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Geometric phase in open systems: Beyond the Markov approximation and weak-coupling limit

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Beyond the quantum Markov approximation and the weak-coupling limit, we present a general theory to calculate the geometric phase for open systems with and without conserved energy. As an example, the geometric phase for a two-level system coupling both dephasingly and dissipatively to its environment is calculated. Comparison with the results from quantum trajectory analysis is presented and discussed.

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I. INTRODUCTION

About 20 years ago, Berry discovered [1] that a state of a quantum system can acquire a phase of purely geometric origin when the Hamiltonian of the system undergoes a cyclic adiabatic change. Since then, there have been numerous proposals for generalizations, including the geometric phase for nonadiabatic, noncyclic, and nonunitary evolution [2], the geometric phase for mixed states [3–5], the geometric phase in systems with driven quantum field and vacuum-induced effects [6], as well as the geometric phase in coupled bipartite systems [7].

Recently, much attention has been devoted to the study of geometric phase in open systems. This is motivated in part by the fact that all realistic system is coupled, at least weakly, to their environment. From the perspective of possible real application, the use of geometric phases in the implementation of fault-tolerant quantum gates [8–11] requires the study of geometric phases for more realistic systems. For instance, the system which carries information may devolve from a quantum superposition into statistical mixtures, and this effect, called decoherence, is the most important limiting factor for a quantum computing.

The study of geometric phase in open systems may be traced as far back as the eighties, when Garrison and Wright [12] first touched on this issue by describing open system evolution in terms of non-Hermitian Hamiltonian. This is a pure-state analysis, so it did not address the problem of geometric phases for mixed states. For the geometric phase for mixed states in open systems, several approaches have been proposed including the solution of a master equation of the system [13–17], employment of quantum trajectory analysis [18,19] or the study of Krauss operators [20], and a perturbative expansions method [21,22] with adiabatic approximations [23]. These works have yield several interesting results which may be briefly summarized as follows: Non-Hermitian Hamiltonian leads to a modification of Berry's phase [12,21]; stochastically evolving magnetic fields produce both energy shift and broadening [22]; phenomenological weakly dissipative Liouvillians alter Berry's phase through the introduction of an imaginary correction term [15] or through damping and mixing of the density matrix elements [16]. However, nearly all these studies have been studied for dissipative systems under various approximations; thus, the representations are only approximately true for systems whose energy is not conserved. Quantum trajectory analysis [18,19] based on the quantum jump approach is available for open systems with conserved energy. Its starting point, however, is the master equation, a result within the quantum Markov approximation and in the weak coupling limit. Beyond the quantum Markov approximation and the weak-coupling limit, the geometric phase of a two-level system *with quantum field driving* has been analyzed [24], where the whole system (the two-level system plus the quantum field) was subjected to dephasing. This is an ideal situation to show the vacuum effects on the geometric phase of the subsystem, as well as the decoherence effects on the geometric phases regardless of its feasibility of experimental realization. However, beyond the Markov approximation and the weak-coupling limit, the geometric phase for a dissipative system remains untouched. In this paper, we will deal with the geometric phase in open systems, beyond the Markov approximation and weak-coupling limit.

The structure of this paper is organized as follows. In Sec. II the exact solution and calculation of the geometric phase of a system dephasingly coupled to its environment are presented. An example to detail the representation and a discussion on physical realization are given in Sec. III. In Sec. IV, we present an example to show the calculation of geometric phases in dissipative systems. Finally we conclude in Sec. V.

II. GEOMETRIC PHASE IN DEPHASING SYSTEMS: GENERAL FORMULATION

In this section, we investigate the behavior of geometric phase of a quantum system under decoherence. In order to make a comparison with the results based on the quantum jump approach, we consider the quantum system without any driving field save for the environment. So, it is not directly relevant to our previous study [24]. The environment that leads to decoherence may originate form the vacuum fluctuations or the background radiation. Here, we restrict ourselves to the case where the system-environment coupling H_I commutes with the free system Hamiltonian H_S. This constitutes our dephasing model in which exact analytical dynamics may be obtained. On the other hand, the evolution of a system with such properties may be described by the master equation when the Markov approximation and the weak-coupling assumption apply; such decoherence would not change the geometric phase of the quantum system by the quantum jump approach [18,19]. However, as will be seen, this is not the case from the perspective of interferometry methods considered in this paper.

