and β is even. We can use (3.1a), (3.1b) and (3.1d) to write T_1 in (3.17) as

$$\begin{aligned} T_1 &= (-1)^{\beta/2}\alpha\eta(8p\tau)^3\left(\sum_{v\in\mathbb{Z}}(-1)^{v+\beta/2}q^{4p(2v+p\beta)^2}\right)+0 \\ &= (-1)^{\beta/2}\alpha\eta(8p\tau)^3\left(\frac{\eta(16p\tau)^2}{\eta(32p\tau)}\right). \end{aligned} \tag{3.20}$$

Altogether, (3.20), (3.16) and (3.4) give us

$$\begin{aligned} \sum_{n\geq 0}\tilde{a}(pn)q^{pn} &= \tilde{a}(p)\left(\frac{\eta(8p\tau)^3\eta(16p\tau)^2}{\eta(32p\tau)}\right)-p\sum_{n\geq 0}\tilde{a}(n)q^{p^2n} \\ &= \tilde{a}(p)\sum_{n\geq 0}\tilde{a}(n)q^{pn}-p\sum_{n\geq 0}\tilde{a}(n)q^{p^2n}. \end{aligned} \tag{3.21}$$

We can now replace q^p by q in both (3.19) and (3.21) to complete the proof of (1.3b).

4. Proof of Theorem 1.6

In this section, we provide the details of the proof of Theorem 1.6.

Lemma 4.1. *The following identities hold:*

$$\frac{\eta(24\tau)^5}{\eta(48\tau)^2} = \sum_{n\in\mathbb{Z}}(6n+1)q^{(6n+1)^2}; \tag{4.1a}$$

$$\frac{\eta(3\tau)^2}{\eta(6\tau)} = \sum_{n\in\mathbb{Z}}(-1)^n q^{3n^2}; \tag{4.1b}$$

$$\frac{\eta(6\tau)^5}{\eta(3\tau)^2} = \sum_{n\in\mathbb{Z}}(-1)^n(3n+1)q^{(3n+1)^2}; \tag{4.1c}$$

$$\frac{\eta(16\tau)^2}{\eta(8\tau)} = \sum_{n\in\mathbb{Z}}q^{(4n+1)^2} = \sum_{n\in\mathbb{Z}}q^{(4n+3)^2} = \frac{1}{2}\sum_{n\in\mathbb{Z}}q^{(2n+1)^2}. \tag{4.1d}$$

Moreover, these identities are invariant under the transformation $n \mapsto n+k$ for any integer k.

Identities (4.1a) and (4.1c) are consequences of the quintuple product identity. An informative discussion on these two identities can be found in [2, Section 9.1]. The other two identities appeared in Lemma 3.1 and are re-stated here for the convenience of the reader.

Using Lemma 4.1, we can rewrite the infinite products in (1.10) as

$$\sum_{n\geq 0}g(n)q^n = \sum_{\substack{x\equiv 1 \pmod 6\\ y\in\mathbb{Z}}}(-1)^y x q^{x^2+3y^2} \tag{4.2a}$$

and

$$\sum_{n\geq 0}\tilde{g}(n)q^n = \sum_{\substack{x\equiv 1 \pmod 3\\ y\equiv 1 \pmod 4}}(-1)^{(x-1)/3} x q^{x^2+3y^2}. \tag{4.2b}$$

Any prime $p \equiv 1 \pmod 3$ can be written in the form

$$p = \mu^2 + 3\nu^2.$$

Furthermore, exactly one of μ or ν is even. In the latter case, we rewrite $p = \alpha^2 + 12\beta^2$, and it can be seen that α is odd and $3 \nmid \alpha$. Hence, we can fix $\alpha \equiv 1 \pmod 6$. Since there is no constraint on the variable y in the sum (4.2a), we observe that the coefficient of q^p is

$$g(p) = g(\alpha^2 + 3(2\beta)^2) = 2(-1)^{2\beta}\alpha = 2\alpha. \tag{4.3}$$

On the other hand, if μ is even, we have $p = 4\alpha^2 + 3\beta^2$, with β odd and $3 \nmid \alpha$. We can then choose α and β such that $\alpha \equiv 2 \pmod 3$ and $\beta \equiv 1 \pmod 4$. It follows that the coefficient of q^p in the sum (4.2b) is

$$\tilde{g}(p) = \tilde{g}((2\alpha)^2 + 3\beta^2) = (-1)^{(2\alpha-1)/3} 2\alpha = -2\alpha. \tag{4.4}$$

