The Mathematics Educator
2013, Vol. 15, No. 1, 98-120

Infusing Problem Solving into Mathematics Content Course for Pre-service Secondary School Mathematics Teachers

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Abstract: This paper presents a re-design of an undergraduate mathematics content course on Introductory Differential Equations for pre-service secondary school mathematics teachers. Based on the science practical paradigm, mathematics practical lessons emphasizing problem-solving processes via the undergraduate content knowledge were embedded within the curriculum delivered through the traditional lecture-tutorial system. The pre-service teachers' performance in six mathematics practical lessons and the mathematics practical test was examined. They were able to respond to the requirements of the mathematics practical to go through the entire process of problem solving and to carry out "Look Back" at their solution: checking the correctness of their solution, offering alternative solutions, and expanding on the given problem. The use of Mathematics Practical has altered the pre-service teachers' approach in tackling mathematics problems in a positive direction.

Keyword: Mathematical problem solving; Undergraduate mathematics teaching; Mathematics practical; Polya's model

Introduction

Mathematics content courses for pre-service mathematics teachers in an undergraduate programme are typically taught by mathematicians. It is not unusual that the mathematicians are primarily concerned with the learning of the mathematics content, be it Number Theory, Algebra, or Differential Equations. The opportunity to model pedagogical methods of teaching mathematics in the background is a secondary goal. With regard to the teaching of problem solving, this seems to be a missed opportunity. If the pedagogy of problem solving can be infused into content courses, we envisage the dual benefit of stronger learning of the content and better understanding among the pre-service teachers of how to teach problem solving in the future. Toh et al. (in press) have reported an attempt to infuse the learning of problem solving in the teaching of Number Theory, and this paper deals with a similar approach for Differential Equations.

This Differential Equations course is for pre-service teachers in the BA (Education) and BSc (Education) programmes in the National Institute of Education (NIE), Singapore. The infusion involved a re-design of the course based on the paradigm of mathematics “practical.” This is similar to science practical, which is used to infuse the processes of doing science within the science curriculum, so that mathematics practical is used to infuse problem solving into the mathematics curriculum. In these practical lessons, mathematical problem solving processes are introduced by means of a practical worksheet. The worksheet takes the pre-service teachers through the problem-solving processes, engages their metacognitive control of problem solving, and encourages them to learn from the problem solving effort.

The authors have trialed this approach in a research project for secondary school students to learn problem solving (Leong et al, 2011a, 2011b; Toh, Quek & Tay, 2008). In this study, we attempted to answer these research questions:

1. Are pre-service teachers able to apply problem solving heuristics taught in the methods course in attempting to answer undergraduate mathematics problems?
2. To what extent would these pre-service teachers adopt Polya’s (1945) “Look Back” in solving mathematics problems?

Literature Review

The authors face challenges as tutors of undergraduate-level mathematics courses at NIE in helping pre-service teachers develop mathematical thinking and dispositions. These pre-service teachers tend to view mathematics primarily as a set of rules to apply. The literature on subject matter knowledge of pre-service teachers in other jurisdictions suggests that they typically enter teacher education programmes with rather narrow conceptions of mathematics as a set of rules and conventions (e.g., Ball 1990a, 1990b; Quinn 1997; Taylor 2002; Wilson & Ball, 1996). Specifically, they do not have adequate conceptual understanding of fundamental K-12 mathematics concepts (Frykholm, 1999a; Morris, 2001). Indeed, conceptual understanding is difficult when pre-service teachers view mathematics as a collection of concrete procedures (Wilson, 1994). This limitation, together with traditional teaching models that they have experienced in pre-university and university mathematics courses, becomes an obstacle to their acquisition of reform-based philosophies and practices in mathematics education (Frykholm, 1999b). The literature also strongly suggests pre-service teachers’ experiences in university mathematics courses as one main obstacle to their professional development (Ball, 1990a). Mathematics teacher education reforms at university level must challenge pre-service teachers’ perceptions of mathematics as rules and mathematics teaching as telling, which have been reinforced as the way of thinking about mathematical knowledge and mathematics education (Ma, Millan, & Wells, 2008).

To effect reforms at pre-service mathematics teacher education, it is reasonable to relate teaching of university level mathematics content courses to the pedagogy that these future teachers have to adopt. Indeed, the Mathematical Sciences Education Board (1996) of the United States pointed out that “there is increased evidence that pre-service teachers can learn about teaching mathematics from studying the practice of mathematics teaching” (p. 7). Frykholm (1999b) argued that creating opportunities for pre-service teachers to engage in concept-based mathematics discourse throughout their teacher education programs was beneficial in developing understanding of not only mathematical content knowledge but also pedagogical content knowledge. Thus, all courses in teacher education programmes have to highlight proactively various aspects of mathematics teaching and learning. In this study, we used this approach to introduce “reform” at the mathematics content course for pre-service mathematics teachers by infusing mathematical problem solving in the course. This is justifiable since the centrality of mathematical problem solving is emphasized in the Singapore mathematics curriculum, from primary to pre-university levels.

In a project undertaken by Silver et al. (2005) that focused primarily on equipping teachers to use an innovative mathematics curriculum based heavily on problem solving, teachers who participated in the project took part in solving the mathematics problems, performed case analysis of other teachers’ attempts at teaching those problems, and completed several cycles of the Lesson Study process: “selecting a target lesson, using a structured set of questions to assist in collaborative lesson planning, teaching a lesson, and discussing their lessons with colleagues” (p. 290). In another project, Leikin and Kawass (2005) also presented teachers with a problem to solve and followed this with a video showing how a pair of students solved the same problem, from incorrect approaches to finally solving it correctly. In these two projects, the researchers reported significant shifts in teachers’ practice, which included planning and expectations of students’ abilities with respect to problem solving. Thus, it is logical to begin with innovation on problem solving at the teacher education level if one desires to have successful implementation at the school level.

Research in mathematical problem solving might have reached its zeitgeist during the 1980s, so why are we focusing on it now? Schoenfeld (1992), Lester (1994), and Stacey (2005) pointed out that findings on problem solving have been less conclusive than desired. Further, although international comparative studies such as the TIMSS studies have shown that Singapore achieves a high level of competence in mathematics in schools, these studies have also noted relatively weaker performance of Singapore students on the problem solving items. This is a matter of concern because Singapore teachers have been provided with pre-service preparation or professional development in teaching problem solving. Closer scrutiny by the authors reveals that the resources used in these training courses tend to emphasize the learning of heuristics and do not focus on the mathematics content at a deep level or on the kind of mathematical thinking used by mathematicians, such as conjecturing and proving (e.g., Quek, et al., 2010; Toh, et. al., 2011). Finally, according to Ho and Hedberg (2005), teacher is one important factor in the success of implementing problem solving in Singapore classrooms. Teachers' professional development and beliefs are critical in any effort to bring about significant success in teaching problem solving and these have indeed been the subject of recent research. It is against this backdrop that the authors pursue new ways of infusing problem solving in mathematics teacher education courses.