We consider a situation described by a Hamiltonian of the form

$$H = H_S + H_B + H_I, \tag{1}$$

where H_S describes the free Hamiltonian of the system, H_B stands for the Hamiltonian of the environment, and H_I represents the system-environment couplings. The environment and the system Hamiltonian may be arbitrarily taken but with constraints $[H_S, H_I]=0$. Let us suppose that the interaction Hamiltonian H_I has the form (setting $\hbar=1$)

$$H_I = \sum_m X_m(\Gamma_m^\dagger + \Gamma_m), \tag{2}$$

where the X_m, $(m=1,2,\ldots,M)$ are the system operators satisfying $[H_S, X_m]=0$, and the Γ_m represent environment operators that may take any form in general. The relation $[H_S, X_m]=0$ enables us to write the time evolution operator for the whole systems (system+environment) as

$$U(t) = e^{-iHt} = e^{-iH_S t}\sum_m U_m(t)|E_m\rangle\langle E_m|, \tag{3}$$

with $U_m(t)$ a function of environment operators satisfying

$$i\frac{\partial}{\partial t}U_m(t) = H_{e,m}U_m(t),$$

$$H_{e,m} = H_B + \sum_n e_n^m(\Gamma_n + \Gamma_n^\dagger). \tag{4}$$

Here, $|E_m\rangle$ stands for the eigenstate of H_S with eigenvalue E_m [27], while e_n^m denotes the eigenvalue of X_n corresponding to eigenstate $|E_m\rangle$. For a specific Γ_m, $U_m(t)$ may be expressed in factorized form, which is shown later through the spin-boson model. Furthermore, we assume that the environment and the system are initially independent, such that the total density operator factorizes into a direct product,

$$\rho(0) = \rho_S(0) \otimes \rho_B(0) = \sum_{mn} \rho_{mn}(0)|E_m\rangle\langle E_n| \otimes \rho_B(0). \tag{5}$$

At time t, the reduced density operator of the system is given by

$$\begin{aligned}\rho_S(t) &= \mathrm{Tr}_B[U(t)\rho_S(0)\otimes\rho_B(0)U^\dagger(t)]\\ &\equiv \sum_{mn}\rho_{mn}(0)e^{-i(E_m-E_n)t}|E_m\rangle\langle E_n|F_{mn}(t),\end{aligned} \tag{6}$$

where $F_{mn}(t)$ is defined as $\mathrm{Tr}_B[U_m(t)\rho_B(0)U_n^\dagger(t)]$. Equation (6) shows that the diagonal elements of the reduced density matrix ρ_{mm} are time independent, while the off-diagonal elements evolve with time involving contributions from the environment-system couplings. For most cases, this would lead to a decay in the off-diagonal elements, and eventually results in vanishing of these matrix elements. Now, we turn to study the geometric phase of the open system. For an open system, the state in general is not pure and the evolution of the system is not unitary. For nonunitary evolution, the geometric phase can be calculated as follows. First, solve the eigenvalue problem for the reduced density matrix $\rho(t)$ and obtain its eigenvalues $\varepsilon_k(t)$ as well as the corresponding eigenvectors $|\psi_k(t)\rangle$; second, substitute $\varepsilon_k(t)$ and $|\psi_k(t)\rangle$ into

$$\Phi_g = \arg\left(\sum_k \sqrt{\varepsilon_k(0)\varepsilon_k(T)}\langle\psi_k(0)|\psi_k(T)\rangle e^{-\int_0^T\langle\psi_k(t)|\partial/\partial t|\psi_k(t)\rangle dt}\right). \tag{7}$$

Here, Φ_g is the geometric phase for the system undergoing nonunitary evolution [25]; T is the total evolution time. The geometric phase Eq. (7) is gauge invariant and can be reduced to the well-known results in the unitary evolution. It is experimentally testable. The geometric phase factor defined by Eq. (7) may be understood as a weighted sum over the phase factors pertaining to the eigenstates of the reduced density matrix; thus, the detail of analytical expression for the geometric phase would depend on the digitalization of the reduced density matrix Eq. (6).