We shall first prove identity (1.11a). For a fixed prime $p \equiv 1 \pmod 3$, which is either of the form $p = \alpha^2 + 12\beta^2$ or $p = 4\alpha^2 + 3\beta^2$, we extract the terms in (4.2a) where the power of q is a multiple of p.

$$\begin{aligned}
&\sum_{\substack{x\equiv 1 \pmod 6\\ y\in\mathbb{Z}\\ x^2+3y^2\equiv 0 \pmod p}} (-1)^y x q^{x^2+3y^2} = \sum_{\substack{x\equiv 1 \pmod 6\\ y\in\mathbb{Z}\\ x\equiv sy \pmod p}} (-1)^y x q^{x^2+3y^2} + \sum_{\substack{x\equiv 1 \pmod 6\\ y\in\mathbb{Z}\\ x\equiv -sy \pmod p}} (-1)^y x q^{x^2+3y^2} \\
&\qquad\qquad - \sum_{\substack{x\equiv 1 \pmod 6\\ y\in\mathbb{Z}\\ x\equiv y\equiv 0 \pmod p}} (-1)^y x q^{x^2+3y^2} \qquad (4.5)\\
&\qquad = 2T_1 - T_3. \qquad (4.6)
\end{aligned}$$

In the above, the first sum is equal to the second sum via the transformation $y \mapsto -y$. T_3 can be simplified by writing $x = pm, y = pn$. Since $x \equiv 1 \pmod 6$ and $y \in \mathbb{Z}$, we have $m \equiv 1 \pmod 6$ and $n \in \mathbb{Z}$. Thus

$$T_3 = \sum_{\substack{m\equiv 1 \pmod 6\\ n\in\mathbb{Z}}} (-1)^{pn} pm q^{p^2(m^2+3n^2)} = p \sum_{n\geq 0} g(n) q^{p^2 n}. \tag{4.7}$$

To calculate T_1 for $p = \alpha^2 + 12\beta^2$, we set $k = 3$, $a = 1$, $b = 2$, $A = 6$, $B = 1$, $x_0 = 1$, $y_0 = 0$, $E = 6$, $F = 1$ in Theorem 2.1. In addition, since $p\alpha \equiv 1 \pmod 6$ and α is odd, we thus have

$$\begin{aligned}
T_1 &= \sum_{u,v\in\mathbb{Z}} (-1)^{-12\beta u+\alpha v}(\alpha(6u+p\alpha) + 6\beta(v+2p\beta)) q^{p(6u+p\alpha)^2+3p(v+2p\beta)^2} \\
&= \alpha \sum_{u\in\mathbb{Z}} (6u+p\alpha) q^{p(6u+p\alpha)^2} \sum_{v\in\mathbb{Z}} (-1)^v q^{3p(v+2p\beta)^2} \\
&\quad + 6\beta \sum_{v\in\mathbb{Z}} (-1)^v (v+2p\beta) q^{3p(v+2p\beta)^2} \sum_{u\in\mathbb{Z}} q^{p(6u+p\alpha)^2} \\
&= \alpha \sum_{u\in\mathbb{Z}} (6u+1) q^{p(6u+1)^2} \sum_{v\in\mathbb{Z}} (-1)^v q^{3pv^2} + 6\beta \sum_{v\in\mathbb{Z}} (-1)^v v q^{3pv^2} \sum_{u\in\mathbb{Z}} q^{p(6u+1)^2} \qquad (4.8)\\
&= \alpha \frac{\eta(24p\tau)^5 \eta(3p\tau)^2}{\eta(48p\tau)^2 \eta(6p\tau)}, \qquad (4.9)
\end{aligned}$$

where Lemma 4.1 is used and the sum over v in the second term of (4.8) is 0.

Combining the results in (4.6), (4.7), (4.9) and (4.3), we obtain that for $p = \alpha^2 + 12\beta^2$,

$$\sum_{n\geq 0} g(pn)q^{pn} = g(p)\frac{\eta(3p\tau)^2\eta(24p\tau)^5}{\eta(6p\tau)\eta(48p\tau)^2} - p\sum_{n\geq 0} g(n)q^{p^2n},$$

proving the first assertion of (1.11a) when q^p is replaced by q.