On how problem solving can be infused in the mathematics content courses of pre-service teachers, it is essential to identify the role of mathematical problem solving in these courses. In this regard, the distinctions made by Schroeder and Lester (1989) are still widely used in the literature (e.g., Ho & Hedberg, 2005; Stacey, 2005), namely

- Teaching *for* problem solving
- Teaching *about* problem solving
- Teaching *through* problem solving

Each of these perspectives offers different affordances which, when any one aspect is left out, will result in a lack of vital elements in the overall strategy to carry out a successful problem solving agenda. Each of these conceptions is feasible within any mathematics content course:

(a) the knowledge of mathematical content built up through the years of schooling and continually expanded, serves as a resource for teaching *for* problem solving;
(b) there is a need to model and teach explicitly to students the language and strategies used in problem solving (*about* problem solving); and
(c) when there is familiarity with the problem solving processes, the problem solving approach to instruction can become a means (*through* which) to teach standard mathematical content.

Re-designing a Mathematics Content Course: Differential Equations

The first author has taught this course to Year 3 pre-service teachers, and from his experience, it is apparent that most of the pre-service teachers were resistant to following any model of problem solving in solving non-routine problems. Even the mathematically better pre-service teachers who could solve a given mathematics problem refused to make the extra effort to finally check and extend the problem or to think of alternative solutions to the problem. They were generally very concerned with the correctness of their final answers or search for solution, and they often neglected to examine the soundness of their approach or look for alternative methods. Indeed, there is room for enhancing their skills and raising their metacognitive awareness during problem solving.

In the re-design of this course, we subscribe to a model of curricular innovation that does not disregard the realistic constraints of teaching in university settings as we are interested in research that can be adopted by other teacher educators in similar situations. In particular, we consciously work within the constraints of conventional structures of university courses, such as the lecture-tutorial format and the fixed number of contact hours allocated for this course.

Mathematics practical lessons

As mentioned above, several mathematics practical lessons were designed to help pre-service teachers develop the mental habit of following through with a problem solving model. It is especially desirable that the pre-service teachers habitually put on a problem solving mindset when they are "stuck" with a problem. The Polya's model was chosen as it is well-known and highlighted in the Singapore mathematics curriculum at all levels; thus, the pre-service teachers would see the relevance of this model when they later teach in schools.

The traditional lecture-tutorial mode of delivery of the lessons consisted of 24 hours of lectures and 12 hours of tutorial lessons. This mode was modified to 16 hours of lectures, 8 hours of practical lessons, and 12 hours of tutorial. The 8-hour practical lessons adopted the mode of teaching *about* problem solving, drawing from the "resources" taught in the 16-hour lecture segment. In each practical lesson, the tutor first introduced one aspect of problem solving (see Toh, et. al., 2011b for the detailed lesson plan) and engaged the students to solve a relatively challenging problem on differential equations, based on the lecture. They were given 40 minutes to solve a given problem. The tutor then went over the solution of the problem while the pre-service teachers performed peer marking. Templates of the Mathematics Practical worksheets are shown in Figures 1 and 2. We hoped to achieve a paradigm shift in the way students looked at these relatively challenging problems which had to be done during the practical lessons (see Toh, et al., 2011b). The six problems used for the practical lessons are included in Appendix A.

Assessment

Any effort to meet the challenges of teaching mathematical problem solving would call for a curriculum that emphasizes the process (while not neglecting the product) of problem solving and an assessment strategy to match it in order to drive teaching and learning. One of the challenges in implementing a problem solving lesson is that students may lack motivation to learn it, as conventional assessment does not explicitly test the problem solving processes. We think that directly assessing the problem solving processes provides an indication of what is valued in mathematics education. Riding on the local norms that students value and work hard on what is formally assessed, we think that assessing the problem solving processes will motivate students to take problem solving seriously. Given this consideration, assessment of the mathematics practical lessons was a critical part of the re-design. The students were informed that their performance in the mathematics practical lessons constituted a significant portion of their continual assessment of the course.

Peer assessment

Prior to this Differential Equations course, the pre-service teachers had already been exposed to the Polya's model in their methods course. Before the first practical lesson, the tutor revised the stages of Polya's model and demonstrated how the stages could be applied to solve a problem in differential equations. The assessment rubric (Appendix B, see below) was introduced at the beginning of the first practical lesson. Each practical lesson was centered on one particular problem called Problem of the Day. The pre-service teachers were to assess their peers' solutions of the Problem of the Day because research has shown that any opportunity for pre-service teachers to assess their own understanding of mathematical knowledge and that of their peers could be beneficial in their early professional development (McTighe & Wiggins, 2004). This would also help them appreciate the importance of the various processes of problem solving. The practical worksheets were collected and moderated by the tutor.

Assessment rubric

The rubric used to score the practical worksheets was based on that used in our earlier study with five Singapore secondary schools (Toh et al., 2011c). The criteria for different facets of problem solving were guided by the question:

> What must students do or show to suggest that they have used Polya's approach to solve the given mathematics problems, that they have made use of heuristics, that they have exhibited "control" over the problem-solving process, and that they have checked the solution and extended the problem solved (learnt from it)? (Toh et al., 2011b, p. 23)

The rubric has four main components covering the following crucial aspects:

- Applying Polya's 4-stage approach to solving mathematics problems
- Making use of heuristics
- Exhibiting "control" during problem solving
- Checking and expanding the problem solved

The pre-service teachers were encouraged to go through the Polya's stages and to return to one of the three earlier stages when they failed to devise a plan. Those who exhibited control over the problem solving process earned marks. The categories used to code the Practical Worksheet are described in Toh, Quek, Leong, Dindyal, and Tay (2011).

Each problem was marked out of 20. Up to 70% of the total mark could be earned for a correct solution, but this fell short of distinction (75%) for the problem. The remaining 30% would be awarded for *Checking and Expanding*, an area of instruction in problem solving that has been largely unsuccessful (Silver et al., 2005). This is given special attention in the following discussion about the findings.

Participants

The entire cohort of 51 Year 3 pre-service teachers was required to participate in this study in 2012. This cohort was comparable with earlier cohorts at their entry level, namely performance in Years 1 and 2 mathematics courses.

Findings and Discussion

Overall performance

Table 1 shows the results for the six problems. The emphasis of *Checking and Expanding* was to develop the awareness that this Polya's stage involved a correct application of all the constituents: checking, finding alternative solutions, and generalization. As such, the instrument was not calibrated to differentiate fine-grained differences in the attempts at this stage.

Under "correctness of solution," the pre-service teachers were generally able to respond with the appropriate use of heuristics. Only one pre-service teacher presented an *incorrect solution without use of appropriate heuristics* for Problem Three. Problems Three and Six were the two most difficult problems as they involved mathematical proofs. Despite this, most pre-service teachers were able to "struggle" along and arrive at a correct approach.