III. GEOMETRIC PHASE IN DEPHASING SYSTEM: EXAMPLE

To be specific, we now present a detailed model to illustrate the idea in Sec. II. The system under consideration consists of a two-level system coupled to its environment with interaction strengths $\{g_i\}$. The Hamiltonian which governs the evolution of such a system may be expressed as

$$\begin{aligned}H = {}&\frac{\omega}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + \frac{1}{2}(|e\rangle\langle e| - |g\rangle\langle g|)\sum_i g_i(a_i^\dagger + a_i)\\ &+ \sum_i \omega_i a_i^\dagger a_i,\end{aligned} \tag{8}$$

where $a_i^\dagger$, a_i are the creation and annihilation operators of the environment bosons, and $|e\rangle$, $|g\rangle$ denote the excited and ground states, respectively, of the two-level system with Rabi frequency ω. This Hamiltonian corresponds to $X_m=X=\frac{1}{2}(|e\rangle\langle e|-|g\rangle\langle g|)$, and $\Gamma_m=\Gamma=\Sigma_i g_i a_i$ in the general model Eq. (2). Generally speaking, the choice of the coupling between the system and the environment determines the effect of the environment. For example, the choice of the system operator X_m that does not change the good quantum number of H_S would result in dephasing of the system, but not a relaxation of the energy. The system-environment coupling taken in this section is exactly of this kind.

By the procedure presented above, the reduced density matrix in basis $\{|e\rangle, |g\rangle\}$ for the open system follows [26]:

$$\rho_S = \begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}\sin\theta F_{12}(t) \\ \frac{1}{2}\sin\theta F_{21}(t) & \sin^2\frac{\theta}{2}\end{pmatrix}, \tag{9}$$

where an initial state of $(\cos\frac{\theta}{2}|e\rangle + \sin\frac{\theta}{2}|g\rangle)\otimes|0\rangle_B$ for the total system was assumed in the calculation, and

$$F_{12}(t) = F^*_{21}(t) = F(t) = e^{-i\omega t}e^{-\Sigma_j \eta_j(t)}, \qquad (10)$$

with $\eta_j(t)=4|\frac{g_j}{\omega_j}|^2(1-\cos\omega_j t)$, and $|0\rangle_B$ denoting the vacuum state of the environment. Some remarks on the reduced density matrix are now in order. For any j, $\eta_j(t)\geq 0$, so as t tends to infinity (with respect to the system's coherence time), $|F(t)|$ tends to zero. This indicates that the off-diagonal elements would vanish on a long time scale with respect to the decoherence time, and hence the open system would not acquire geometric phase when time is longer than the decoherence time. This is different from the results concluded in the previous work, where the subsystem may acquire geometric phase even for the whole system in its pointer states [24]. To calculate the geometric phase pertaining to Eq. (9), we first write the eigenstate and its corresponding eigenvalue for the reduced density matrix ρ_S as

$$|\varepsilon_\pm(t)\rangle = C[\Theta_\pm(t)]|e\rangle + S[\Theta_\pm(t)]|g\rangle,$$

$$\varepsilon_\pm(t) = \frac{1}{2}(1 \pm \sqrt{\cos^2\theta + \sin^2\theta|F(t)|^2}), \qquad (11)$$

with

$$C[\Theta_\pm(t)] = \frac{\sin\theta F(t)}{\sqrt{\sin^2\theta|F(t)|^2 + 4\left(\varepsilon_\pm(t) - \cos^2\frac{\theta}{2}\right)^2}}, \qquad (12)$$