We now consider the second assertion of (1.11a) for $p = 4\alpha^2 + 3\beta^2$. In this case, to calculate T_1, we substitute $a = 2$, $b = 1$, $A = 6$, $B = 1$, $x_0 = 1$, $y_0 = 0$, $E = 3$, $F = 2$ in Theorem 2.1. Recall that we chose α such that $\alpha \equiv 2 \pmod 3$ which means $2p\alpha \equiv 1 \pmod 3$. We have

$$\begin{aligned}
T_1 &= \sum_{u,v\in\mathbb{Z}} (-1)^{-3\beta u+4\alpha v}(2\alpha(3u+2p\alpha)+3\beta(2v+p\beta))q^{p(3u+2p\alpha)^2+3p(2v+p\beta)^2}\\
&= 2\alpha\sum_{u\in\mathbb{Z}}(-1)^u(3u+2p\alpha)q^{p(3u+2p\alpha)^2}\sum_{v\in\mathbb{Z}}q^{3p(2v+p\beta)^2}\\
&\quad + 3\beta\sum_{v\in\mathbb{Z}}(2v+p\beta)q^{3p(2v+p\beta)^2}\sum_{u\in\mathbb{Z}}(-1)^u q^{p(3u+2p\alpha)^2}\\
&= (-1)^{(2p\alpha-1)/3}2\alpha\sum_{u\in\mathbb{Z}}(-1)^u(3u+1)q^{p(3u+1)^2}\sum_{v\in\mathbb{Z}}q^{3p(2v+1)^2} \qquad (4.10)\\
&\quad + 3\beta\sum_{v\in\mathbb{Z}}(2v+1)q^{3p(2v+1)^2}\sum_{u\in\mathbb{Z}}(-1)^u q^{p(3u+1)^2}\\
&= \tilde{g}(p)\frac{\eta(6p\tau)^5}{\eta(3p\tau)^2}\cdot 2\frac{\eta(48p\tau)^2}{\eta(24p\tau)}, \qquad (4.11)
\end{aligned}$$

where Lemma 4.1 is used and the sum over v in the second term of (4.10) reduces to 0. In addition, the coefficient in (4.11) is due to (4.4).

As a result of (4.5), (4.7) and (4.11), we have for $p = 4\alpha^2 + 3\beta^2$,

$$\sum_{n\geq 0} g(pn)q^{pn} = 4\tilde{g}(p)\frac{\eta(6p\tau)^5\eta(48p\tau)^2}{\eta(3p\tau)^2\eta(24p\tau)} - p\sum_{n\geq 0} g(n)q^{p^2n},$$

proving the second assertion of (1.11a).

We now turn to identity (1.11b). Similar to (4.5), we can write

$$\begin{aligned}
&\sum_{\substack{x\equiv 1 \ (\mathrm{mod}\ 3)\\ y\equiv 1 \ (\mathrm{mod}\ 4)\\ x^2+3y^2\equiv 0 \ (\mathrm{mod}\ p)}} (-1)^{(x-1)/3}xq^{x^2+3y^2} \qquad (4.12)\\
&= \sum_{\substack{x\equiv 1 \ (\mathrm{mod}\ 3)\\ y\equiv 1 \ (\mathrm{mod}\ 4)\\ x\equiv sy \ (\mathrm{mod}\ p)}} (-1)^{(x-1)/3}xq^{x^2+3y^2} + \sum_{\substack{x\equiv 1 \ (\mathrm{mod}\ 3)\\ y\equiv 1 \ (\mathrm{mod}\ 4)\\ x\equiv -sy \ (\mathrm{mod}\ p)}} (-1)^{(x-1)/3}xq^{x^2+3y^2}\\
&\quad - \sum_{\substack{x\equiv 1 \ (\mathrm{mod}\ 3)\\ y\equiv 1 \ (\mathrm{mod}\ 4)\\ x\equiv y\equiv 0 \ (\mathrm{mod}\ p)}} (-1)^{(x-1)/3}xq^{x^2+3y^2}\\
&= T_1 + T_2 - T_3. \qquad (4.13)
\end{aligned}$$

As before, since $p \equiv 1 \pmod 6$, we can simplify T_3 as follows

$$T_3 = \sum_{\substack{pm\equiv 1 \pmod 3\\ pn\equiv 1 \pmod 4}} (-1)^{(pm-1)/3} pm q^{p^2(m^2+3n^2)} \qquad = p\sum_{n\geq 0} \tilde{g}(n) q^{p^2 n}. \tag{4.14}$$