The pre-service teachers were unfamiliar with Checking and Expanding for Problem One, and they managed to attempt Stage IV for Problems Two, Four, and Five. Many of them still did not get to Stage IV for Problems Three and Six, which involved more challenging mathematical proofs.

Table 1
Pre-service Teachers' Performance in Problems One to Six

Correctness of Solution	No. of students for Problem					
	One	Two	Three	Four	Five	Six
Completely correct solution	35	41	24	45	45	22
Partially correct with appropriate use of heuristics	15	9	13	5	6	25
Incorrect solution with appropriate use of heuristics	1	0	13	1	0	4
Incorrect solution without use of appropriate heuristics	0	0	1	0	0	0
Stage IV: Checking and Expanding	**No. of students for Problem**					
	One	Two	Three	Four	Five	Six
No attempt in Stage IV	27	0	31	3	1	11
Attempt to check reasonableness of answer	21	8	4	14	7	16
Attempt to check answer + either alternative solution or generalize the problem	3	11	13	13	7	15
Attempt to check answer + alternative solution + generalize the problem	0	11	3	21	36	9

Students' performance in Problem One

Many pre-service teachers in previous cohorts did not even attempt Problem One. We were particularly interested to see how this cohort would respond to this problem under this novel practical approach, especially as the first problem to solve. In the first lecture, the pre-service teachers were introduced to the rationale of qualitative analysis of a first order differential equation. A brief introduction of the graphical approach was given. The tutor did not give a detailed explanation or practice for sketching a slope field. In the first practical lesson, they were required to solve Problem One. The mean score was 13.5 (see Table 1), and all except four scored at least 10 marks. Most pre-service teachers had obtained at least a partially correct solution. Only one student gave an incorrect solution.

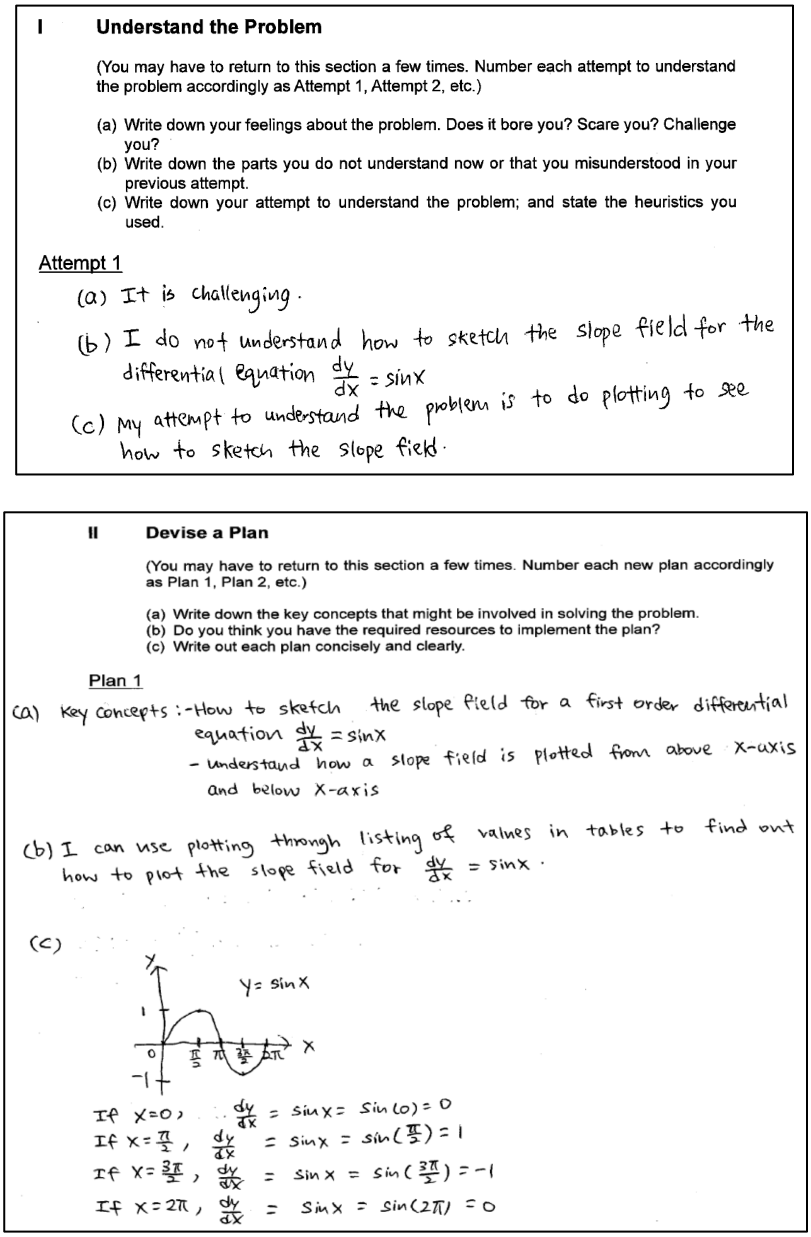
Twenty seven pre-service teachers did not check their solution or expand on the problem, and 21 of them demonstrated how the correctness of their solution was verified. Only three students provided alternative solutions or generalized the problem. Solving Problem One was the first time they were exposed to Polya's

Stage IV throughout their experience as a student or a pre-service teacher, as they informed the first author that Polya's model was not emphasized when they were school students, even though it has been in the school mathematics curriculum since 1990. Thus, it is understandable that relatively few of them embarked on Stage IV.

Three sample solutions are discussed below. These are shown in Figures 1 (Edmund), 2 (Amy), and 3 (Wilfred, all pseudonyms). These samples were selected from the lower, middle, and higher ability groups respectively. The worksheet had encouraged them to write about their problem solving. These written thoughts (e.g., descriptions of Plans 1 and 2) facilitated recall and inspection of their problem solving processes and experiences during problem solving, or at a later stage, in self-reflection. The worksheet also made them aware of metacognitive control during problem solving (e.g., checking on understanding), but not to the extent that was envisaged: Edmund did not write anything, Amy wrote something which the tutor could follow up on, if necessary, and Wilfred wrote more under the *Control* column. These differences could be due to their lack of familiarity with the vocabulary for describing metacognitive behavior or the difficulty of pausing one's thinking processes during execution in order to describe in words these processes. Whether or not this lack could be addressed with training, or more importantly, whether writing down metacognitive thoughts clearly would be of significance (e.g., contributes to the ability to think about thinking) are issues for the future. Wilfred's worksheet showed his attempt to *Check and Expand.* A pertinent question is "would Wilfred have checked and expanded on his solution, without the practical?" It may be argued that the pre-service teachers in this study wrote on the Practical Worksheets because they felt compelled to do so due to the marks allocated to the assessment. We think that this "compulsion" should be seen in a more positive light: through filling in the relevant portions of the worksheet, they were made conscious of the Polya's stages and heuristics. This is especially evident in the following sections as we take a closer look at their responses to the worksheet.

