AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **30** (2004), Pages 3–20

Graph-functions associated with an edge-property

F.M. Dong*

Mathematics and Mathematics Education
National Institute of Education
Nanyang Technological University
Singapore 637616
fmdong@nie.edu.sg

M.D. Hendy K.L. Teo C.H.C. Little

Institute of Fundamental Sciences (Mathematics)
Massey University
Palmerston North
New Zealand

Abstract

Let $\mathcal{P}$ be an edge-property of graphs. For any graph G we construct a polynomial $\Psi(G, \eta, \mathcal{P})$, in an indeterminate η, in which the coefficient of η^r for any $r \geq 0$ gives the number of subsets of $E(G)$ that have cardinality r and satisfy $\mathcal{P}$. An example is the well known matching polynomial of a graph. After studying the properties of $\Psi(G, \eta, \mathcal{P})$ in general, we specialise to two particular edge-properties: that of being an edge-covering and that of inducing an acyclic subgraph. The resulting polynomials, called the edge-cover and acyclic polynomials respectively, are studied and recursive formulae for computing them are derived. As examples we calculate these polynomials for paths and cycles.

1 A graph-function related to an edge-property

The graphs considered in this paper are undirected finite non-null graphs which may contain multiple edges and loops. For a graph G, let $V(G)$, $E(G)$, $v(G)$ and $e(G)$ be the vertex set, edge set, order and size of G respectively. An *edge-property* $\mathcal{P}$ of graphs is a property possessed by some edge sets, provided the following condition is satisfied:

* This paper was mostly completed when Dong was a post-doctoral fellow at Massey University from 1998–2000. He thanks Massey University for its financial support.

for any graphs G_1 and G_2 with $G_1 \cong G_2$, if $E_1 \subseteq E(G_1)$ corresponds to $E_2 \subseteq E(G_2)$ under an isomorphism, then E_1 has property $\mathcal{P}$ in G_1 if and only if E_2 has property $\mathcal{P}$ in G_2.

Examples of edge-properties are property $\mathcal{P}_c$ that the subgraph induced by the edges is spanning, property $\mathcal{P}_a$ that the subgraph induced by the edges is acyclic, and property $\mathcal{P}_m$ that the subgraph induced by the edges is a matching.

The analogous concept of vertex-properties has been studied in [1].

Let $\mathcal{P}$ be any edge-property and G be any graph. Define

$$\mathcal{P}(G) = \{E' \subseteq E(G) | E' \text{ has edge-property } \mathcal{P} \text{ in } G\}.$$

For any integer $n \geq 0$, define $\mathcal{F}(G, n, \mathcal{P})$ to be the set of mappings

$$f: \ \{1, 2, \cdots, n\} \to E(G),$$

subject to the condition that $\{f(1), f(2), \cdots, f(n)\} \in \mathcal{P}(G)$. Note that when $n = 0$, $\{f(1), f(2), \cdots, f(n)\}$ is empty. We write

$$F(G, n, \mathcal{P}) = |\mathcal{F}(G, n, \mathcal{P})|. \tag{1}$$

Observe that $F(G, n, \mathcal{P})$ is a graph-function.

Lemma 1.1 *For any edge-property $\mathcal{P}$ and graph G,*

$$F(G, 0, \mathcal{P}) = \begin{cases} 1, & \text{if } \emptyset \in \mathcal{P}(G), \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

□

An edge-property $\mathcal{P}$ is said to be *inclusive* if $\mathcal{P}(H) \subseteq \mathcal{P}(G)$ for any graph G and spanning subgraph H of G. For $a \in E(G)$, let $G - a$ denote the graph obtained from G by deleting a. We write

$$\mathcal{F}(G, a, n, \mathcal{P}) = \mathcal{F}(G, n, \mathcal{P}) - \mathcal{F}(G - a, n, \mathcal{P}), \tag{3}$$

and

$$F(G, a, n, \mathcal{P}) = |\mathcal{F}(G, a, n, \mathcal{P})|. \tag{4}$$

Note that $\mathcal{F}(G, a, n, \mathcal{P})$ is the set of f in $\mathcal{F}(G, n, \mathcal{P})$ such that $f^{-1}(a) \neq \emptyset$.

Lemma 1.2 *Let $\mathcal{P}$ be an inclusive edge-property. Then*

$$F(G, a, n, \mathcal{P}) = F(G, n, \mathcal{P}) - F(G - a, n, \mathcal{P}). \tag{5}$$

Proof. Since $\mathcal{P}$ is inclusive, we have $\mathcal{F}(G - a, n, \mathcal{P}) \subseteq \mathcal{F}(G, n, \mathcal{P})$. Hence the result holds. □

For two graphs G_1, G_2, let $G_1 \oplus G_2$ be the graph H with a vertex partition $\{V_1, V_2\}$ such that $H[V_i] \cong G_i$ for $i = 1, 2$, and x and y are not adjacent for any

$x \in V_1$ and $y \in V_2$. For a disconnected graph G with two subgraphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \emptyset$ and $E(G) = E(G_1) \cup E(G_2)$, we have $G \cong G_1 \oplus G_2$.

An edge-property $\mathcal{P}$ is said to be *resolvable* if for any graph $G = G_1 \oplus G_2$ and $E' \subseteq E(G)$, $E' \in \mathcal{P}(G)$ if and only if $E' \cap E(G_1) \in \mathcal{P}(G_1)$ and $E' \cap E(G_2) \in \mathcal{P}(G_2)$.

Theorem 1.1 *Let $\mathcal{P}$ be a resolvable edge-property. For graphs G_1 and G_2,*

$$F(G_1 \oplus G_2, n, \mathcal{P}) = \sum_{r=0}^{n} \binom{n}{r} F(G_1, r, \mathcal{P}) F(G_2, n-r, \mathcal{P}). \tag{6}$$

Proof. Let $G = G_1 \oplus G_2$. For any mapping f from $\{1, 2, \cdots, n\}$ into $E(G)$, let

$$N_1 = \{1 \le k \le n | f(k) \in E(G_1)\} \quad \text{and} \quad N_2 = \{1, 2, \cdots, n\} - N_1.$$

Define two mappings $g_1 : N_1 \to E(G_1)$ and $g_2 : N_2 \to E(G_2)$,

$$g_i(k) = f(k), \quad k \in N_i,\ i = 1, 2.$$

Since $\mathcal{P}$ is resolvable, $\{f(1), f(2), \cdots, f(n)\} \in \mathcal{P}(G)$ if and only if $g_1(N_1) \in \mathcal{P}(G_1)$ and $g_2(N_2) \in \mathcal{P}(G_2)$.