We now calculate T_1 for $p = \alpha^2 + 12\beta^2$, where $a = 1$ and $b = 2$. In order to proceed to utilize Theorem 2.1, we rewrite T_1 as two sums. The transformation for each of the two sum uses parameters $a = 1$, $b = 2$, $A = 6$, $B = 4$, $y_0 = 1$, $E = 6$, $F = 4$ but the former requires $x_0 = 1$ while the latter $x_0 = 4$. We have

$$\begin{aligned}
T_1 &= \sum_{\substack{x\equiv 1 \pmod 6\\ y\equiv 1 \pmod 4\\ x\equiv sy \pmod p}} (-1)^{(x-1)/3} x q^{x^2+3y^2} + \sum_{\substack{x\equiv 4 \pmod 6\\ y\equiv 1 \pmod 4\\ x\equiv sy \pmod p}} (-1)^{(x-1)/3} x q^{x^2+3y^2}\\
&= \alpha \sum_{u\in\mathbb{Z}} (6u + p(\alpha-6\beta)) q^{p(6u+p(\alpha-6\beta))^2} \sum_{v\in\mathbb{Z}} q^{3p(4v+p(2\beta+\alpha))^2} \qquad (4.15)\\
&\quad + 6\beta \sum_{v\in\mathbb{Z}} (4v + p(2\beta+\alpha)) q^{3p(4v+p(2\beta+\alpha))^2} \sum_{u\in\mathbb{Z}} q^{p(6u+p(\alpha-6\beta))^2}\\
&\quad - \alpha \sum_{u\in\mathbb{Z}} (6u + p(4\alpha-6\beta)) q^{p(6u+p(4\alpha-6\beta))^2} \sum_{v\in\mathbb{Z}} q^{3p(4v+p(8\beta+\alpha))^2}\\
&\quad - 6\beta \sum_{v\in\mathbb{Z}} (4v + p(8\beta+\alpha)) q^{3p(4v+p(8\beta+\alpha))^2} \sum_{u\in\mathbb{Z}} q^{p(6u+p(4\alpha-6\beta))^2}.
\end{aligned}$$

Repeating this process for T_2, we obtain

$$\begin{aligned}
T_2 &= \sum_{\substack{x\equiv 1 \pmod 6\\ y\equiv 1 \pmod 4\\ x\equiv -sy \pmod p}} (-1)^{(x-1)/3} x q^{x^2+3y^2} + \sum_{\substack{x\equiv 4 \pmod 6\\ y\equiv 1 \pmod 4\\ x\equiv -sy \pmod p}} (-1)^{(x-1)/3} x q^{x^2+3y^2}\\
&= \alpha \sum_{u\in\mathbb{Z}} (6u + p(\alpha+6\beta)) q^{p(6u+p(\alpha+6\beta))^2} \sum_{v\in\mathbb{Z}} q^{3p(4v+p(2\beta-\alpha))^2} \qquad (4.16)\\
&\quad + 6\beta \sum_{v\in\mathbb{Z}} (4v + p(2\beta-\alpha)) q^{3p(4v+p(2\beta-\alpha))^2} \sum_{u\in\mathbb{Z}} q^{p(6u+p(\alpha+6\beta))^2}\\
&\quad - \alpha \sum_{u\in\mathbb{Z}} (6u + p(4\alpha+6\beta)) q^{p(6u+p(4\alpha+6\beta))^2} \sum_{v\in\mathbb{Z}} q^{3p(4v+p(8\beta-\alpha))^2}\\
&\quad - 6\beta \sum_{v\in\mathbb{Z}} (4v + p(8\beta-\alpha)) q^{3p(4v+p(8\beta-\alpha))^2} \sum_{u\in\mathbb{Z}} q^{p(6u+p(4\alpha+6\beta))^2}.
\end{aligned}$$