Incorrect solution with appropriate use of Heuristics: Edmund

The lecture examples on the sketching of slope fields involved simple polynomial functions, whereas this problem involved a trigonometric function. See Figure 1. In Stage I, Edmund did not understand how to solve the problem, although he knew that he was expected to plot points. In Stage II, he attempted to find the values of the slopes for several points. Together with the evidence in Stage III, it was clear that he had misunderstood the difference between the slope field given by the equation $\frac{dy}{dx} = f(x)$ and y = f(x). He appeared to encounter difficulty in reconciling the difference between the two graphs. Edmund did not proceed to Stage IV. He was awarded 8 out of 20 marks.



I Understand the Problem

(You may have to return to this section a few times. Number each attempt to understand the problem accordingly as Attempt 1, Attempt 2, etc.)

(a) Write down your feelings about the problem. Does it bore you? Scare you? Challenge you?
(b) Write down the parts you do not understand now or that you misunderstood in your previous attempt.
(c) Write down your attempt to understand the problem; and state the heuristics you used.

Attempt 1

(a) It is challenging.

(b) I do not understand how to sketch the slope field for the differential equation $\frac{dy}{dx} = \sin x$

(c) My attempt to understand the problem is to do plotting to see how to sketch the slope field.

II Devise a Plan

(You may have to return to this section a few times. Number each new plan accordingly as Plan 1, Plan 2, etc.)

(a) Write down the key concepts that might be involved in solving the problem.
(b) Do you think you have the required resources to implement the plan?
(c) Write out each plan concisely and clearly.

Plan 1

(a) Key concepts :- How to sketch the slope field for a first order differential equation $\frac{dy}{dx} = \sin x$
- understand how a slope field is plotted from above x-axis and below x-axis

(b) I can use plotting through listing of values in tables to find out how to plot the slope field for $\frac{dy}{dx} = \sin x$.

(c)

If $x=0$, $\frac{dy}{dx} = \sin x = \sin(0) = 0$
If $x = \frac{\pi}{2}$, $\frac{dy}{dx} = \sin x = \sin(\frac{\pi}{2}) = 1$
If $x = \frac{3\pi}{2}$, $\frac{dy}{dx} = \sin x = \sin(\frac{3\pi}{2}) = -1$
If $x = 2\pi$, $\frac{dy}{dx} = \sin x = \sin(2\pi) = 0$

Figure 1. Edmund's Solution of Problem One.

III Carry out the Plan

(You may have to return to this section a few times. Number each implementation accordingly as Plan 1, Plan 2, etc., or even Plan 1.1, Plan 1.2, etc. if there are two or more attempts using Plan 1.)

(a) Write down in the *Control* column, the key points where you make a decision or observation, for e.g., go back to check, try something else, look for resources, or totally abandon the plan.
(b) Write out each implementation in detail under the *Detailed Mathematical Steps* column.

Detailed Mathematical Steps	*Control*
Attempt 1	

I will use listing. of values to guide me in sketching slope field of $\frac{dy}{dx} = \sin x$

x	0	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
$\frac{dy}{dx}$	0	1	-1	0	-1

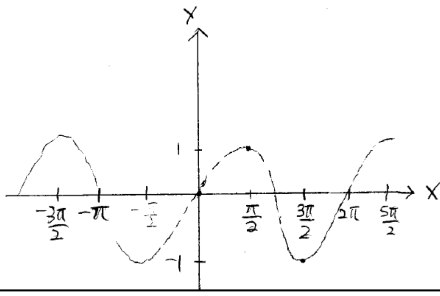


Figure 1. Edmund's Solution of Problem One (continued).

Partially correct with appropriate use of Heuristics*: *Amy

According to Amy, this problem was a little daunting (Stage I in Figure 2). In Stage II, she had apparently decided to identify the independent and dependent variables and attempted to use the heuristics of substituting different values of *x* and *y* to obtain the slopes. In Stage III, she carried out the plan and analysed the slope of the equation in the first quadrant under the *Control* Column. The translation from point plotting and the analysis of the signs in the quadrants were not completely correct as some slopes translated over from the table to the graph were incorrect. Like Edmund, she did not proceed to Stage IV. Despite not being able to provide a completely correct solution, Amy managed to use the problem solving heuristics to analyse part of the problem. She was awarded 12 marks.

I Understand the Problem

(You may have to return to this section a few times. Number each attempt to understand the problem accordingly as Attempt 1, Attempt 2, etc.)

(a) Write down your feelings about the problem. Does it bore you? Scare you? Challenge you?
(b) Write down the parts you do not understand now or that you misunderstood in your previous attempt.
(c) Write down your attempt to understand the problem; and state the heuristics you used.

Attempt 1

a) Sketching the slope field seems okay, but it is a litty scary as the equation is sinx. The question seems to be quite understandable, as there are no complicated parts.

II Devise a Plan - skill

(You may have to return to this section a few times. Number each new plan accordingly as Plan 1, Plan 2, etc.)

(a) Write down the key concepts that might be involved in solving the problem. what is $\frac{dy}{dx}$
(b) Do you think you have the required resources to implement the plan?
(c) Write out each plan concisely and clearly. tabulate values.

Plan 1

Let x be the independent variable, y be the dependent variable.

Plan 2:

Use a few values for x and y, and compute the gradient $\frac{dy}{dx}$ for the x and y values

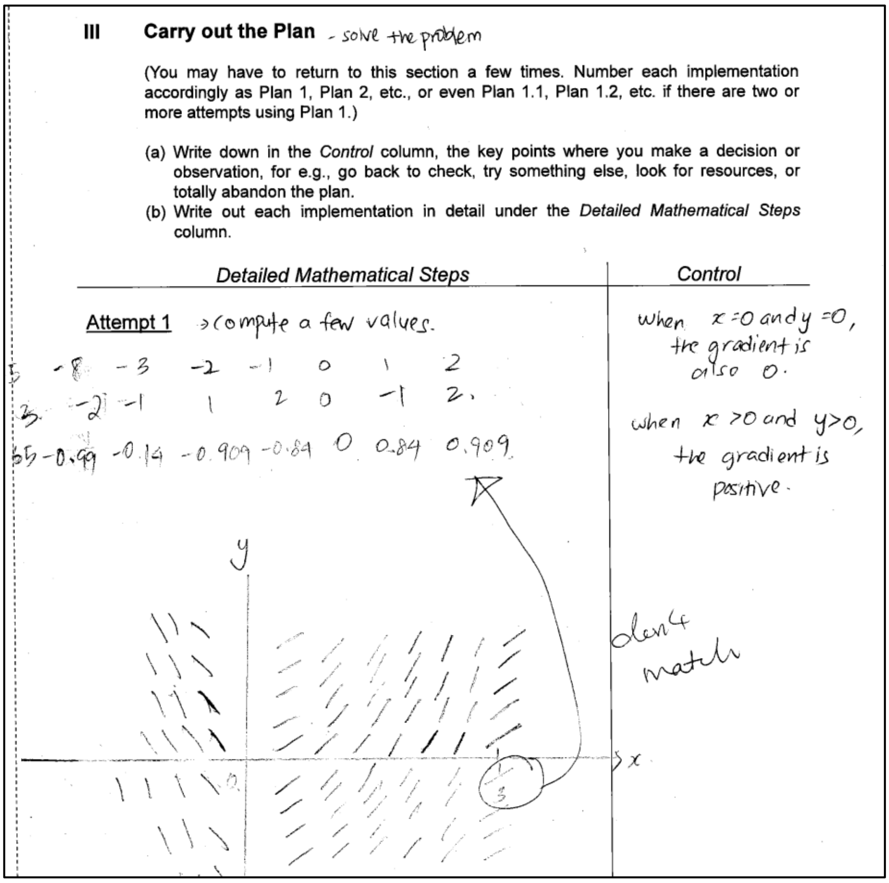
Plan 3:

Find out if the gradient is positive or negative, and sketch on the graph accordingly.

Plan 4:

Try various values and see if the slope gets steeper or not, as the values increase, and sketch the slopes accordingly in each quadrant.

Figure 2. Amy's Solution of Problem One.



III Carry out the Plan - solve the problem

(You may have to return to this section a few times. Number each implementation accordingly as Plan 1, Plan 2, etc., or even Plan 1.1, Plan 1.2, etc. if there are two or more attempts using Plan 1.)

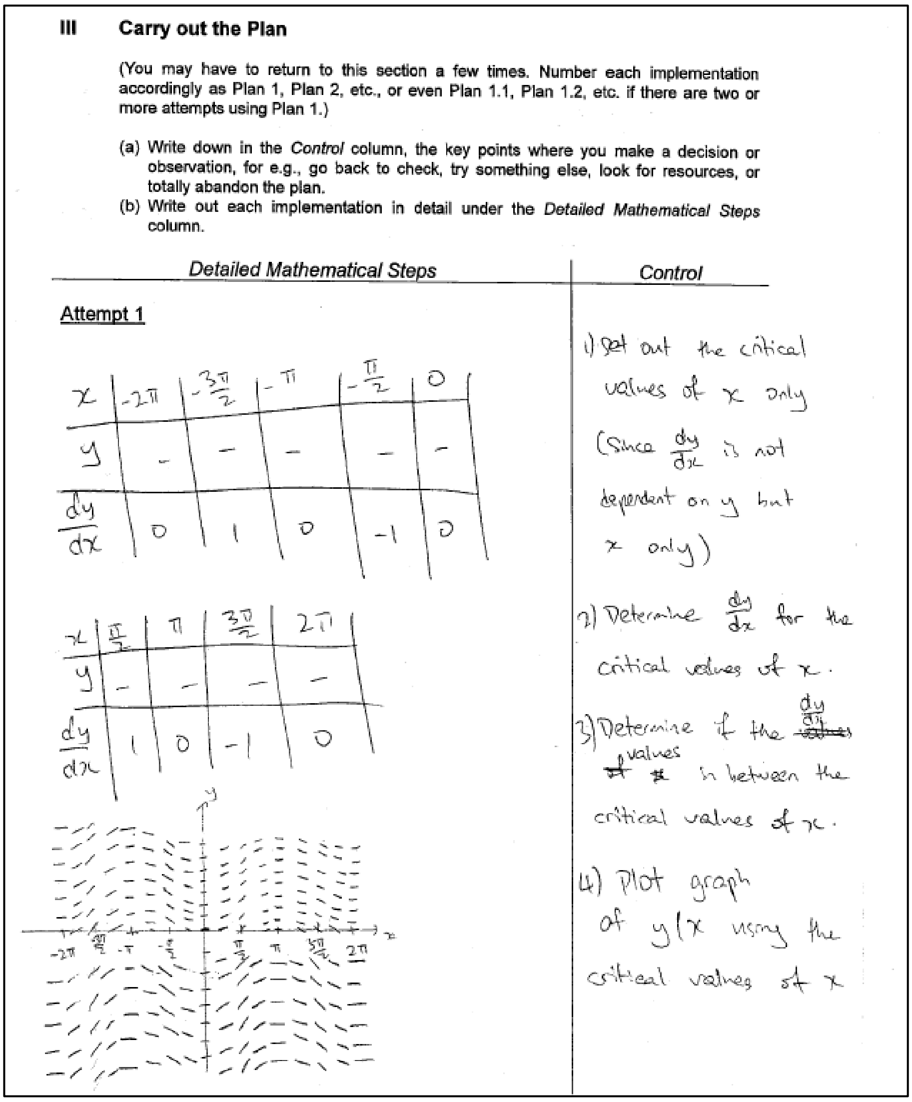
(a) Write down in the *Control* column, the key points where you make a decision or observation, for e.g., go back to check, try something else, look for resources, or totally abandon the plan.
(b) Write out each implementation in detail under the *Detailed Mathematical Steps* column.

Detailed Mathematical Steps	*Control*
<u>Attempt 1</u> → compute a few values.	when $x=0$ and $y=0$, the gradient is also 0.
-8 -3 -2 -1 0 1 2	
-2 -1 1 2 0 -1 2	when $x>0$ and $y>0$, the gradient is positive.
-0.99 -0.14 -0.909 -0.84 0 0.84 0.909	
	doesn't match

Figure 2. Amy's Solution of Problem One (continued).

Correct solution with appropriate use of Heuristics: Wilfred

Wilfred provided a completely correct solution where his thought processes were detailed in the *Control* Column in Figure 3 for Stage III. He did not write anything in the spaces for Stages I and II of the practical worksheet. However, he demonstrated the use of the heuristics clearly (plotting values and using diagrams). He proceeded to Stage IV (Figure 4) and earned 15 marks.



III Carry out the Plan

(You may have to return to this section a few times. Number each implementation accordingly as Plan 1, Plan 2, etc., or even Plan 1.1, Plan 1.2, etc. if there are two or more attempts using Plan 1.)

(a) Write down in the *Control* column, the key points where you make a decision or observation, for e.g., go back to check, try something else, look for resources, or totally abandon the plan.
(b) Write out each implementation in detail under the *Detailed Mathematical Steps* column.

Detailed Mathematical Steps	*Control*

Attempt 1

x	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0
y	–	–	–	–	–
$\frac{dy}{dx}$	0	1	0	−1	0

x	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	–	–	–	–
$\frac{dy}{dx}$	1	0	−1	0

1) Set out the critical values of x only (Since $\frac{dy}{dx}$ is not dependent on y but x only)

2) Determine $\frac{dy}{dx}$ for the critical values of x.

