Coincidence Bell Inequality for Three Three-Dimensional Systems

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We construct a Bell inequality for coincidence probabilities on a three three-dimensional (qutrit) system. We show that this inequality is violated when each observer measures two noncommuting observables, defined by the so-called unbiased six-port beam splitter, on a maximally entangled state of two qutrits. The strength of the violation agrees with the numerical results presented by Kaszlikowski *et al.*, quant-ph/0202019. It is proven that the inequality defines facets of the polytope of local variable models.

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The nonexistence of local-realistic (LR) models that could reproduce the correlations for the experimental outcomes observed in composite quantum systems was shown by Bell in 1964 [1] through the violations of certain constraints, known as Bell inequalities. The Bell inequalities and the Clauser-Horne-Shimony-Holt (CHSH) inequality [2], the latter being cast into a form more amenable for experimental verification, were formulated for the simplest composite quantum system, namely, a system of two two-dimensional particles (or two qubits). Since then, Bell arguments have been generalized to more complicated situations, either for a larger number of particles or for two particles of dimension greater than two.

For three two-dimensional particles, Greenberger, Horne, and Zeilinger presented an elegant argument, also known as GHZ paradox, where the conflict between classical theories and quantum mechanics was shown to be qualitatively stronger in this case than for two qubits [3]. For N ($N > 3$) two-dimensional particles, Mermin, Belinskii, and Klyshko separately generalized the CHSH inequality and proved that the quantum violation of this inequality increases exponentially with the number of particles [4,5].

For two particles of dimension greater than two, it was found that the CHSH inequality can be maximally violated in higher dimensional systems and this violation continues to survive in the limit of infinite dimension [6]. In Ref. [7], the authors showed, using a numerical procedure with two separated observers who can choose between two von Neumann measurements that the contradiction between LR models and quantum mechanics increases with the dimension, d. Their results were later confirmed analytically in [8,9] where a Bell inequality for two d-dimensional particles, also called qudits, was given in [8,9]. Moreover, this inequality was shown to be tight, in the sense that it defines one of the facets of the convex polytope of LR models [10].

The experimental implementation, i.e., the two von Neumann measuring apparatus, needed for the maximal violation of this inequality are relatively easy to construct: they belong to the class of the so-called tritter measurements (or unbiased six-port beam splitter) which are experimentally realizable [11,12]. Surprisingly, the maximal violation of the inequality is not obtained for two-qudit maximally entangled state [13], an unexpected result that still lacks of an intuitive explanation.

Moving to higher dimension, very little is known for N-qudit systems, with $N, d > 2$. GHZ paradoxes have been generalized in [14,15], and some numerical results have been presented in [16] for three- and four-qutrit systems. In this Letter, we present an interesting coincidence Bell inequality for three qutrits in the case for which each observer measures two noncommuting observables. This inequality imposes necessary conditions on the existence of an LR description for the correlations generated by three qutrits. We show the quantum violation of this inequality in a Gedanken experiment whose measurements are performed by the observers using unbiased symmetric six-port beam splitters on a maximally entangled state. The threshold for the violation is the same as predicted numerically in Ref. [16].

We consider the following Bell-type scenario: three space-separated observers, denoted by A, B, and C (or Alice, Bob, and Charlie), can measure two different local observables of three outcomes, labeled by 0, 1, and 2. We denote by X_i the observable measured by party X and by x_i the outcome with $X, = A, B, C$ ($x = a, b, c$). If the observers decide to measure A_1, B_1, and C_2, the result is (0, 2, 1) with probability $p(a_1 = 0, b_1 = 2, c_2 = 1)$. The set of these 8×27 probabilities gives a complete description of any statistical quantity that can be observed in such Gedanken experiment. We denote by $p(a_i + b_j + c_k = r)$ the *coincidence* probability

$$p(a_i + b_j + c_k = r) = \sum_{a,b=0,1,2} p(a_i = a, b_j = b, c_k = r - a - b), \quad (1)$$

where all the equalities are modulo three.