For $p = \alpha^2 + 12\beta^2$, recall that we fixed $\alpha \equiv 1 \pmod 6$. In addition with $p(2\beta+\alpha) \equiv -p(2\beta-\alpha) \pmod 4$, one can show that

$$\begin{aligned}
T_1 + T_2 &= 2\alpha \sum_{u\in\mathbb{Z}} (6u+1) q^{p(6u+1)^2} \sum_{v\in\mathbb{Z}} q^{3p(4v+1)^2} - 2\alpha \sum_{u\in\mathbb{Z}} (6u+4) q^{p(6u+4)^2} \sum_{v\in\mathbb{Z}} q^{3p(4v+1)^2}\\
&= 2\alpha \sum_{u\in\mathbb{Z}} (-1)^u (3u+1) q^{p(3u+1)^2} \sum_{v\in\mathbb{Z}} q^{3p(4v+1)^2} \qquad (4.17)\\
&= g(p) \frac{\eta(6p\tau)^5 \eta(48p\tau)^2}{\eta(3p\tau)^2 \eta(24p\tau)},
\end{aligned}$$

where the coefficient in (4.17) is due to (4.3).

Consequently, (4.13), (4.14) and (4.17) give us

$$\sum_{n\geq 0}\tilde{g}(pn)q^{pn}=g(p)\frac{\eta(6p\tau)^5\eta(48p\tau)^2}{\eta(3p\tau)^2\eta(24p\tau)}-p\sum_{n\geq 0}\tilde{g}(n)q^{p^2n}, \tag{4.18}$$

proving the first assertion of (1.11b).

Finally, we shall prove the second assertion of (1.11b) where $p=4\alpha^2+3\beta^2$. Similar to (4.15), we rewrite T_1 as two sums and apply Theorem 2.1. This time, the parameters involved are $a=2,b=1,A=6,B=4,y_0=1,E=12,F=2$ and the former sum requires $x_0=1$ while the latter $x_0=4$. We then have

$$\begin{aligned}
T_1 &= \sum_{\substack{x\equiv 1 \pmod 6\\ y\equiv 1 \pmod 4\\ x\equiv sy \pmod p}}(-1)^{(x-1)/3}xq^{x^2+3y^2}+\sum_{\substack{x\equiv 4 \pmod 6\\ y\equiv 1 \pmod 4\\ x\equiv sy \pmod p}}(-1)^{(x-1)/3}xq^{x^2+3y^2}\\
&= 2\alpha\sum_{u\in\mathbb{Z}}(12u+p(2\alpha-3\beta))q^{p(12u+p(2\alpha-3\beta))^2}\sum_{v\in\mathbb{Z}}q^{3p(2v+p(\beta+2\alpha))^2}\\
&\quad+3\beta\sum_{v\in\mathbb{Z}}(2v+p(\beta+2\alpha))q^{3p(2v+p(\beta+\alpha))^2}\sum_{u\in\mathbb{Z}}q^{p(12u+p(2\alpha-3\beta))^2}\\
&\quad-2\alpha\sum_{u\in\mathbb{Z}}(12u+p(8\alpha-3\beta))q^{p(12u+p(8\alpha-3\beta))^2}\sum_{v\in\mathbb{Z}}q^{3p(2v+p(4\beta+2\alpha))^2}\\
&\quad-3\beta\sum_{v\in\mathbb{Z}}(2v+p(4\beta+2\alpha))q^{3p(2v+p(4\beta+2\alpha))^2}\sum_{u\in\mathbb{Z}}q^{p(12u+p(8\alpha-3\beta))^2}.
\end{aligned} \tag{4.19}$$

Repeating this process with Theorem 2.1, we arrive at

$$\begin{aligned}
T_2 &= \sum_{\substack{x\equiv 1 \pmod 6\\ y\equiv 1 \pmod 4\\ x\equiv -sy \pmod p}}(-1)^{(x-1)/3}xq^{x^2+3y^2}+\sum_{\substack{x\equiv 4 \pmod 6\\ y\equiv 1 \pmod 4\\ x\equiv -sy \pmod p}}(-1)^{(x-1)/3}xq^{x^2+3y^2}\\
&= 2\alpha\sum_{u\in\mathbb{Z}}(12u+p(2\alpha+3\beta))q^{p(12u+p(2\alpha+3\beta))^2}\sum_{v\in\mathbb{Z}}q^{3p(2v+p(\beta-2\alpha))^2}\\
&\quad+3\beta\sum_{v\in\mathbb{Z}}(2v+p(\beta-2\alpha))q^{3p(2v+p(\beta-2\alpha))^2}\sum_{u\in\mathbb{Z}}q^{p(12u+p(2\alpha+3\beta))^2}\\
&\quad-2\alpha\sum_{u\in\mathbb{Z}}(12u+p(8\alpha+3\beta))q^{p(12u+p(8\alpha+3\beta))^2}\sum_{v\in\mathbb{Z}}q^{3p(2v+p(4\beta-2\alpha))^2}\\
&\quad-3\beta\sum_{v\in\mathbb{Z}}(2v+p(4\beta-2\alpha))q^{3p(2v+p(4\beta-2\alpha))^2}\sum_{u\in\mathbb{Z}}q^{p(12u+p(8\alpha+3\beta))^2}.
\end{aligned} \tag{4.20}$$