On the other hand, for any partition $\{N_1, N_2\}$ of $\{1, 2, \cdots, n\}$ and any mappings g_1, g_2:

$$g_1 : N_1 \to E(G_1) \quad \text{and} \quad g_2 : N_2 \to E(G_2),$$

we can define a mapping $f : \{1, 2, \cdots, n\} \to E(G)$ given by $f(k) = g_i(k)$ if $k \in N_i$ for $i = 1, 2$.

Given a partition $\{N_1, N_2\}$ of $\{1, 2, \cdots, n\}$ with $|N_1| = r$, there are $F(G_1, r, \mathcal{P})$ mappings g_1 from N_1 into $E(G_1)$ and $F(G_2, n-r, \mathcal{P})$ mappings g_2 from N_2 into $E(G_2)$ such that $g_1(N_1) \in \mathcal{P}(G_1)$ and $g_2(N_2) \in \mathcal{P}(G_2)$. Thus there are

$$F(G_1, r, \mathcal{P}) F(G_2, n-r, \mathcal{P})$$

different mappings f from $\{1, 2, \cdots, n\}$ into $E(G)$ such that $\{i | f(i) \in E(G_1)\} = N_1$ and $\{f(1), f(2), \cdots, f(n)\} \in \mathcal{P}(G)$. Hence

$$\begin{aligned} F(G_1 \oplus G_2, n, \mathcal{P}) &= \sum_{\substack{N_1 \cup N_2 = \{1,2,\cdots,n\} \\ N_1 \cap N_2 = \emptyset}} F(G_1, |N_1|, \mathcal{P}) F(G_2, |N_2|, \mathcal{P}) \\ &= \sum_{r=0}^{n} \sum_{\substack{N_1 \subseteq \{1,2,\cdots,n\} \\ |N_1| = r}} F(G_1, r, \mathcal{P}) F(G_2, n-r, \mathcal{P}) \\ &= \sum_{r=0}^{n} \binom{n}{r} F(G_1, r, \mathcal{P}) F(G_2, n-r, \mathcal{P}). \end{aligned}$$

□

2 $F(G, n, \mathcal{P})$

Recall that the number of surjections from an n-set to an r-set is given by

$$\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}i^n = r!S(n,r), \tag{7}$$

where $S(n,r)$ is the *Stirling number* of the *second kind*. (See [3].)

For any edge-property $\mathcal{P}$, graph G and integer $r \geq 0$, define

$$\mathcal{S}(G,r,\mathcal{P}) = \{E' \in E(G)|\ E' \in \mathcal{P}(G),\ |E'| = r\}. \tag{8}$$

We write

$$s(G,r,\mathcal{P}) = |\mathcal{S}(G,r,\mathcal{P})|. \tag{9}$$

Lemma 2.1 *For any edge-property $\mathcal{P}$ and graph G,*

$$s(G,0,\mathcal{P}) = F(G,0,\mathcal{P}). \tag{10}$$

□

In general, we have

Theorem 2.1 *For any edge-property $\mathcal{P}$, graph G and integer $n \geq 0$,*

$$F(G,n,\mathcal{P}) = \sum_{r=0}^{e(G)} s(G,r,\mathcal{P})r!S(n,r). \tag{11}$$

Proof. By Lemma 2.1, the result holds for $n = 0$. Now let $n \geq 1$. For any $f \in \mathcal{F}(G,n,\mathcal{P})$, $\{f(1), f(2), \cdots, f(n)\} \in \mathcal{S}(G,r,\mathcal{P})$ for some r with $1 \leq r \leq e(G)$. Since $S(n,0) = 0$, it suffices to prove that for any $E' \in \mathcal{S}(G,r,\mathcal{P})$ with $1 \leq r \leq e(G)$, there are exactly $r!S(n,r)$ mappings $f \in \mathcal{F}(G,n,\mathcal{P})$ with $\{f(1), f(2), \cdots, f(n)\} = E'$. Observe that there are exactly $r!S(n,r)$ mappings from $\{1, 2, \cdots, n\}$ onto E'. Since $E' \in \mathcal{S}(G,r,\mathcal{P})$, all such mappings are contained in $\mathcal{F}(G,n,\mathcal{P})$. Hence the result holds. □

3 A generating function

For any edge-property $\mathcal{P}$ and graph G, define

$$\Phi(G,\mu,\mathcal{P}) = \sum_{n=0}^{\infty}\frac{F(G,n,\mathcal{P})}{n!}\mu^n, \tag{12}$$

where μ is a real number.

Theorem 3.1 *Let $\mathcal{P}$ be a resolvable edge-property. For graphs G_1 and G_2,*

$$\Phi(G_1 \oplus G_2,\mu,\mathcal{P}) = \Phi(G_1,\mu,\mathcal{P})\Phi(G_2,\mu,\mathcal{P}). \tag{13}$$

Proof. Observe that

$$\begin{aligned}
& \Phi(G_1 \oplus G_2, \mu, \mathcal{P}) \\
= & \sum_{n=0}^{\infty} \frac{F(G_1 \oplus G_2, n, \mathcal{P})}{n!}\mu^n \\
= & \sum_{n=0}^{\infty}\sum_{r=0}^{n} \binom{n}{r} F(G_1, r, \mathcal{P})F(G_2, n-r, \mathcal{P})\frac{\mu^n}{n!} \qquad \text{(by Theorem 1.1)} \\
= & \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{F(G_1, r, \mathcal{P})}{r!} \cdot \frac{F(G_2, n-r, \mathcal{P})}{(n-r)!}\mu^n \\
= & \left(\sum_{n=0}^{\infty} \frac{F(G_1, n, \mathcal{P})}{n!}\mu^n\right)\left(\sum_{n=0}^{\infty} \frac{F(G_2, n, \mathcal{P})}{n!}\mu^n\right) \\
= & \Phi(G_1, \mu, \mathcal{P})\Phi(G_2, \mu, \mathcal{P}).
\end{aligned}$$

□

Lemma 3.1 ([7]) *For any integer $r \geq 0$,*

$$\sum_{n=0}^{\infty} \frac{r!S(n,r)}{n!}\mu^n = (e^{\mu} - 1)^r. \tag{14}$$

Proof. See (3.6.2) in [7]. □

Theorem 3.2 *For any edge-property $\mathcal{P}$ and graph G, we have*

$$\Phi(G, \mu, \mathcal{P}) = \sum_{r=0}^{e(G)} s(G, r, \mathcal{P})(e^{\mu} - 1)^r. \tag{15}$$