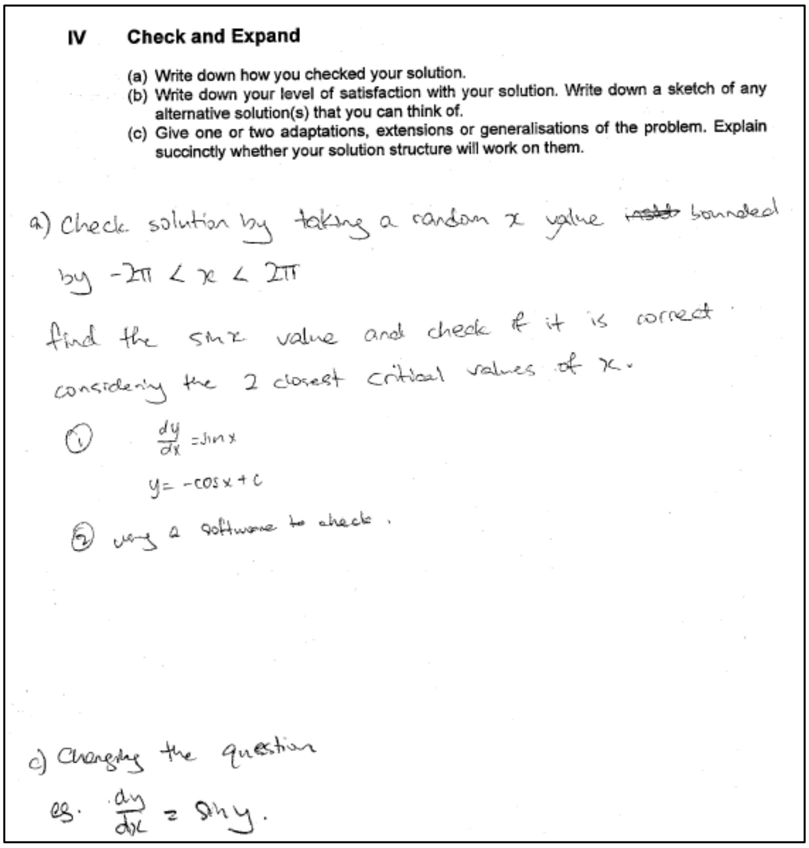
3) Determine if the $\frac{dy}{dx}$ ~~values~~ values ~~of x~~ in between the critical values of x.

4) Plot graph of y/x using the critical values of x

Figure 3. Wilfred's Solution of Problem One.

Wilfred provided three ways of checking the correctness of his solution (Figure 4): (1) substituting other values of *x* within the specified domain; (2) performing integration and checking the function against the graph (which he performed in

Stage III under Figure 3); and (3) using a graphing software. Although he had not indicated how the software could be used, it was apparent that he was able to link what he had learnt in the methods course to this mathematics content course by suggesting the use of a graphing software. For (2), he had utilized his prior knowledge on integration as the reverse of differentiation to check his answer. Wilfred further demonstrated a changing of the problem from $\frac{dy}{dx} = \sin x$ to $\frac{dy}{dx} = \sin y$. However, there was no indication that he was able to solve this problem, as finding the solution of a general autonomous differential equation analytically had not been taught at this stage.



IV Check and Expand

(a) Write down how you checked your solution.
(b) Write down your level of satisfaction with your solution. Write down a sketch of any alternative solution(s) that you can think of.
(c) Give one or two adaptations, extensions or generalisations of the problem. Explain succinctly whether your solution structure will work on them.

a) Check solution by taking a random x value ~~inside~~ bounded by $-2\pi < x < 2\pi$

find the $\sin x$ value and check if it is correct.

considering the 2 closest critical values of x.

① $\frac{dy}{dx} = \sin x$

$y = -\cos x + c$

② using a software to check.

c) Changing the question

eg. $\frac{dy}{dx} = \sin y$.

Figure 4. Wilfred's Stage IV of Problem One.

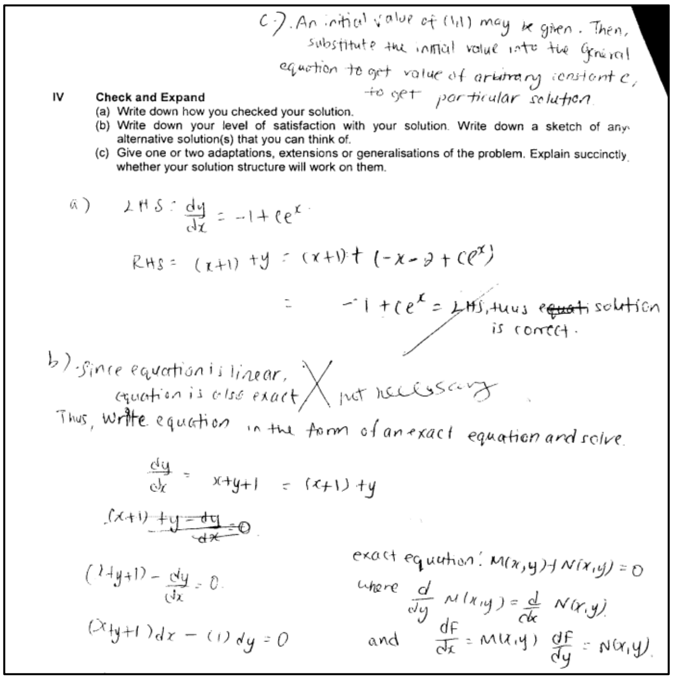
Mathematics practical test

A final mathematics practical test was administered at the end of the course. The test problem is as follows, also awarded out of 20 marks:

Find the general solution of the differential equation $\frac{dy}{dx} = x + y + 1 \cdot$

All the pre-service teachers had proceeded to Stage IV, showing that they became clearer about its place in the overall Polya's model as the course proceeded. The quality of Amy's and Edmund's responses is discussed below, as this is instructive because they did not make an attempt in Stage IV earlier.

Amy's attempt. As shown in Figure 5, Amy attempted to check the correctness of the solution, which was correct (not shown in the diagram). However, she was not able to provide an alternative solution because she had confused linear equation with exact equations. She provided a "trivial" adaptation by converting the problem of finding general solution to an initial value problem. Together with her correct solution of the original problem, she was awarded 15 marks.



c). An initial value of (1,1) may be given. Then, substitute the initial value into the General equation to get value of arbitrary constant c, to get particular solution

IV **Check and Expand**

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(c) Give one or two adaptations, extensions or generalisations of the problem. Explain succinctly whether your solution structure will work on them.

a) LHS : $\frac{dy}{dx} = -1 + ce^x$

RHS = $(x+1) + y = (x+1) + (-x-2+ce^x)$

$= -1 + ce^x$ = LHS, thus ~~equati~~ solution is correct.

b). Since equation is linear, equation is also exact ✗ not necessary

Thus, write equation in the form of an exact equation and solve.