It can be observed that the second and the last term in each of (4.19) and (4.20) reduces to 0. In addition, recall that for $p=4\alpha^2+3\beta^2$, we have fixed α, β such that $\alpha\equiv 2 \pmod 3$ and $\beta\equiv 1 \pmod 4$. One can then use elementary methods to show that

(i) if $p(2\alpha-3\beta)\equiv 1 \pmod{12}$, then $p(2\alpha+3\beta)\equiv 7 \pmod{12}$; and
(ii) if $p(2\alpha-3\beta)\equiv 7 \pmod{12}$, then $p(2\alpha+3\beta)\equiv 1 \pmod{12}$.

Moreover, we can also prove that $p(8\alpha-3\beta)\equiv 7 \pmod{12}$ and $p(8\alpha+3\beta)\equiv 1 \pmod{12}$. These results are useful in adding (4.19) and (4.20) as follows

$$\begin{aligned}T_1+T_2 &= 2\alpha\sum_{u\in\mathbb{Z}}\left((12u+1)q^{p(12u+1)^2}+(12u+7)q^{p(12u+7)^2}\right)\sum_{v\in\mathbb{Z}}\left(q^{3p(2v+1)^2}-q^{3p(2v)^2}\right)\\ &= -2\alpha\sum_{u\in\mathbb{Z}}(6u+1)q^{p(6u+1)^2}\sum_{v\in\mathbb{Z}}(-1)^v q^{3v^2}\\ &= \tilde{g}(p)\frac{\eta(24p\tau)^5\eta(3p\tau)^2}{\eta(48p\tau)^2\eta(6p\tau)},\end{aligned} \tag{4.21}$$

where the coefficient in (4.21) comes from (4.4).

With (4.13), (4.14) and (4.21), we obtain that for $p=4\alpha^2+3\beta^2$,

$$\sum_{n\ge0} g(pn)q^{pn}=\tilde{g}(p)\frac{\eta(24p\tau)^5\eta(3p\tau)^2}{\eta(48p\tau)^2\eta(6p\tau)}-p\sum_{n\ge0} g(n)q^{p^2n}, \tag{4.22}$$

proving the second assertion of (1.11b).

5. Concluding Remarks

In this article, we derived a general transformation for theta series associated with the quadratic form x^2+ky^2 and used this transformation to provide elementary proofs of many striking identities that interlinked coefficients of such theta series. It appears that there are many such identities that can be proved using our method and thus we did not attempt to be exhaustive and only presented five new infinite families of identities. Much more can be done. For example, in a recent paper by Mahadeva Naika and Gireesh [4], four identities analogous to (1.2) were proved. Using our general transformation, it is fairly straight forward to prove the following generalization.

Theorem 5.1. *Let j be a positive integer such that $p=2j+1$ is prime. If*

$$\sum_{n\ge0} s_j(n)q^n=\sum_{\substack{x\equiv1 \pmod 4\\ y\equiv1 \pmod 4}} q^{x^2+2jy^2}=\frac{\eta(16\tau)^2\eta(32j\tau)^2}{\eta(8\tau)\eta(16j\tau)}, \tag{5.1a}$$

$$\sum_{n\ge0} t_j(n)q^n=\sum_{\substack{x\equiv1 \pmod 4\\ y\equiv0 \pmod 2}} q^{x^2+2jy^2}=\frac{\eta(16\tau)^2\eta(16j\tau)^5}{\eta(8\tau)\eta(8j\tau)^2\eta(32j\tau)^2}, \tag{5.1b}$$

then

$$\sum_{n\ge0} s_j(pn)q^n+\sum_{n\ge0} s_j(n)q^{pn}=\sum_{n\ge0} t_j(n)q^n; \tag{5.2a}$$

$$\sum_{n\ge0} t_j(pn)q^n+\sum_{n\ge0} t_j(n)q^{pn}=4\sum_{n\ge0} s_j(n)q^n. \tag{5.2b}$$

Identity (5.2a) is equivalent to [4, Equation (1.1)] while identity (5.2b) is a generalization of [4, Equations (1.2) to (1.4)] which were stated and proved only for the cases $j=1,2$ and 3, corresponding to the primes $p=3,5$ and 7. Furthermore, if we keep j fixed, our technique allows us to generalize each case from the prime $p=2j+1$ to all primes that can be represented as $p=\alpha^2+2j\beta^2$. We illustrate this for the cases of $j=1,2$ and 3.