Proof. Observe that

$$\begin{aligned}
\Phi(G, \mu, \mathcal{P}) &= \sum_{n=0}^{\infty} \frac{F(G, n, \mathcal{P})}{n!}\mu^n \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{e(G)} s(G, r, \mathcal{P})r!S(n,r)\frac{\mu^n}{n!} \qquad \text{(by Theorem 2.1)} \\
&= \sum_{r=0}^{e(G)} s(G, r, \mathcal{P}) \sum_{n=0}^{\infty} \frac{r!S(n,r)\mu^n}{n!} \\
&= \sum_{r=0}^{e(G)} s(G, r, \mathcal{P})(e^{\mu} - 1)^r \qquad \text{(by Lemma 3.1).}
\end{aligned}$$

□

For a graph G and a real number $\eta > -1$, we write

$$\Psi(G, \eta, \mathcal{P}) = \Phi(G, \log(1+\eta), \mathcal{P}). \tag{16}$$

By Theorems 3.1 and 3.2, we have

Theorem 3.3 *Let $\mathcal{P}$ be a resolvable edge-property. For graphs G_1 and G_2,*

$$\Psi(G_1 \oplus G_2, \mu, \mathcal{P}) = \Psi(G_1, \mu, \mathcal{P})\Psi(G_2, \mu, \mathcal{P}). \tag{17}$$

□

Theorem 3.4 *For any edge-property $\mathcal{P}$ and graph G,*

$$\Psi(G, \eta, \mathcal{P}) = \sum_{r=0}^{e(G)} s(G, r, \mathcal{P})\eta^r. \tag{18}$$

□

Remarks:

(i) From Theorem 3.4, we observe that $\Psi(G, \eta, \mathcal{P})$ is a polynomial in η of degree at most $e(G)$, and the coefficient of η^r is the number of subsets of $E(G)$ that have cardinality r and satisfy $\mathcal{P}$. We may consider η to be an indeterminate in the polynomial $\Psi(G, \eta, \mathcal{P})$.

(ii) $\Psi(G, \eta, \mathcal{P}_m)$ is the matching polynomial of G, which has been studied by Farrell [4] and others. In this paper, we obtain recursive formulae for computing $\Psi(G, \eta, \mathcal{P}_c)$ and $\Psi(G, \eta, \mathcal{P}_a)$. In the latter case the formula is akin to the well known recursive formula for the chromatic polynomial of a graph in that both formulae are based on deletions and contractions of edges. In the former case the contraction operation is replaced by a different operation in which two vertices x and y are deleted and the edges in the cocycle $\partial\{x, y\}$ are replaced by loops.

4 Edge-covers

For a vertex x in G, let $E_G(x)$ (or simply $E(x)$) be the set of edges incident with x, and let $N_G(x)$ (or simply $N(x)$) be the set of vertices y such that $E(x) \cap E(y) \neq \emptyset$. By the definition of $\mathcal{P}_c$, for any graph G,

$$\mathcal{P}_c(G) = \{E' \subseteq E(G) | E' \cap E_G(x) \neq \emptyset, \forall x \in V(G)\}. \tag{19}$$

For $E' \subseteq E(G)$, E' is called an *r-edge-cover* of G if $|E'| = r$ and $E' \cap E_G(x) \neq \emptyset$ for all $x \in V(G)$. Hence $s(G, r, \mathcal{P}_c)$ is the number of r-edge-covers of G.

Lemma 4.1 *$\mathcal{P}_c$ is resolvable and inclusive.* □

We write

$$\begin{aligned} F_c(G, n) &= F(G, n, \mathcal{P}_c), \\ \Phi_c(G, \mu) &= \Phi(G, \mu, \mathcal{P}_c), \\ \Psi_c(G, \eta) &= \Psi(G, \eta, \mathcal{P}_c). \end{aligned}$$

Since $\mathcal{P}_c$ is resolvable and inclusive, the results in section 1, 2 and 3 hold for $F_c(G,n)$, $\Phi_c(G,\mu)$ and $\Psi_c(G,\eta)$. We are particularly interested in $\Psi_c(G,\eta)$ which, by Theorem 3.4, is a polynomial in η of degree at most $e(G)$. In this section, we provide a recursion for computing $\Psi_c(G,\eta)$. To determine $\Psi_c(G,\eta)$, we need to use the other two functions: $F_c(G,n)$ and $\Phi_c(G,\mu)$.

Lemma 4.2 *Let G be a graph. If $E_G(x)=\emptyset$ for some $x\in V(G)$, then $F_c(G,n)=0$ for all $n\geq 0$.*

Proof. Since $E_G(x)=\emptyset$, we have $\mathcal{P}_c(G)=\emptyset$. Thus $s(G,r,\mathcal{P}_c)=0$ for all $r\geq 0$. The result is then clear by Theorem 2.1. □

Lemma 4.3 *For any graph G and integer $n\geq 0$, if $v(G)>2n$, then*

$$F_c(G,n)=0. \tag{20}$$

Proof. For $f\in\mathcal{F}(G,n,\mathcal{P}_c)$, let x_i,y_i be the end-vertices of $f(i)$, $i=1,2,\cdots,n$. Since $E_G(x)\cap\{f(1),f(2),\cdots,f(n)\}\neq\emptyset$ for all $x\in V(G)$, we have

$$V(G)=\{x_i,y_i|\ i=1,2,\cdots,n\}.$$

But $|\{x_i,y_i|\ i=1,2,\cdots,n\}|\leq 2n<v(G)=|V(G)|$, a contradiction. Hence $\mathcal{F}(G,n,\mathcal{P}_c)=\emptyset$, i.e., $F_c(G,n)=0$. □

Corollary *For any graph G, $F_c(G,0)=0$.* □

For integer $k\geq 0$, let L_k be the graph with one vertex and k loops and let B_k be the graph with order 2, size k and no loops.