and $|C[\Theta_\pm(t)]|^2+|S[\Theta_\pm(t)]|^2=1$. We first discuss two limiting cases with $g_i=0$ and small damping rate γ. The discussions for general cases will be presented in the next paragraph. Clearly, for a closed system, namely $g_i=0$, $F(t)=e^{-i\omega t}$, the eigenvalues reduce to $\varepsilon_\pm(t)=1,0$, and $C[\Theta_+(t)]=\cos\frac{\theta}{2}e^{-i\omega t}$, $S[\Theta_+(t)]=\sin\frac{\theta}{2}$, $C[\Theta_-(t)]=-\sin\frac{\theta}{2}$, $S[\Theta_-(t)]=\cos\frac{\theta}{2}e^{i\omega t}$. These relations yield the well-known geometric phase $\Phi_g^{(0)}=(1+\cos\theta)$. Equation (11) and (12) are the exact results for the open two-level system; the geometric phase would depend on how $F(t)$ varies with time. For a continuous spectrum of environmental modes with constant spectral density $\sigma(\omega)=\epsilon$, $F(t)=e^{-i\omega t}e^{-\gamma t}$ with $\gamma=2\pi\epsilon|g|^2$, where $g_i=g$ was assumed. Up to first order in γ, the geometric phase $\Phi_g^{(1)}$ at time $T=2\pi/\omega$ is given by

$$\Phi_g^{(1)} = \pi(1+\cos\theta) - \frac{\gamma}{\omega}\pi^2\sin^2\theta. \qquad (13)$$

This result can be easily understood as follows. The geometric phase factor for mixed states is defined as a weighted sum over the phase factors associated with the eigenstates of the reduced density matrix. The dephasing that leads to decays in the off-diagonal elements would change the phase factors acquired by each eigenstate of the reduced density matrix. This modifies the geometric phase. This is different from the definition in the quantum jump approach [19], in which the problem of defining Berry's phase for mixed states was avoided by approaching the dynamics of open system from a sequence of pure states, leading to the result that the geometric phase is unaffected by dephasing, even though it lowers the observed visibility in any interference measurements. As a mixed state, the evolution of the system depends on the many trajectories with the corresponding probabilities. Thus, the geometric phase is defined as a weighted sum over those trajectories that the system undergoes.

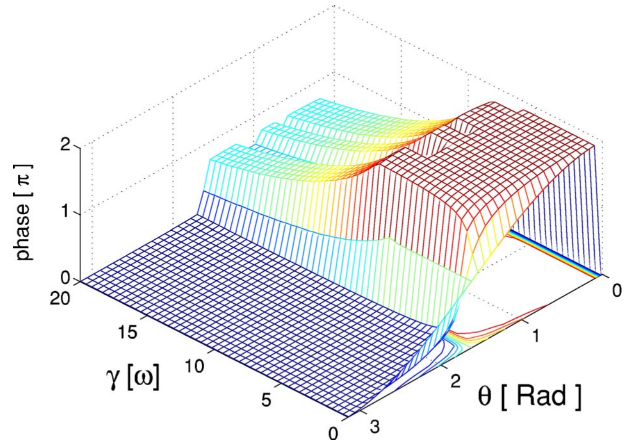

FIG. 1. (Color online) An illustration of the geometric phases of the two-level system coupling to the environment at time $T=2\pi/\omega$. The phase was calculated in units of π, and θ was in units of radians.

Now, we are in a position to discuss the geometric phase acquired at time $T=2\pi/\omega$ by the two-level system. Substituting Eqs. (11) and (12) into Eq. (7), we obtain

$$\Phi_g = \int_0^T \omega|C[\Theta_+(t)]|^2 dt. \qquad (14)$$

It is worth noting that Eq. (14) is the geometric phase beyond the quantum Markov approximation and the weak-coupling limit. In this sense it provides us more insight into the geometric phase for dephasing system. The numerical results for Eq. (14) are presented in Fig. 1, where the dependence of the geometric phase is illustrated as a function of the azimuthal angle θ and the damping rate γ. The spectrum of environmental modes was taken to be $\sigma(\omega)=\varepsilon$ in this plot. Clearly, the two-level system acquires zero geometric phase with γ tending to infinity; this indicates that the two-level system acquires no geometric phase after the decoherence time. The representation in this paper may be used to analyze and estimate the error in the holonomic quantum computation due to decoherence [28], in which the key error occurs within the degenerate subspace. The Hamiltonian that describes such a system reads $H=\Sigma_{i,j,\alpha}g^\alpha_{i,j}(a^\dagger_\alpha+a_\alpha)|i\rangle\langle j|+\Sigma_i\omega_i a^\dagger_i a_i$, where the degenerate energy was assumed to be zero, and $\{|i\rangle, i=1,\ldots,N\}$ denotes the degenerate levels coupled to the environment with coupling constants $\{g^\alpha_{i,j}\}$. This Hamiltonian can be rewritten as $H=\Sigma_{\beta,\alpha}g^\alpha_\beta(a^\dagger_\alpha+a_\alpha)|\beta\rangle\langle\beta|+\Sigma_i\omega_i a^\dagger_i a_i$, with an appropriate choice of $|\beta\rangle=\Sigma^N_{i=1}c_i|i\rangle$. This is exactly the case discussed in Secs. II and III.