Any LR description of the gedanken experiment, must satisfy some constraints, known as Bell inequalities. Some of these constraints, such as normalization and the no-signaling condition, are trivial, in the sense that these conditions are also true for quantum mechanics. Thus the latter are useless for performing a Bell test. Nevertheless, it is possible to find more refined inequalities that do allow one to check whether one can describe quantum correlations by a classical model. For simplicity, we consider coincidencelike inequalities, where only terms such as (1) appear. Using the same ideas as in Refs. [8,9] one can see that the following condition is satisfied by all LR theories,

$$\begin{aligned} &p(a_1 + b_1 + c_1 = 0) + p(a_1 + b_2 + c_2 = 1) + p(a_2 + b_1 + c_2 = 1) + p(a_2 + b_2 + c_1 = 1) + \\ &p(a_2 + b_2 + c_2 = 0) - p(a_2 + b_1 + c_1 = 2) - p(a_1 + b_2 + c_1 = 2) - p(a_1 + b_1 + c_2 = 2) \le 3. \end{aligned} \quad (2)$$

In order to derive this bound, we can restrict our considerations to deterministic local models. This is because any probabilistic model can be transformed into a deterministic one by simply adding some additional variables [17]. Any of these models is completely specified by fixing the outcome of all the six local observables, i.e., a six-component vector $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)$. The corresponding probabilities are

$$p(a_i = a, b_j = b, c_k = c) = \delta_{a,\alpha_i}\delta_{b,\beta_j}\delta_{c,\gamma_k}, \quad (3)$$

where $a, b, c = 0, 1, 2$ and $i, j, k = 1, 2$. The rest of LR models correspond to convex combinations of these 3^6 points, i.e., they are the generators of the polytope of LR models. Coming back to the previous inequality, if one takes equal to one as many of the positive terms as possible, trying to beat the bound, the local-realistic (LR) constraints force some of the terms with negative sign to take the value one. For instance, consider the case in which the first term in each line is satisfied. This means that $a_1 + b_1 + c_1 + a_2 + b_2 + c_2 = 0$. Therefore, if one of the other terms in the first line is fulfilled, the corresponding term in the second line with negative sign must be also one. After some simple algebra, one can easily show that for all the deterministic models saturating the previous inequality, all the terms must be zero except three of the four terms in the first line. In particular, the fifth term with positive sign, $p(a_2 + b_2 + c_2 = 0)$, cannot be one if the inequality is saturated. In other words, Eq. (2) gives at most 2 for all the models where $a_2 + b_2 + c_2 = 0$. This suggests that one can increase its weight without changing the bound, the new inequality being

$$\begin{aligned} &p(a_1 + b_1 + c_1 = 0) + p(a_1 + b_2 + c_2 = 1) + p(a_2 + b_1 + c_2 = 1) + p(a_2 + b_2 + c_1 = 1) + \\ &2p(a_2 + b_2 + c_2 = 0) - p(a_2 + b_1 + c_1 = 2) - p(a_1 + b_2 + c_1 = 2) - p(a_1 + b_1 + c_2 = 2) \le 3. \end{aligned} \quad (4)$$

This is the final form for our three-qutrit Bell inequality. We should stress at this point that the above inequality is a member of the set of inequalities that can obtained from (4) by permutations of the indices enumerating the outcomes of the measurements as well as by permutations of the indices enumerating the observables. One can see that this inequality is tight, i.e., it gives one of the facets of the polytope of LR models. Indeed, the number of linearly independent generators that saturate the inequality turns out to be equal to the number of linearly independent generators, see Eq. (3), minus 1. This condition is satisfied only by tight inequalities [10].

Taking $c_1 = c_2 = 0$ in Eq. (4), one derives the two-qutrit inequality

$$\begin{aligned} &p(a_1 + b_1 = 0) + p(a_1 + b_2 = 1) + p(a_2 + b_1 = 1) + p(a_2 + b_2 = 0) - \\ &p(a_1 + b_1 = 2) - p(a_2 + b_1 = 2) - p(a_1 + b_2 = 2) - p(a_2 + b_2 = 2) \le 2. \end{aligned} \quad (5)$$

Interestingly, this is just the inequality recently derived in [8] for two-qutrit systems. As mentioned previously, this inequality is known to be tight [10,18]. Moreover, if one restricts the considerations to coincidence probabilities as in Eq. (1), it is a necessary and sufficient condition for the existence of a classical description (see [10] for more details). Unfortunately, we were not able to derive from Eq. (5) the Mermin-Belinskii-Klyshko inequality.