Lemma 4.4 *For integers $k\geq 0$ and $G\in\{L_k,B_k\}$,*

$$\begin{aligned} F_c(G,n) &= k^n, \qquad n\geq 1, &(21)\\ \Phi_c(G,\mu) &= e^{k\mu}-1, &(22)\\ \Psi_c(G,\eta) &= (1+\eta)^k-1. &(23)\end{aligned}$$

Proof. Consider the case $G=L_k$. By Lemma 4.2, (21) holds for $k=0$. When $k\geq 1$, $f\in\mathcal{F}(G,n,\mathcal{P}_c)$ for any mapping f from $\{1,2,\cdots,n\}$ into $E(G)$. Thus (21) holds for $k\geq 1$. By the corollary to Lemma 4.3, $F_c(G,0)=0$. By (12) and (21), (22) is obtained. By (16) and (22), (23) is obtained.

For $G=B_k$, the result can be obtained in the same way. □

4.1 Recursive expressions for $\Psi_c(G,\eta)$

If $v(G)=1$ or $v(G)=2$ and G contains no loops, $\Psi_c(G,\eta)$ is given by (23). We now consider the general case.

For $x,y\in V(G)$, let $B_G(x,y)$ or simply $B(x,y)$ denote the set of edges with end-vertices x and y, and let $b_G(x,y)=|B_G(x,y)|$ (or simply write $b(x,y)$ for $b_G(x,y)$). We write $L(x)$ for $B(x,x)$ and $l(x)$ for $b(x,x)$.

For $V'\subseteq V(G)$, let $G-V'$ be the graph obtained from G by deleting all vertices in V' and all edges in $\bigcup_{x\in V'}E_G(x)$. When $v(G)\geq 3$, for $x,y\in V(G)$ with $x\neq y$, let $G\star xy$ be the graph obtained from $G-\{x,y\}$ by adding $b(x,u)+b(y,u)$ more loops at each $u\in N_G(x)\cup N_G(y)-\{x,y\}$. When $v(G)\geq 2$, let $G\star x$ be the graph obtained from $G-\{x\}$ by adding $b(x,u)$ more loops at each $u\in N_G(x)-\{x\}$.

Lemma 4.5 *For any graph G with $v(G)\geq 2$ and loop a at $x\in V(G)$,*

$$F(G,a,n,\mathcal{P}_c)=F(L_{l(x)}\oplus(G\star x),a',n,\mathcal{P}_c), \tag{24}$$

where a' is a loop in $L_{l(x)}$.

Proof. Let $a_1,a_2,\cdots,a_k$ be the edges in $E_G(x)$ which are not loops. Construct a graph G' from G by replacing each a_i with end-vertices x and u_i by a loop a_i' at u_i. Let Q be the mapping from $E(G)$ onto $E(G')$ defined by:

$$Q(b)=\begin{cases} b, & \text{if } b\in E(G)-\{a_1,a_2,\cdots,a_k\},\\ a_i', & \text{if } b=a_i.\end{cases}$$

Observe that Q is a one-to-one correspondence between $E(G)$ and $E(G')$.

For each mapping f from $\{1,2,\cdots,n\}$ to $E(G)$, define a mapping g from $\{1,2,\cdots,n\}$ to $E(G')$:

$$g(i)=Q(f(i)),\quad i=1,2,\cdots,n.$$

Observe that the above equation gives a one-to-one relation between the mappings from $\{1,2,\cdots,n\}$ to $E(G)$ and the mappings from $\{1,2,\cdots,n\}$ to $E(G')$. We also observe that $f\in\mathcal{F}(G,n,\mathcal{P}_c)$ with $f^{-1}(a)\neq\emptyset$ if and only if $g\in\mathcal{F}(G',n,\mathcal{P}_c)$ with $g^{-1}(a)\neq\emptyset$. Hence

$$F(G,a,n,\mathcal{P}_c)=F(G',a,n,\mathcal{P}_c).$$

Since $G'\cong L_{l(x)}\oplus(G\star x)$, the result is obtained. □

Lemma 4.6 *For a graph G with $v(G)\geq 2$ and $x\in V(G)$,*

$$F_c(G,n)=F_c(G-L(x),n)+F_c(L_{l(x)}\oplus(G\star x),n). \tag{25}$$

Proof. If $L(x)=\emptyset$, then $l(x)=0$ and $F_c(L_{l(x)}\oplus(G\star x),n)=0$ by Lemma 4.2. Hence (25) holds when $L(x)=\emptyset$. Now we assume that $L(x)\neq\emptyset$.

Let a be a loop at x in G and a' be a loop in the component $L_{l(x)}$ of the graph $L_{l(x)} \oplus (G \star x)$. By (24), we have

$$F(G, a, n, \mathcal{P}_c) = F(L_{l(x)} \oplus (G \star x), a', n, \mathcal{P}_c),$$

i.e.,

$$\begin{aligned} & F(G, n, \mathcal{P}_c) - F(G - a, n, \mathcal{P}_c) \\ = \; & F(L_{l(x)} \oplus (G \star x), n, \mathcal{P}_c) - F(L_{l(x)-1} \oplus (G \star x), n, \mathcal{P}_c). \end{aligned} \tag{26}$$

By applying (26) repeatedly, we have

$$\begin{aligned} & F(G, n, \mathcal{P}_c) \\ = \; & F(G - L(x), n, \mathcal{P}_c) + F(L_{l(x)} \oplus (G \star x), n, \mathcal{P}_c) - F(L_0 \oplus (G \star x), n, \mathcal{P}_c) \\ = \; & F(G - L(x), n, \mathcal{P}_c) + F(L_{l(x)} \oplus (G \star x), n, \mathcal{P}_c). \end{aligned}$$

□

Theorem 4.1 *For any graph G with $v(G) \geq 2$ and $x \in V(G)$,*

$$\Phi_c(G, \mu) = \Phi_c(G - L(x), \mu) + \left(e^{l(x)\mu} - 1\right) \Phi_c(G \star x, \mu), \tag{27}$$

$$\Psi_c(G, \eta) = \Psi_c(G - L(x), \eta) + \left((1+\eta)^{l(x)} - 1\right) \Psi_c(G \star x, \eta). \tag{28}$$

Proof. Equation (27) follows from (25), (12), (13) and (22), and equation (28) follows from (27) and (16). □

When $v(G) \geq 2$, for $x, y \in V(G)$ with $x \neq y$, let $G \odot xy$ be the graph obtained from G by identifying x and y so that $l_{G \odot xy}(w) = b(x, y) + l_G(x) + l_G(y)$, where w is the vertex produced when identifying x and y. (See Figure 1.)