$\frac{dy}{dx} = x+y+1 = (x+1)+y$

$\cancel{(x+1)+y-\frac{dy}{dx}=0}$

$(x+y+1) - \frac{dy}{dx} = 0$

$(x+y+1)dx - (1)dy = 0$

exact equation: $M(x,y) + N(x,y) = 0$

where $\frac{d}{dy} M(x,y) = \frac{d}{dx} N(x,y)$

and $\frac{dF}{dx} = M(x,y)$ $\frac{dF}{dy} = N(x,y)$

Figure 5. Amy's Stage IV of Mathematics Practical Test.

Edmund's attempt. As shown in Figure 6, Edmund managed to check the correctness of his solution. In addition, he gave an alternative solution to the first order linear differential equation by using substitution. This was quite impressive as using substitution was not treated as the standard approach of solving a first order linear equation. He further provided a generalization and, interestingly, he "discovered" that the general first order linear equation of the form $\frac{dy}{dx} = nx + my + a$ can be solved by the linear substitution. This was very encouraging as it shows that, by going through Stage IV, it is possible for students to "discover" interesting results in the process of checking and expanding on a given problem. In fact, more than half the class found a similar alternative solution of this special class of linear differential equations.

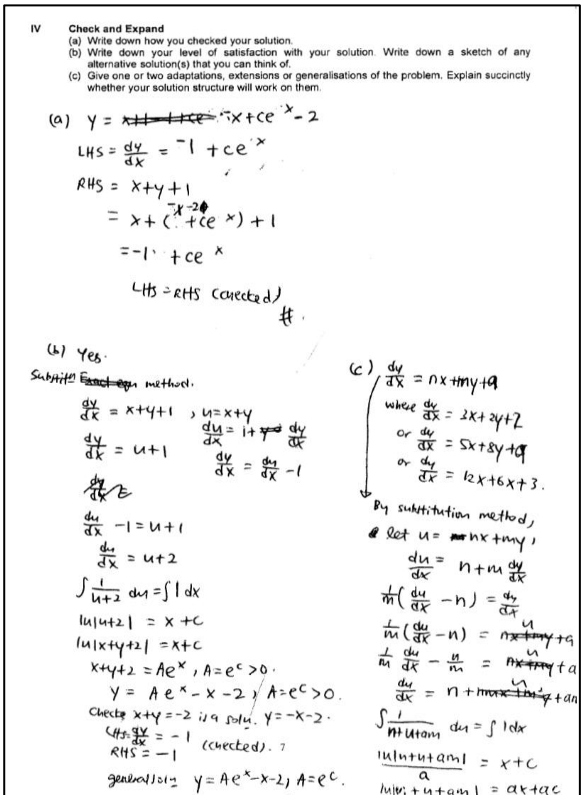

Figure 6. Edmund's Stage IV of Mathematics Practical Test.

Conclusion

The re-design of the pre-service teacher mathematics content course had included features that would help them appreciate and experience a problem solving curriculum as well as build confidence as problems solvers and teachers of problem solving. These features were carefully considered during planning and implementation to ensure that the mathematics content covered in the revised content course was not less than what was done under the traditional lecture-tutorial mode. Thus, the pre-service teachers in this study were able to acquire the same mathematics content knowledge through the traditional way of teaching (with some lectures replaced by practical lessons), but they also demonstrated the ability to use problem solving heuristics to solve problems and to apply these skills to learn university-level mathematics knowledge. We were especially encouraged to see that they were able to respond to the change in expectation in relation to mathematical problem solving. They could apply appropriate heuristics, even though they might not always obtain completely correct solutions. Furthermore, they not only proceeded to Stage IV toward the end of the course, but also provided substantive expansions after they had obtained the mathematical solution to a problem. We believed that this study has shown a satisfactory progression towards helping pre-service teachers gain competence in mathematical problem solving.

Taken together with another similar study of using Practical Worksheet in the teaching of Number Theory course in NIE (Toh et al., in press), the findings in the present study indicate that this teaching mathematics *through* problem solving approach is a promising way forward in the design and structuring of university-level mathematics content courses for pre-service teachers. However, problem solving experience over only two courses throughout the entire undergraduate experience may not be sufficient to embed in the pre-service teachers a natural problem solving disposition. There is room to explore how other content courses can be taught in this problem solving approach in order help them reach a state of habitual problem solving mindset whenever they come across unfamiliar problems.

How is this consideration of problem solving among pre-service teachers relevant to the enactment of problem solving in actual classroom settings? We have worked closely with several schools using a similar design consisting of the Mathematics Practical (Leong et al., 2011b) and found that a critical factor in the success of this implementation is teacher preparation (Leong et al., 2011a). In particular, teacher confidence and sufficient experience in problem solving is vital to model problem solving processes in their classroom in convincing ways. Unless there is problem solving competence in teachers, there is limited vision about problem solving taking a prominent role in actual mathematics classrooms. Instead of confronting the challenges of problem solving only in schools, we think that prospective teachers should start their problem solving learning journey during pre-service training. This study is located at the beginning of this vision.

References

Ball, D. L. (1990a). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal, 90*, 449-466.

Ball, D. L. (1990b). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21*, 132-144.

Frykholm, J. A. (1999a). The impact of reform: Challenges for mathematics teacher preparation. *Journal of Mathematics Teacher Education, 1*, 79-105.

Frykholm, J. A. (1999b, January). *Elementary mathematics: A missing piece in secondary mathematics teacher education?* Paper presented at the annual meeting of the Association of Mathematics Teacher Educators. Chicago, IL.

Ho, K. F., & Hedberg, J. G. (2005). Teachers' pedagogies and their impact on students' mathematical problem solving. *Journal of Mathematical Behavior, 24*(3-4), 238-252.

Leikin, R., & Kawass, S. (2005). Planning teaching an unfamiliar mathematics problem: The role of teachers' experience in solving the problem and watching pupils solving it. *Journal of Mathematical Behavior, 24*(3-4), 253-274.

Leong, Y.H., Dindyal, J., Toh, T.L., Quek, K.S., Tay, E.G., & Lou, S.T. (2011a). Teacher education for a problem-solving curriculum in Singapore. *ZDM: The International Journal on Mathematics Education, 43*(6-7), 819-831.

Leong, Y.H., Tay, E.G., Toh, T.L., Quek, K.S., & Dindyal, J, (2011b). Reviving Polya's "look back" in a Singapore school. *Journal of Mathematical Behavior, 30*(3), 181-193.

Lester, F. K. (1994). Musing about mathematical problem-solving research: 1970-1994. *Journal of Research in Mathematics Education, 25*, 660-676.

Mathematical Sciences Education Board. (1996). *Mathematics and science education around the world: What can we learn from the survey of mathematics and science opportunities (SMSO) and the Third International Mathematics and Science Study (TIMSS)?* Washington, DC: National Research Council.

McTighe, J., & Wiggins, G. (2004). *Understanding by design: Professional development workbook*. Alexandria, VA: Association for Supervision and Curriculum Development.

Morris, H. (2001). Issues raised by testing trainee primary teachers' mathematical knowledge. *Mathematics Teacher Education and Development, 3*, 37-47.