After deriving the Bell inequality, our next step will be to look for quantum states and measurements violating it. In the general approach to this problem, the initial three-qutrit pure state shared by the parties is not fixed and they can apply the most general quantum measurements, i.e., the so-called positive operators valued measures (POVM). Of course, this is a very hard problem. Indeed, even if we restrict the possible measurements to projective ones (or von Neumann), each measurement is defined by six real parameters. If we add the 52 real parameters needed for specifying a three-qutrit pure state, we have

$6 \times 6 + 52 = 88$ real parameters. Not all the parameters are independent. Moreover, even if we consider local unitary transformations to remove some parameters, it is still not a tractable problem. We need to make some assumptions.

First, as initial state, we take

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle), \tag{6}$$

which can be regarded as a generalization of the maximally entangled state of two qutrits [19]. Next, we restrict to the experimentally feasible tritter measurements, or unbiased symmetric six-port beam splitters [11,12]. The action of these devices in the computational basis is as follows: first a phase factor is applied depending on the initial state, i.e., $|j\rangle \to e^{i\phi_j}|j\rangle$, where $j = 0, 1, 2$. Following this, a Fourier transform is performed and the resulting state is measured in the computational basis. Therefore, any of these measurements is defined by a three-phase vector $\vec{\phi} = (\phi_0, \phi_1, \phi_2)$, and the corresponding unitary transformation,

$$[U_{\rm QFT} U(\vec{\phi})]_{ij} = \frac{1}{\sqrt{3}} \exp\left[i\frac{2\pi}{3}(i-1)(j-1) \right] \exp[i\phi_{(i-1)}]. \tag{7}$$

The probability of obtaining the outcome (a, b, c), given a measurement apparatus for Alice, Bob, and Charlie specified by the three-phase vectors $\vec{\phi}_A, \vec{\phi}_B, \vec{\phi}_C$, and an initial state $|\Phi\rangle \in \mathbb{C} \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ is

$$p(a, b, c) = |\langle abc| U_{\rm QFT} U(\vec{\phi}_A) \otimes U_{\rm QFT} U(\vec{\phi}_B) \otimes U_{\rm QFT} U(\vec{\phi}_C) |\Phi\rangle|^2. \tag{8}$$

Note that one can take, without loss of generality, the first term in each phase vector equal to zero, i.e., $\vec{\phi} = (0, \phi, \phi')$. The phase vectors can be changed by the observers; they represent the local macroscopic parameters available to them. For the coincidence terms appearing in our Bell inequality and the state (6), tritter measurements give simple and nice expressions, as it happens in the bipartite case. It is easy to see that

$$p(a_i + b_j + c_k = r) = \frac{1}{9}\left[3 + 2\cos\left(\varphi + \frac{2r\pi}{3}\right) + 2\cos\left(\varphi' + \frac{4r\pi}{3}\right) + 2\cos\left(\varphi' - \varphi + \frac{2r\pi}{3}\right)\right], \tag{9}$$

where $\varphi = \phi_{A_i} + \phi_{B_j} + \phi_{C_k}$ and analogously for φ'. Indeed, all the probabilities appearing in Eq. (1) are equal due to the symmetry in the state and the measurements, i.e., $p(a_i + b_j + c_k = r) = 9p(a_i = 0, b_j = 0, c_k = r)$.

We can now look for the maximal violation of the inequality, under the given assumptions. The optimal settings correspond to the following phase vectors (the first component of the phase vectors is always zero):

$$\begin{aligned} &\vec{\phi}_{A_1} = (0, 0), \qquad \vec{\phi}_{B_1} = \left(-\frac{2\pi}{3}, -\frac{\pi}{3}\right), \\ &\vec{\phi}_{C_1} = \left(\frac{\pi}{3}, 0\right), \qquad \vec{\phi}_{A_2} = \left(0, \frac{2\pi}{3}\right), \\ &\vec{\phi}_{B_2} = \left(-\frac{2\pi}{3}, \frac{\pi}{3}\right), \qquad \vec{\phi}_{C_2} = \left(\frac{\pi}{3}, \frac{2\pi}{3}\right). \end{aligned} \tag{10}$$

For this choice of settings, and the state (6), all the probabilities terms with a positive sign are equal to $7/9$, while the terms with negative sign are equal to $1/9$, so the inequality gives $6 \times 7/9 - 3 \times 1/9 = 39/9 \simeq 4.33 > 3$.