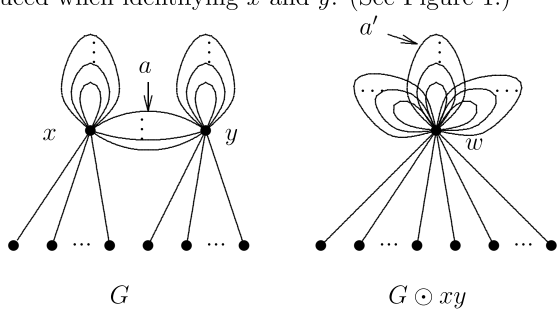


Figure 1

Lemma 4.7 *Let G be a graph and $x, y \in V(G)$ with $x \neq y$. Let a be an edge with end-vertices x and y, and let a' be any loop at w in $G \odot xy$, where w is the new vertex in $G \odot xy$ produced by identifying x and y. For any $n \geq 0$, we have*

$$F(G, a, n, \mathcal{P}_c) = F(G \odot xy, a', n, \mathcal{P}_c), \quad n \geq 0. \tag{29}$$

Proof. Let $B_G(x,y) = \{e_1, e_2, \cdots, e_k\}$ and let e_i' be the corresponding edge of e_i in $G \odot xy$, for $i = 1, 2, \cdots, k$.

For each mapping f from $\{1, 2, \cdots, n\}$ to $E(G)$, define

$$g(t) = \begin{cases} f(t), & \text{if } f(t) \notin \{e_1, \cdots, e_k\}, \\ e_j', & \text{if } f(t) = e_j, \end{cases}$$

for each t such that $1 \le t \le n$. Observe that g is a mapping from $\{1, 2, \cdots, n\}$ to $E(G \odot xy)$. Notice that $f \in \mathcal{F}(G, n, \mathcal{P}_c)$ with $f^{-1}(e_1) \neq \emptyset$ if and only if $g \in \mathcal{F}(G \odot xy, n, \mathcal{P}_c)$ with $g^{-1}(e_1') \neq \emptyset$. Thus

$$F(G, e_1, n, \mathcal{P}_c) = F(G \odot xy, e_1', n, \mathcal{P}_c).$$

Note that $F(G, e_1, n, \mathcal{P}_c) = F(G, a, n, \mathcal{P}_c)$ for any $a \in B_G(x,y)$ and $F(G \odot xy, e_1', n, \mathcal{P}_c) = F(G \odot xy, a', n, \mathcal{P}_c)$ for any loop a' at w. The result follows. $\square$

Lemma 4.8 *For a graph G with $v(G) \ge 3$ and $x, y \in V(G)$ with $x \neq y$,*

$$\begin{aligned} F_c(G, n) &= F_c(G - B_G(x,y), n) + F_c(L_s \oplus (G \star xy), n) \\ &\quad - F_c(L_{s'} \oplus (G \star xy), n), \qquad n \ge 0, \end{aligned} \tag{30}$$

where $s = l_G(x) + l_G(y) + b_G(x,y)$ and $s' = l_G(x) + l_G(y)$.

Proof. Equation (30) holds when $B(x,y) = \emptyset$. For any edge $a \in B(x,y)$, by (29) and (24), we have

$$F_c(G, n) - F_c(G - a, n) = F_c(L_s \oplus (G \star xy), n) - F_c(L_{s-1} \oplus (G \star xy), n). \tag{31}$$

Equation (30) is then obtained by using (31) repeatedly. $\square$

Theorem 4.2 *For a graph G with $v(G) \ge 3$ and $x, y \in V(G)$ with $x \neq y$,*

$$\Phi_c(G, \mu) = \Phi_c(G - B(x,y), \mu) + \left(e^{s\mu} - e^{s'\mu}\right) \Phi_c(G \star xy, \mu), \tag{32}$$

$$\Psi_c(G, \eta) = \Psi_c(G - B(x,y), \eta) + \left((1+\eta)^s - (1+\eta)^{s'}\right) \Psi_c(G \star xy, \eta), \tag{33}$$

where $s = b(x,y) + l(x) + l(y)$ and $s' = l(x) + l(y)$.

Proof. Equation (32) follows from (30), (12), (13) and (22), and equation (33) follows from (32) and (16). $\square$

Remark: If either $v(G) = 1$ or $v(G) = 2$ and G contains no loops, the function $\Psi_c(G, \eta)$ is given by (23). Now let G be a graph with $v(G) \ge 2$, assuming that G contains loops if $v(G) = 2$.

(a) If G is disconnected with components $G_1, G_2, \cdots, G_k$, then by (17),

$$\Psi_c(G, \eta) = \prod_{i=1}^{k} \Psi_c(G_i, \eta);$$

(b) if $L_G(x) \neq \emptyset$ for some $x \in V(G)$, a recursive expression is given by (28);

(c) if $v(G) \ge 3$ and $B(x,y) \neq \emptyset$ for some $x, y \in V(G)$ with $x \neq y$, then a recursive expression is given by (33).

4.2 Examples

Let P_k be the path graph with order k ($k \geq 1$) and C_k be the cycle graph with order k ($k \geq 1$). For $k \geq 2$, let P'_k be the graph obtained from P_k by adding a loop at one end-vertex of the path P_k, and let P''_k be the graph obtained from P_k by adding a loop at each end-vertex of the path P_k. For $k = 1$, we define $P'_k \cong L_1$ and $P''_k \cong L_2$. We shall find formulae for $\Psi_c(P_k, \eta)$ and $\Psi_c(C_k, \eta)$.

Theorem 4.3 *For a graph G with $v(G) \geq 3$ and $x \in V(G)$, if $|E_G(x)| = 1$ and $N_G(x) = \{y\}$ for some $y \neq x$, then*

$$\Psi_c(G, \eta) = \eta(1+\eta)^{l(y)}\Psi_c(G \star xy, \eta). \tag{34}$$

Proof. It follows from Theorem 4.2. □

Theorem 4.4 *For $k \geq 1$, we have*

$$\Psi_c(P_k, \eta) = \sum_{i=0}^{k-1} \binom{i-1}{k-i-1}\eta^i = \sum_{i=\lceil k/2 \rceil}^{k-1} \binom{i-1}{k-i-1}\eta^i. \tag{35}$$

