Reflections on HCF and LCM: a variety of Mathematical Connections

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This article attempts to provide an integration of commonly perceived ideas of the concepts of HCF (highest common factor) and LCM (least common multiple) with other ideas which have not been widely circulated. For example, many previous articles have tended to focus on procedural aspects of GCF and LCM (Adams 1982, Adkins 1981, Henry 1978, Lamb and Hutcherson 1984, McLellan 1985, Moeckel 1986, Roy 1978, Stern 1984 and 1985) and have not sufficiently emphasized two intuitive basic notions repeated subtraction and repeated addition. Furthermore, it has rarely been noticed that there are two possible interpretations of the GCF of a pair of numbers and the recognition of these interpretations will enlarge the conceptual understanding of the concept.

The article is not concerned with providing 'how to teach' guidelines for the teaching of HCF and LCM but aims to show the variety of mathematical connections which can and should be recognised by a teacher. Wherever appropriate, comments will be made to indicate how and when the relationships can be developed, and how a teacher might help students make connections between their current understanding of the concepts and extensions of that understanding.

Basic ideas and representations

Some basic ideas and representations that are commonly used to introduce the concepts of HCF and LCM as well as to obtain their computation are summarised below.

Given any two non-zero whole numbers r and s, there is a highest or greatest factor that they have in common; this factor is called the HCF. By a factor of a number is meant a divisor of the number which leaves a remainder of zero upon division. The technicality of the definition can be illuminated by using the idea of repeated subtraction. For example, 3 is a factor of 15 because repeated subtraction of 15 by 3 up to five times leaves a remainder of zero.

Often neglected from mention is the observation that if r and s are equal then the HCF is r (or s); but if r and s are unequal (say r is less than s), then since 1 is always a factor of any number, so the HCF is either 1 or a number smaller than or equal to r.

Examples:

(a) For the two numbers r = 15 and s = 15,
the HCF = 15.

(b) For the two numbers r = 7 and s = 15,
the HCF = 1. Such numbers that have 1 as HCF are said to be *relatively prime* to each other.

(c) For the two numbers r = 12 and s = 15,
the HCF = 3.

(d) For the two numbers r = 5 and s = 15,
the HCF = 5.

Given any two non-zero whole numbers r and s, there is a smallest or least multiple that they have in common; this multiple is called the LCM. By a multiple of a number is meant the number obtained by repeated addition. For example, 15 is a multiple of 3 because repeated addition of 3 up to five times gives the number. Conversely, 3 is said to be a factor of 15. Hence, the terms 'multiple' and 'factor' are related in the sense that if r is a multiple of s then s is a factor of r. This relation can be illustrated by the idea that, for example, if 15 can be reduced to zero by repeated subtraction of 3 up to five times, then repeated addition of 3 up to the same number of times will reproduce the original number 15.

Often neglected from mention is the observation that if r and s are equal then the LCM is r (or s); but if r and s are unequal (say r is less than s), then repeated additions of each of the numbers respectively will always produce a common multiple. The last assertion can be illuminated by the teacher by getting the students to build 'trains' of equal length with two different lengths of number rods.

Examples:

(a) For the two numbers $r = 15$ and $s = 15$,
the LCM = 15. Other common multiples are 30, 45, ...

(b) For the two numbers $r = 3$ and $s = 5$,
the LCM = 15 since five groups of 3 and three groups of 5 will produce the common multiple of 15.

(c) For the two numbers $r = 6$ and $s = 10$,
the LCM = 30 since (once again) five groups of 6 and three groups of 10 will produce the common multiple.

The conceptual knowledge of HCF and LCM as outlined above is usually preceded by appropriate activities involving identifying and recognizing factors and multiples of numbers.

Examples: Play activity with cards: What am I?

I am • less than 20 • greater than 10 • a multiple of 6 • a factor of 24	I am • less than 30 • greater than 20 • a common multiple of 7 and 14

All too frequently however conceptual knowledge is soon supplanted by procedural knowledge which is more concerned with computing HCF and LCM. The following are commonly used methods.