In Ref. [7], the so-called resistance to noise was proposed as a measure of the strength of quantum correlations for violating local realism. It specifies the amount of white noise to be added to a system such that it looses its nonclassical correlations. For a three-qutrit pure state $|\Phi\rangle$, it is equal to the value of λ such that the state

$$\rho(\lambda, |\Phi\rangle) = (1 - \lambda)|\Phi\rangle\langle\Phi| + \lambda\frac{\mathbb{1}}{27} \tag{11}$$

admits a classical description. Note that $0 \leq \lambda \leq 1$. It roughly estimates the distance between the quantum probabilities and the polytope given by all the LR models. It is a good measure of nonclassical condition due to its simplicity and therefore it is easily computable when the system is not too complex. However, it may lead to some unexpected (and in some sense unwanted) results [13]. For instance, for two qutrits and two projective measurements per party, it is maximized for a nonmaximally entangled state [13]. This result seems to hold for arbitrary dimension [13], and it was generalized to the multipartite scenario and tritter measurements in Ref. [16]. There, it was also numerically shown that the resistance to noise for the state (6), when any observer can choose between two tritter measurements, is $\lambda = 0.4$. Remarkably, our inequality reproduces this result, since $\rho(\lambda, |\Psi\rangle)$ does not violate it for $\lambda > 0.4$.

With some measurements fixed, and depending on the form of the Bell inequality, one can construct the so-called Bell operator [20], B. For a quantum state ρ, the function ${\rm Tr}(B\rho)$ gives its Bell value. The maximal violation of the inequality (with the initially chosen measurements) corresponds to the maximal eigenvalue of this operator. For our inequality (4) and the settings specified by Eqs. (10), the Bell operator has a simple structure, with blocks of 3×3 matrices of nonzero entries. The maximal eigenvalue of B is given by the maximal eigenvalue of the 3×3 matrix $M_{ij} = 1 + \delta_{i1}(1 - \delta_{j1})$ for the subspace spanned by $|000\rangle, |111\rangle, |222\rangle$. Therefore, the maximal violation is equal to $(3 + \sqrt{33})/2 \simeq 4.37$, slightly larger than the violation for the maximally entangled state. The corresponding state, $|\Psi_{\rm mv}\rangle$, has the same form as the state (6) but the coefficient of the $|000\rangle$ term is larger. This is quite similar to what happens in the case of two qutrits [13]. We have performed a numerical search for the maximal violation of our inequality using tritter measurements, and the result is given by the state $|\Psi_{mv}\rangle$ and the phase vectors of Eqs. (10). This means that the resistance to noise for the best Bell test using our inequality and tritter measurements is $\simeq 0.407$. It was shown in [16] that there exists a three-qutrit state for which the resistance to noise, using tritters, is equal to 0.571. Therefore, and contrary to what happens in the bipartite case, our inequality does not allow one to reproduce this maximal resistance to noise for tritter measurements, although it does it for the maximally entangled state.

For more general measurements (von Neumann type), we have numerically found that the maximal violation of our inequality is indeed 4.3723. However, the resistance to noise for the state (6) increases up to $\lambda \simeq 0.527$, which cannot be predicted by our inequality. This value is greater than $1/2$, the maximal resistance to noise for three-qubit states and two von Neumann measurements per observer, that is obtained for the GHZ state, $|\mathrm{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$. Finally, it can be shown that our qutrit inequality can be violated with a three-qubit GHZ state and three-outcome measurements (of course, not projective ones).

There are few results extending the Bell argument to N-qudit systems ($N, d > 2$) where the disagreement with LR models becomes more pronounced. In this article, we have presented a coincidence Bell inequality for three-qutrit states, where the parties can choose between two noncommuting observables. Our inequality is optimized for the case in which the initial state is equal to (6) and tritter measurements are applied. Indeed it reproduces the numerical predictions in Ref. [16]. Moreover, this inequality defines facets of the LR models. When restricted to the bipartite case, it reduces to the Collins-Gisin-Linden-Massar-Popescu inequality, which is also known to be tight. Unfortunately, we could not extend the structure of our inequality to a higher number of parties or dimension and this problem remains as an interesting open question.

Tritter measurements have played a significant role in the analysis of nonlocal correlations in bipartite quantum systems. They suffice for revealing the states whose nonclassical correlations are most resistant against noise. They are also optimal for N-qubit systems, since they maximize the Bell violation for the Mermin inequality and the GHZ state [4,21]. It follows from our previous discussion that they are no longer optimal for more complex systems (three qutrits), and more general measurements are needed.

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Note added in proof.—After completion of this work, we found that the presented Bell inequality reduces to the Bell inequality of Chen *et al.* [22]when one restricts to two outcomes. Numerical calculations indicate that this second inequality is violated by all three-qubit pure entangled states.

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