Proof. If $k \leq 2$, (35) follows from (23). Now assume that $k \geq 3$. Let x be an end-vertex of P_k, and let $N(x) = \{y\}$. By Theorem 4.3,

$$\Psi_c(P_k, \eta) = \eta\Psi_c(P'_{k-2}, \eta).$$

Thus $\Psi_c(P_k, \eta) = \eta^2$ for $k = 3$, which implies the result holds for $k = 3$. Now consider the case $k \geq 4$. By Theorem 4.1, we have

$$\begin{aligned}
\Psi_c(P'_{k-2}, \eta) &= \Psi_c(P_{k-2}, \eta) + \eta\Psi_c(P'_{k-3}, \eta) \\
&= \Psi_c(P_{k-2}, \eta) + \eta\Psi_c(P_{k-3}, \eta) + \cdots + \eta^{k-3}\Psi_c(P_1, \eta) + \eta^{k-2}.
\end{aligned}$$

Hence

$$\Psi_c(P_k, \eta) = \eta\Psi_c(P_{k-2}, \eta) + \eta^2\Psi_c(P_{k-3}, \eta) + \cdots + \eta^{k-2}\Psi_c(P_1, \eta) + \eta^{k-1}.$$

By induction, for $2 \leq r \leq k-1$,

$$\begin{aligned}
\eta^{r-1}\Psi_c(P_{k-r}, \eta) &= \eta^{r-1}\sum_{i=0}^{k-r}\binom{i-1}{k-r-i-1}\eta^i \\
&= \sum_{i=0}^{k-r}\binom{i-1}{k-r-i-1}\eta^{i+r-1} \\
&= \sum_{j=r-1}^{k-1}\binom{j-r}{k-j-2}\eta^j \\
&= \sum_{j=0}^{k-1}\binom{j-r}{k-j-2}\eta^j.
\end{aligned}$$

By the identity,

$$\sum_{i=0}^{m} \binom{i}{n} = \binom{m+1}{n+1}, \quad m \geq n \geq 0,$$

we have

$$\sum_{r=2}^{k-1} \binom{j-r}{k-j-2} = \sum_{r=0}^{j-2} \binom{r}{k-j-2} = \begin{cases} \binom{j-1}{k-j-1}, & \text{if } j \leq k-2, \\ 0, & \text{if } j = k-1. \end{cases}$$

Therefore

$$\begin{aligned} \Psi_c(P_k, \eta) &= \eta^{k-1} + \sum_{r=2}^{k-1} \eta^{r-1} \Psi_c(P_{k-r}, \eta) \\ &= \eta^{k-1} + \sum_{r=2}^{k-1} \sum_{j=0}^{k-1} \binom{j-r}{k-j-2} \eta^j \\ &= \eta^{k-1} + \sum_{j=0}^{k-1} \sum_{r=2}^{k-1} \binom{j-r}{k-j-2} \eta^j \\ &= \eta^{k-1} + \sum_{j=0}^{k-2} \binom{j-1}{k-j-1} \eta^j \\ &= \sum_{i=0}^{k-1} \binom{i-1}{k-i-1} \eta^i. \end{aligned}$$

This completes the proof. □

Theorem 4.5 *For $k \geq 2$, we have*

$$\Psi_c(C_k, \eta) = \sum_{i=\lceil k/2 \rceil}^{k} \frac{k}{i} \binom{i}{k-i} \eta^i. \tag{36}$$

Proof. Observe that $C_2 = B_2$. For $k = 2$, the result follows from (23). Now let $k \geq 3$. By (33),

$$\Psi_c(C_k, \eta) = \Psi_c(P_k, \eta) + \eta \Psi_c(P''_{k-2}, \eta).$$

By (34),

$$\Psi_c(P_{k+2}, \eta) = \eta \Psi_c(P'_k, \eta) = \eta^2 \Psi_c(P''_{k-2}, \eta).$$

Hence

$$\Psi_c(C_k, \eta) = \Psi_c(P_k, \eta) + \frac{1}{\eta} \Psi_c(P_{k+2}, \eta).$$

Theorem 4.5 is then obtained. □

5 Acyclic spanning subgraphs

Let G be a graph. For $E' \subseteq E(G)$, let $G[E']$ be the graph with vertex set $V(G)$ and edge set E'. A graph is said to be *acyclic* if it contains no cycles. Observe that an acyclic graph has no loops or multiple edges. Let $\mathcal{P}_a$ be the edge-property that for any graph G,

$$\mathcal{P}_a(G) = \{E' \subseteq E(G) |\ G[E'] \text{ is acyclic}\}. \tag{37}$$

Lemma 5.1 *$\mathcal{P}_a$ is inclusive and resolvable.* □

We write that

$$\begin{aligned} F_a(G,n) &= F(G,n,\mathcal{P}_a), \\ \Phi_a(G,\mu) &= \Phi(G,\mu,\mathcal{P}_a), \\ \Psi_a(G,\eta) &= \Psi(G,\eta,\mathcal{P}_a). \end{aligned}$$

By Theorem 3.4, $\Psi_a(G,\eta)$ is a polynomial in indeterminate η. Observe that $s(G,r,\mathcal{P}_a) = 0$ when $r \geq v(G)$. Thus $\Psi_a(G,\eta)$ is a polynomial of degree at most $v(G)-1$. In fact, it is easy to show that the degree of the polynomial $\Psi_a(G,\eta)$ is $v(G)-c(G)$, where $c(G)$ is the number of components of G, and the coefficient of η^r is the number of acyclic spanning subgraphs of size r in G. We also observe that if G is connected, then the coefficient of $\eta^{v(G)-1}$ in the polynomial $\Psi_a(G,\eta)$ is the number of spanning trees in G.

Since $\mathcal{P}_a$ is resolvable and inclusive, the results in section 1, 2 and 3 hold for $F_a(G,n)$, $\Phi_a(G,\mu)$ and $\Psi_a(G,\eta)$. In this section, we shall develop a method to compute $\Psi_a(G,\eta)$.

Lemma 5.2 *For any graph G, $F_a(G,0) = 1$.* □

Lemma 5.3 *For any graph G, we have*

$$\begin{aligned} F_a(G,n) &= F_a(G',n), \quad n \geq 0, & (38) \\ \Phi_a(G,\mu) &= \Phi_a(G',\mu), & (39) \\ \Psi_a(G,\eta) &= \Psi_a(G',\eta), & (40) \end{aligned}$$

where G' is the graph obtained from G by deleting all loops in G.