Quek, K.S., Toh, T.L., Dindyal, J., Leong, Y.H., Tay, E.G., & Lou, S.T. (2010). Resources for teaching problem solving: A problem to discuss. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education* (pp. 753-756). Fremantle, Australia: Mathematics Education Research Group of Australasia.

Quek, K.S., Dindyal, J., Toh, T.L., Leong, Y.H., & Tay, E.G. (2011). Problem solving for everyone: A design experiment. *Journal of the Korea Society of Mathematical Education Series D: Research in Mathematical Education, 15*(1), 31-44.

Quinn, R. J. (1997). Effects of mathematical methods courses on the mathematical attitudes and content knowledge of preservice teachers. *Journal of Educational Research, 91*, 108-119.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York, NY: Macmillan.

Shroeder, T., & Lester, F. (1989). Developing understanding in mathematics via problem solving. In P. Traffon & A. Shulte (Eds.), *New directions for elementary school mathematics: 1989 Yearbook* (pp. 31-42). Reston, VA: National Council of Teachers of Mathematics.

Silver, E. A., Ghousseini, H., Gosen, D., Charalambous, C., & Strawhun, B. T. F. (2005). Moving from rhetoric to praxis: Issues faced by teachers in having students consider multiple solutions for problems in the mathematics classroom. *Journal of Mathematical Behavior, 24*(3-4), 287-301.

Stacey, K. (2005). The place of problem solving in contemporary mathematics curriculum documents. *Journal of Mathematical Behavior*, *24*(3-4), 341-350.

Taylor, P. M. (2002). Implementing the standards: Keys to establishing positive professional inertia in preservice mathematics teachers. *School Science and Mathematics*, *102*, 137-142.

Toh, T.L., Quek, K.S., & Tay, E.G. (2008). Mathematical problem solving – a new paradigm. In J. Vincent, J. Pierce, & J. Dowsey (Eds.), *Connected Maths: MAV Yearbook 2008* (pp. 356-365). Melbourne: Mathematical Association of Victoria.

Toh, T.L., Quek, K.S., Tay, E.G., Leong, Y.H., & Dindyal, J. (2011a). Enacting a problem solving curriculum in a Singapore school. In L.A. Bragg (Ed.), *Maths is multi-dimensional* (pp. 77-86). Melbourne: Mathematical Association of Victoria.

Toh, T.L., Quek, K.S., Leong, Y.H., Dindyal, J., & Tay, E.G. (2011b). *Making mathematics practical: An approach to problem solving.* Singapore: World Scientific.

Toh, T.L., Quek, K.S., Leong, Y.H., Dindyal, J., & Tay, E.G. (2011c). Assessing problem solving in the mathematics curriculum: A new approach. In B. Kaur, & K.Y. Wong (Eds.), *Assessment in the mathematics classroom: AME Yearbook 2011* (pp. 33- 66). Singapore: World Scientific.

Toh, P.C., Leong, Y.H., Toh, T.L., Dindyal, J., Quek, K.S., Tay, E.G., & Ho, F.H. (In press). The problem solving approach in the teaching of Number Theory. *International Journal of Mathematical Education in Science and Technology*. DOI: 10.1080/0020739X.2013.822580

Wilson, S., & Ball, D. L. (1996). Helping teachers meet the standards: New challenges for teacher educators. *Elementary School Journal, 97*, 121-138.

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Appendix A: Problems for the Mathematics Practical Lessons

Problem One. Sketch the slope field for the differential equation $\frac{dy}{dx} = \sin x$

Problem Two. Find the general solution of the differential equation $\frac{dy}{dx} = xy$

Problem Three. Must a first order separable equation always be exact? Must an exact differential equation always be separable?

Problem Four. Find the general solution of the differential equation $\frac{dy}{dx} = 2(2x - y)$

Problem Five. Find the general solution of the differential equation $\frac{dy}{dx} = 1 + x + y + xy$

Problem Six. Find the general solution of the differential equation $\frac{d^2y}{dx^2} + m\frac{dy}{dx} + 4y = 0$, where m is a real number

Appendix B: Scoring Rubric

Polya's Stages		
	Descriptors/Criteria *(evidence suggested/indicated on practical sheet or observed by teacher)*	**Marks Awarded**
Correct Solution		
Level 3	Evidence of complete use of Polya's Stages – UP + DP + CP*; and when necessary, appropriate loops. [10]	
Level 2	Evidence of trying to understand the problem and having a clear plan – UP + DP + CP*. [9]	
Level 1	No evidence of attempt to use Polya's Stages. [8]	
Partially Correct Solution *(solve significant part of the problem or lacking rigour)*		
Level 3	Evidence of complete use of Polya's Stages – UP + DP + CP*; and when necessary, appropriate loops. [8]	
Level 2	Evidence of trying to understand the problem and having a clear plan – UP + DP + CP*. [7]	
Level 1	No evidence of attempt to use Polya's Stages. [6]	
Incorrect Solution		
Level 3	Evidence of complete use of Polya's Stages – UP + DP + CP*; and when necessary, appropriate loops. [6]	
Level 2	Evidence of trying to understand the problem and having a clear plan – UP + DP + CP*. [5]	
Level 1	No evidence of attempt to use Polya's Stages. [0]	

Heuristics		
	Descriptors/Criteria *(evidence suggested/indicated on practical sheet or observed by teacher)*	**Marks Awarded**
Correct Solution		
Level 2	Evidence of appropriate use of heuristics. [4]	
Level 1	No evidence of heuristics used. [3]	
Partially Correct Solution *(solve significant part of the problem or lacking rigour)*		
Level 2	Evidence of appropriate use of heuristics. [3]	
Level 1	No evidence of heuristics used. [2]	
Incorrect Solution		
Level 2	Evidence of appropriate use of heuristics. [2]	
Level 1	No evidence of heuristics used. [0]	

Note: *UP-DP-CP represents the first three Polya's Stages: Understanding the Problem (UP), Devise Plan (DP) and Carry out the plan (CP)

Checking and Expanding		
	Descriptors/Criteria *(evidence suggested/indicated on practical sheet or observed by teacher)*	**Marks Awarded**
Checking		
Level 2	Checking done – mistakes identified and correction attempted by cycling back to UP, DP, or CP, until solution is reached. [1]	
Level 1	No checking, or solution contains errors. [0]	
Alternative Solutions		
Level 3	Two or more correct alternative solutions. [2]	
Level 2	One correct alternative solution. [1]	
Level 1	No alternative solution. [0]	
Extending, Adapting & Generalizing		
Level 4	More than one related problem with suggestions of correct solution methods/strategies; or one **significant** related problem, with suggestion of correct solution method/strategy; or one **significant** related problem, with explanation why method of solution for original problem cannot be used. [3]	
Level 3	One related problem with suggestion of correct solution method/strategy. [2]	
Level 2	One related problem given but without suggestion of correct solution method/strategy. [1]	
Level 1	None provided. [0]	