LISTING OF ELEMENTS

A: Order factors by size	B: Order multiples by size
Examples:	
30: **1**, **2**, 5, **6**, 10, 30	30: 30, 60, **90**, 120, ...**180**
18: **1**, **2**, 3, **6**, 9, 18	18: 18, 36, ... **90**, ... **180**
HCF (18, 30) = 6	LCM (18, 30) = 90

LONG DIVISION

C: Product of common factors

Examples:

2	18	30
3	9	15
	3	5

HCF (18, 30) = 2 x 3 = 6

D: Product of all factors

2	18	30
3	9	15
	3	**5**

LCM (18, 30) = 2 x 3 x 3 x 5
= 90

PRIME FACTORIZATION

E: Product of common factors

30 = **2** x **3** x 5

18 = **2** x 3 x **3**

HCF = 2 x 3 = 6

F: Product of all factors

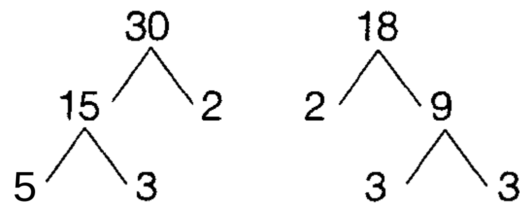
30 = 2 x 3 x 5

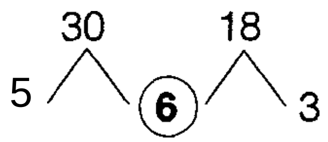
18 = 2 x 3 x 3

LCM = **2** x **3** x **3** x **5**

TREE DIAGRAMS

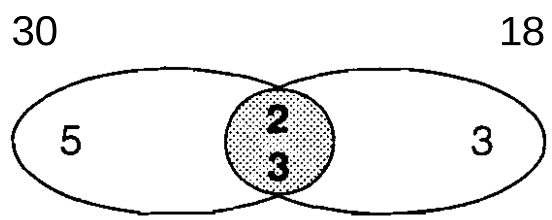
Examples:



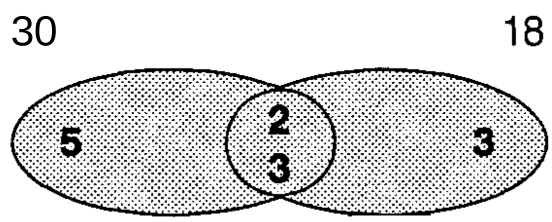


HCF = 6
LCM = 5 x 6 x 3 = 90

VENN DIAGRAMS



HCF = 6 (set intersection)



LCM = 90 (set union)

A number of observations should be made concerning the above methods. The method of Listing of Elements exhibits the meanings of the words 'common', 'factor', 'greatest', 'highest' 'multiple' and 'least' quite clearly and thus it facilitates understanding of the underlying ideas and makes precise their expression. All the methods require either by trial and error or through experience the determination of all factors or all prime factors of a number; thus, only for fairly simple numbers will the methods be practicable. A common classroom practice is to complement the method of Listing of Elements with either the Long Division or Prime Factorization method. The methods using Tree or Venn Diagrams have the advantage of providing pictorial representations and lend themselves to simple interpretations of the other methods. A further advantage of these methods is that they provide students with necessary experiences for the appreciation in later years of the fundamental theorem of arithmetic *(Every non-zero whole number can be factorized as a product of prime factors in one and only one way.)* (Greenleaf and Wisner 1959, Rogers Jr. 1970) A disadvantage of these methods is that they rely entirely on numerical manipulations and do not appeal to the basic notions of repeated subtraction or addition which underlie the concepts of HCF and LCM.

Problem representations,

Translation problems that reflect real world situations can and should be used to motivate students and show the relevance and application of the concepts of HCF and LCM. Solving of some of the problems does not have to depend upon acquiring the procedural knowledge outlined above. The following problems have been selected to show certain important features of the concepts.