Proof. Since $\mathcal{P}_a$ is inclusive, $\mathcal{P}_a(G') \subseteq \mathcal{P}_a(G)$. For any $E' \in \mathcal{P}_a(G)$, E' contains no loops and thus $E' \subseteq E(G')$, which implies that $\mathcal{P}_a(G) \subseteq \mathcal{P}_a(G')$. Hence $\mathcal{P}_a(G) = \mathcal{P}_a(G')$. By the definition of $F(G,n,\mathcal{P}_a)$, (38) follows. It is clear that (39) follows from (12) and (38), and (40) follows from (16) and (39). □

Lemma 5.4 *For any $k \geq 0$,*

$$\begin{aligned} F_a(L_k,n) &= \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{otherwise}, \end{cases} & (41) \\ \Phi_a(L_k,\mu) &= 1, & (42) \\ \Psi_a(L_k,\eta) &= 1. & (43) \end{aligned}$$

Proof. It is clear that (41) holds for $k = 0$. By (38), (41) holds for $k \geq 1$. Equation (42) follows from (12) and (41), and (43) follows from (16) and (42). $\square$

Lemma 5.5 *For any graph G with $v(G) \geq 2$, if x is a vertex in G with $E_x = \emptyset$, then*

$$\begin{aligned} F_a(G, n) &= F_a(G - x, n), \qquad n \geq 0, &(44)\\ \Psi_a(G, \eta) &= \Psi_a(G - x, \eta). &(45) \end{aligned}$$

Proof. Observe that $G = (G - x) \oplus L_0$. Since $\mathcal{P}_a$ is resolvable, by Theorem 1.1 and (41), we have

$$F_a(G, n) = \sum_{r=0}^{n} \binom{n}{r} F_a(G - x, r) F_a(L_0, n - r) = F_a(G - x, n).$$

Equation (45) then follows from (44) and (12). $\square$

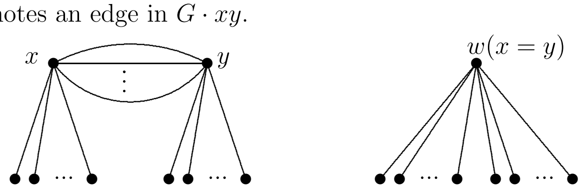
Corollary *For any empty graph G, $\Psi_a(G, \eta) = 1$.*
Proof. By (45), $\Psi_a(G, \eta) = \Psi_a(L_0, \eta)$. The result then follows from (43). $\square$

Lemma 5.6 *For $k \geq 0$, we have*

$$F_a(B_k, n) = k, \qquad n \geq 1. \tag{46}$$

Proof. For $k = 0$, it follows from (44) and (41). Now let $k \geq 1$. Observe that $\mathcal{P}_a(B_k) = \{\emptyset\} \cup \{\{e\} | e \in E(B_k)\}$. Since $n \geq 1$, for any $f \in \mathcal{F}(G, n, \mathcal{P}_a)$, $\{f(1), f(2), \cdots, f(n)\} = \{e\}$ for some $e \in E(B_k)$. Hence $F_a(B_k, n) = k$. $\square$

Let G be a graph without loops. For any vertices x and y in G with $x \neq y$, let $G \cdot xy$ be the graph obtained from $G \odot xy$ by deleting all loops. Let w be the new vertex in $G \cdot xy$ produced by identifying x and y. Figure 2 displays the local structure of $G \cdot xy$. Assume that $b(x, y) = 0$. Observe that $G \cdot xy$ has the same size as G, and for each edge e in G, there is a corresponding edge e^* in $G \cdot xy$ such that e^* has the same end-vertices as e if $e \notin E_x \cup E_y$, and e^* has end-vertices w and u if e has end-vertices x and u or y and u. Hence, if e denotes an edge in G, we assume that e also denotes an edge in $G \cdot xy$.



G $\qquad$ $G \cdot xy$

Figure 2

Lemma 5.7 *Let G be any graph without loops. Let $x, y \in V(G)$ and $e \in B(x, y)$. Then*

$$F(G, e, n, \mathcal{P}_a) = F(B_1 \oplus (G \cdot xy), e', n, \mathcal{P}_a), \qquad n \geq 1, \tag{47}$$

where e' is the edge in B_1.

Proof. For any mapping $f \in \mathcal{F}(G, e, n, \mathcal{P}_a)$, $f^{-1}(e^*) = \emptyset$ for each $e^* \in B(x, y) - \{e\}$. Let G' be the graph obtained from G by deleting all edges in $B(x, y) - \{e\}$. Then $\mathcal{F}(G, e, n, \mathcal{P}_a) \subseteq \mathcal{F}(G', e, n, \mathcal{P}_a)$. Since $\mathcal{P}_a$ is inclusive, $\mathcal{F}(G', e, n, \mathcal{P}_a) \subseteq \mathcal{F}(G, e, n, \mathcal{P}_a)$. Therefore,

$$F(G', e, n, \mathcal{P}_a) = F(G, e, n, \mathcal{P}_a).$$

Thus for the lemma, we need to consider only the case when $b(x, y) = 1$.

Let G_0 denote the graph $B_1 \oplus (G \cdot xy)$. We need to show only that there is a one-to-one correspondence between $\mathcal{F}(G, e, n, \mathcal{P}_a)$ and $\mathcal{F}(G_0, e', n, \mathcal{P}_a)$.

For each mapping f from $\{1, 2, \cdots, n\}$ to $E(G)$, construct a mapping g from $\{1, 2, \cdots, n\}$ to $E(G_0)$ such that

$$g(i) = \begin{cases} f(i), & \text{if } f(i) \neq e, \\ e', & \text{otherwise.} \end{cases}$$

Observe that this construction for g gives a one-to-one correspondence between the mappings from $\{1, 2, \cdots, n\}$ to $E(G)$ and the mappings from $\{1, 2, \cdots, n\}$ to $E(G_0)$. Note that for any graph H and $u, v \in V(H)$ with $b(u, v) = 1$, $H \cdot uv$ is acyclic if and only if H is acyclic. Hence $f \in \mathcal{F}(G, e, n, \mathcal{P}_a)$ if and only if $g \in \mathcal{F}(G_0, e', n, \mathcal{P}_a)$. The result is then obtained. □

Theorem 5.1 *Let G be any graph without loops. Let $x, y \in V(G)$ with $x \neq y$. Then for any $n \geq 0$,*

$$\begin{aligned} F_a(G, n) &= b(x, y)\left(F_a(B_1 \oplus (G \cdot xy), n) - F_a(G \cdot xy, n)\right) \\ &\quad + F_a(G - B(x, y), n). \end{aligned} \tag{48}$$