Theorem 5.2. *For any prime* $p \equiv 1$ *or* $3 \pmod 8$*, we have*

$$\sum_{n\geq 0} s_1(pn)q^n + \sum_{n\geq 0} s_1(n)q^{pn} = \begin{cases} t_1(p)\displaystyle\sum_{n\geq 0} s_1(n)q^n & \text{if } p \equiv 1 \pmod 8, \\ s_1(p)\displaystyle\sum_{n\geq 0} t_1(n)q^n & \text{if } p \equiv 3 \pmod 8; \end{cases} \tag{5.3a}$$

$$\sum_{n\geq 0} t_1(pn)q^n + \sum_{n\geq 0} t_1(n)q^{pn} = \begin{cases} t_1(p)\displaystyle\sum_{n\geq 0} t_1(n)q^n & \text{if } p \equiv 1 \pmod 8, \\ 4s_1(p)\displaystyle\sum_{n\geq 0} s_1(n)q^n & \text{if } p \equiv 3 \pmod 8. \end{cases} \tag{5.3b}$$

For any prime $p \equiv 1 \pmod 4$*, we have*

$$\sum_{n\geq 0} s_2(pn)q^n + \sum_{n\geq 0} s_2(n)q^{pn} = \begin{cases} t_2(p)\displaystyle\sum_{n\geq 0} s_2(n)q^n & \text{if } p \equiv 1 \pmod 8, \\ s_2(p)\displaystyle\sum_{n\geq 0} t_2(n)q^n & \text{if } p \equiv 5 \pmod 8; \end{cases} \tag{5.4a}$$

$$\sum_{n\geq 0} t_2(pn)q^n + \sum_{n\geq 0} t_2(n)q^{pn} = \begin{cases} t_2(p)\displaystyle\sum_{n\geq 0} t_2(n)q^n & \text{if } p \equiv 1 \pmod 8, \\ 4s_2(p)\displaystyle\sum_{n\geq 0} s_2(n)q^n & \text{if } p \equiv 5 \pmod 8. \end{cases} \tag{5.4b}$$

For any prime $p \equiv 1$ *or* $7 \pmod{24}$*, we have*

$$\sum_{n\geq 0} s_3(pn)q^n + \sum_{n\geq 0} s_3(n)q^{pn} = \begin{cases} t_3(p)\displaystyle\sum_{n\geq 0} s_3(n)q^n & \text{if } p \equiv 1 \pmod{24}, \\ s_3(p)\displaystyle\sum_{n\geq 0} t_3(n)q^n & \text{if } p \equiv 7 \pmod{24}; \end{cases} \tag{5.5a}$$

$$\sum_{n\geq 0} t_3(pn)q^n + \sum_{n\geq 0} t_3(n)q^{pn} = \begin{cases} t_3(p)\displaystyle\sum_{n\geq 0} t_3(n)q^n & \text{if } p \equiv 1 \pmod{24}, \\ 4s_3(p)\displaystyle\sum_{n\geq 0} s_3(n)q^n & \text{if } p \equiv 7 \pmod{24}. \end{cases} \tag{5.5b}$$

References

[1] B. C. Berndt, *Number theory in the spirit of Ramanujan*, Amer. Math. Soc., Providence, Rhode Island, 2006.

[2] S. Cooper, *The quintuple product identity,* Int. J. Number Theory, **2** (2006), 115–161.

[3] M. D. Hirschhorn, *Some interesting q-series identities,* Ramanujan J., **36** (2015) 297–304.

[4] M. S. Mahadeva Naika, D. S. Gireesh, *Arithmetic properties arising from Ramanujan's theta functions,* Ramanujan J., **42** (2017) 601–615.

[5] P. C. Toh, *On certain pairs of q-series identities,* Ramanujan J., **40** (2016), 359–365.

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