1 At a birthday party, each child receives the same number of cookies. Each child also receives the same number of candies. There are 36 cookies and 45 candies. What is the greatest number of children who could attend the party? Could fewer children have attended the party and still meet the conditions?

2 Alice has a blue ribbon which is 45 cm long and a red ribbon which is 36 cm long. She wants to cut each ribbon into an equal number of smaller pieces. Each piece whether blue or red must be the same length. What is the longest size for each of the pieces?

3 A rectangular piece of cardboard is 30 cm long and 12 cm wide. It is to be cut up into an exact number of equal squares. Find the least possible number of squares.

4 This Sunday Jimmy mowed the lawn and trimmed the hedge. The lawn needs to be mowed every 7 days and the hedge to be trimmed every 9 days. When will Jimmy have to do both jobs again on the same day?

5 A collection of toys can be separated into 14 equal groups in one way, and into 18 equal groups in another way. What is the smallest number of toys in the collection?

6 A length of stiff wire is measured in whole number of cm units. It can be bent into 7 equal parts in one way. It can also be bent into 9 equal parts in another way. What is the shortest length of wire in cm units?

Both Problems 1 and 2 have the same solution 9 which is the HCF of 36 and 45. There are however two interpretations to this solution: in Problem 1 it refers to the number of children while in Problem 2 it refers to the length of a piece of, ribbon. In the first case, the HCF may be regarded as a 'counting' factor as it alludes to the number of children receiving cookies or candies; thus, each of 9 children will receive 4 cookies and 5 candies. The numbers 4 and 5 however represent the basic partitions of the set of cookies or candies. In the second case, the HCF may be said to be a 'base' factor as it alludes to the basic common length of a piece of ribbon; that is, there are 5 basic pieces of blue ribbon and 4 basic pieces of red ribbon after the partition. Computational methods outlined previously do not distinguish between 'counting' and 'base' factors.

The idea of HCF as a 'base' factor leads to an alternative and more general method of computing the HCF. For example, in order to answer Problem 3, one needs to determine the largest square that can be subtracted from the sides of 12 cm and 30 cm. This can be done by first finding the difference between 30 and 12 which is 18, then the difference between 18 and 12 which is 6; but 6 will go into 12 an even 2 times and go into 30 exactly 5 times without remainder. Hence the HCF of 12 and 30 is 6. Thus, the least possible number of squares (which is 10) is found by taking the largest possible square which has side 6 cm.

The next three problems involve the LCM. Both Problems 4 and 6 have the same solution 63 which is the LCM of 7 and 9. The solution is obtained by taking the product of 7 and 9. This is tantamount to repeated addition of 7 nine times and of 9 seven times. The building of equal 'trains' using '7' and '9' number rods provides a concrete representation for these operations.

7 7 7 7 7 7 7 7 7	a train of '7' rods
9 9 9 9 9 9 9	a train of '9' rods

Notice that 7 and 9 are relatively prime, i.e. they have no common factor except the number 1.

In Problem 5, the solution 126 is not the product of 14 and 18 but the product of 2, 7 and 9. The reason is that 2 is the HCF of 14 and 18 and hence 2 x 7 x 9 is the same as 14 x 9 or 18 x 7. An explanation in concrete terms is illustrated as follows.

A train of '14' rods repeated *eighteen* times

is as long as

A train of '18' rods repeated *fourteen* times

Notice that since 2 is the HCF of 14 and 18, it follows that the pair of trains above can be separated into two sets of shorter equal trains as shown below.

A train of '14' rods repeated *nine* times

is as long as

A train of '18' rods repeated *seven* times

Alternative representations

The discussion of selected problems points to two important mathematical connections. Firstly, the 'base' nature of the HCF as exemplified in Problems 2 and 3 corresponds to the idea'of repeated subtraction which can be used to obtain the HCF of two numbers. Generally, suppose there are two non-zero whole numbers r and s with r less than s. For example, r = 312 and s = 441. To find the HCF of r and s means to find the largest number that will go an exact number of times into each of r and s. The HCF can be found by the following process.