Proof. By Lemma 5.2, (48) holds for $n = 0$. Now assume that $n \geq 1$. Let e be any edge in $B(x, y)$. We have

$$F(G, e, n, \mathcal{P}_a) = F_a(G, n) - F_a(G - e, n).$$

Let e' be the edge in B_1. Note that

$$F(B_1 \oplus (G \cdot xy), e', n, \mathcal{P}_a) = F_a(B_1 \oplus (G \cdot xy), n) - F_a(B_0 \oplus (G \cdot xy), n).$$

By (47) and (44),

$$F_a(G, n) = F_a(G - e, n) + F_a(B_1 \oplus (G \cdot xy), n) - F_a(G \cdot xy, n). \tag{49}$$

By using (49) repeatedly, (48) is obtained. □

Theorem 5.2 *Let G be any graph having no loops and with $v(G) \geq 2$. Let $x, y \in V(G)$ with $x \neq y$. Then*

$$\Psi_a(G, \eta) = \Psi_a(G - B(x, y), \eta) + b(x, y)\eta\Psi_a(G \cdot xy, \eta). \tag{50}$$

Proof. By Lemma 5.6, $\Phi_a(B_1, \mu) = e^{\mu}$. By Theorem 3.1,

$$\Phi_a(B_1 \oplus (G \cdot xy), \mu) = e^{\mu}\Phi_a(G \cdot xy, \mu).$$

Thus by (12) and (48), we have

$$\Phi_a(G, \mu) = \Phi_a(G - B(x, y), \mu) + b(x, y)(e^{\mu} - 1)\Phi_a(G \cdot xy, \mu).$$

Therefore, by (16), (50) is obtained. □

Lemma 5.3 shows that when we consider the function $\Psi_a(G, n)$, G can be supposed to have no loops. By the corollary to Lemma 5.5, $\Psi_a(G, \eta) = 1$ if G is an empty graph. For any graph G with $v(G) \geq 2$ and $e(G) \geq 1$, Theorem 5.2 gives a recursive expression for $\Psi_a(G, \eta)$. In the following, we give some properties of $\Psi_a(G, \eta)$ and determine the function $\Psi_a(G, \eta)$ for some special graphs. By Theorem 5.2, we have

Lemma 5.8 *Let G be a graph and $x \in V(G)$ with $N_G(x) = \{y\}$ for some $y \neq x$. Then*

$$\Psi_a(G, \eta) = (1 + b(x, y)\eta)\Psi_a(G - x, \eta). \tag{51}$$

□

Theorem 5.3 *For any forest T with size k,*

$$\Psi_a(T, \eta) = (1 + \eta)^k. \tag{52}$$

Proof. Equation (52) holds when $k = 0$, by the corollary to Lemma 5.5. If $k \geq 1$, T contains a vertex x with $N_T(x) = \{y\}$ for some vertex y. By Lemma 5.8,

$$\Psi_a(T, \eta) = (1 + \eta)\Psi_a(T - x, \eta).$$

Observe that $T - x$ is a forest with size $k - 1$. Then by induction, (52) is obtained. □

Lemma 5.9 *Let G_1 and G_2 be two subgraphs of G such that $V(G_1) \cup V(G_2) = V(G)$, $V(G_1) \cap V(G_2) = \{x\}$ and $E(G_1) \cup E(G_2) = E(G)$. Then*

$$\Psi_a(G, \eta) = \Psi_a(G_1, \eta)\Psi_a(G_2, \eta). \tag{53}$$

Proof. If $e(G_1) = 0$, then (53) holds by (45) and the corollary to Lemma 5.5. Let k be any positive integer. Assume that (53) holds if $e(G_1) < k$. Now suppose that $e(G_1) = k$. Let $u, v \in V(G_1)$ with $b(u,v) \geq 1$. By Theorem 5.2,

$$\Psi_a(G_1, \eta) = \Psi_a(G_1 - B(u,v), \eta) + b(u,v)\eta\Psi_a(G_1 \cdot uv, \eta).$$

Observe that $e(G_1 - B(u,v)) = k - b(u,v) < k$ and $e(G_1 \cdot uv) = k - b(u,v) < k$. Thus by induction,

$$\begin{aligned} \Psi_a(G - B(u,v), \eta) &= \Psi_a(G_1 - B(u,v), \eta)\Psi_a(G_2, \eta), \\ \Psi_a(G \cdot uv, \eta) &= \Psi_a(G_1 \cdot uv, \eta)\Psi_a(G_2, \eta). \end{aligned}$$

By Theorem 5.2 again,

$$\begin{aligned} \Psi_a(G, \eta) &= \Psi_a(G - B(u,v), \eta) + b(u,v)\eta\Psi_a(G \cdot uv, \eta) \\ &= \Psi_a(G_1 - B(u,v), \eta)\Psi_a(G_2, \eta) + b(u,v)\eta\Psi_a(G_1 \cdot uv, \eta)\Psi_a(G_2, \eta) \\ &= \left(\Psi_a(G_1 - B(u,v), \eta) + b(u,v)\eta\Psi_a(G_1 \cdot uv, \eta)\right)\Psi_a(G_2, \eta) \\ &= \Psi_a(G_1, \eta)\Psi_a(G_2, \eta). \end{aligned}$$

□

Corollary *If G has k blocks $G_1, G_2, \cdots, G_k$, then*

$$\Psi_a(G, \eta) = \prod_{i=1}^{k} \Psi_a(G_i, \eta).$$

□

Theorem 5.4 *For the cycle C_n, where $n \geq 1$,*

$$\Psi_a(C_n, \eta) = (1 + \eta)^n - \eta^n. \tag{54}$$

Proof. Note that $C_1 \cong L_1$. By Theorem 5.3, the result holds for $n = 1$. By Lemma 5.8, $\Psi_a(C_2, \eta) = \Psi_a(B_2, \eta) = 1 + 2\eta$. Hence (54) holds when $n = 2$. Now let $n \geq 3$. By Theorems 5.2 and 5.3 and by induction, we have

$$\begin{aligned} \Psi_a(C_n, \eta) &= \Psi_a(P_n, \eta) + \eta\Psi_a(C_{n-1}, \eta) \\ &= (1 + \eta)^{n-1} + \eta((1 + \eta)^{n-1} - \eta^{n-1}) \\ &= (1 + \eta)^n - \eta^n. \end{aligned}$$

□

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(Received 19 Apr 2002; revised 19 Jan 2004)