> 441 - 312 = 129. We can take 129 twice from 312, leaving 54 as remainder. We can take 54 twicefrom 129, leaving 21 as remainder. We can take 21 twice from 54, leaving 12 as remainder. We can take 12 once from 21, leaving 9 as remainder. We can take 9 once from 12, leaving 3 as remainder. We can take 3 thrice from 9, leaving 0 as remainder. The last non-zero remainder 3 gives the value of the HCF. That is, 441 = 3 x 147 and 312 = 3 x 104 and 147 and 104 have no factor in common except 1.

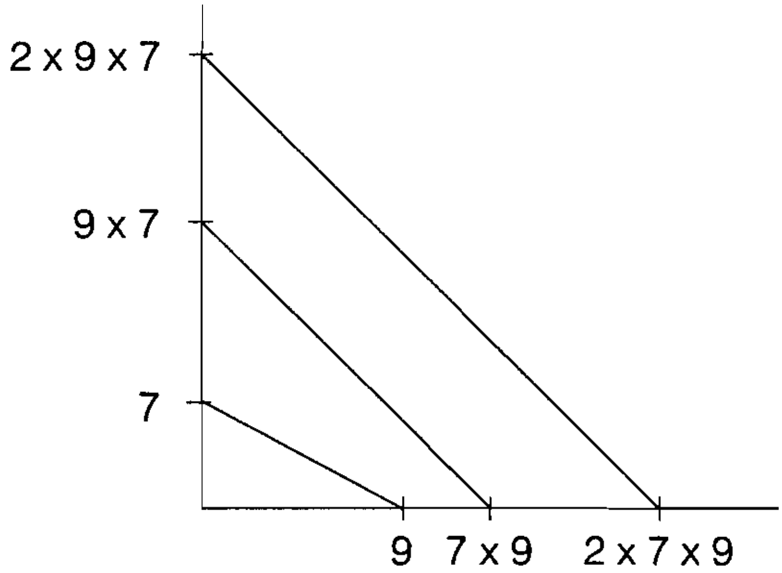
Note that in the above process the number of times we can take a number p from a number q depends on the relative sizes of-the two numbers. Note also that the sequence of remainders 129, 54, … 3, 0 is strictly decreasing and hence must terminate at zero. This ensures that the process is finite. The process of repeated subtraction used in this way has long been known as the *Euclidean Algorithm* (Adkins 1981, Davies 1980, Rogers Jr. 1970). Often the Euclidean Algorithm is treated as a mathematical curiosity and relegated as an enrichment or challenge activity when in fact it represents afundamental perspective on the concept of HCF.

Secondly, it has been shown that if a pair of numbers (r, s) is reduced to its lowest terms (p, q) so that p and q are relatively prime, then the LCM of r and s is related to the LCM of p and q. In fact, the LCM of r and s is the product of the HCF of r and s and the LCM of p and q (Davies 1980). For example, the solution to Problem 5 may be seen like this:

$$\frac{14}{18} = \frac{7}{9}$$

LCM of 14 and 18
= 14 x 9 or 18 x 7
= 2 x 7 x 9 or 2 x 9 x 7
where 2 is the HCF of 14 and 18.

The LCM of 14 and 18 can be represented by an isosceles triangle since it is necessary that 14 x 9 must be equal to 18 x 7. The 'cross-product' idea appears because 7 and 9 are relatively prime to each other.



Conclusion

This article represents personal reflections on the concepts of HCF and LCM. It has emphasized the use of problem representations to elucidate the basic notions of repeated subtraction and addition which underlie the concepts. An examination of the structure of selected problems has led to the Euclidean Algorithm as the most general method for finding the HCF. In a similar manner, an examination of the connections between HCF and LCM has led to the 'equivalent fraction' method for finding the LCM.

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