The effect of open system on the geometric phase may be observed with a combination of the engineering reservoir technique [29] and the Mach-Zehnder atom interferometer [30,31], in which each of the arms consists of an atom in a dark state. Dark states can be realized in the atom-light system that consists of cesium atoms interacting with light resonant with the $F=3\rightarrow F'=3$ transitions of the $D1$ $6^2s_{1/2}$

$\rightarrow 6^2p_{1/2}$ line. It makes the dynamical phase negligible with respect to the geometric phase. A dephasing engineering reservoir in one arm of the interferometer may be simulated by variations of the light fields and gives a relative phase to the atom passing though the arm. The output interference pattern then yields the geometric phase of the atom system.

One of the key assumptions in our discussion is the dephasing condition, i.e., $[H_S, H_I]=0$. Exact formulation can be achieved by using ground states of a quantum system as the qubits. Suppose now there is an additional small term in H_I, $H_I'=\Sigma_m Y_m(\Gamma_m+\Gamma_m^\dagger)$. Simple algebra shows that the transition probability between $|E_m\rangle$ and $|E_n\rangle$ due to coupling $Y_p(\Gamma_p+\Gamma_p^\dagger)$ is proportional to $|\gamma_p\langle E_n|Y_p|E_m\rangle|^2/|E_m-E_n|^2$, where γ_p denotes the maximum of average values of $(\Gamma_p+\Gamma_p^\dagger)$. In the case of $|\gamma_p\langle E_n|Y_p|E_m\rangle| \ll |E_m-E_n|$, the open system may be treated as a dephasing system, because the transition between any different eigenstates of the system may be ignored. The case where this transition could not be ignored will be discussed in the next section.

IV. GEOMETRIC PHASE IN DISSIPATIVE SYSTEMS: EXACTLY SOLVABLE MODEL

In this section, we will consider a spin-$\frac{1}{2}$ particle interacting with an environment formed by N independent spins through the Hamiltonian

$$H = \Delta\sigma_x + \frac{1}{2}\sum_{k=1}^{N} g_k \sigma_z \sigma_z^{(k)}, \tag{15}$$

where $\sigma_i^{(k)}$ and σ_i, $i=x,y,z$ denote Pauli operators for the environment and spin-$\frac{1}{2}$ particle, respectively. g_k, $k=1,2,\ldots,N$, are coupling constants, term with Δ stands for the self-Hamiltonian of the particle. This model is interesting because the pointer states do not coincide with the eigenstates of the interaction Hamiltonian. Rather, they can take the form of coherent states or eigenstates of the system's Hamiltonian determined through the interplay between the self-Hamiltonian and the interaction with the environment. We will calculate the geometric phase gained by the particle beyond the Markov approximation and the weak-coupling limit. The dynamics governed by Hamiltonian Eq. (15) can be exactly solved by a standard procedure [32]; it yields the reduced density matrix of the particle as

$$\rho(t) = [I + \vec{p}(t)\cdot\vec{\sigma}]/2, \tag{16}$$

where $\vec{p}(t)$ is the polarization vector given by $\vec{p}(t)=\int \vec{p}(t,B)\eta(B)dB$ with $\Omega_B^2=\Delta^2+B^2$ and

$$\eta(B) = \frac{1}{\sqrt{2\pi s_N^2}} e^{-B^2/2s_N^2},$$

$$p_x(t,B) = p_x(0)\frac{\Delta^2 + B^2\cos(2\Omega_B t)}{\Omega_B^2} - p_y(0)\frac{B}{\Omega_B}\sin(2\Omega_B t) + p_z(0)\frac{2\Delta B}{\Omega_B^2}\sin^2(\Omega_B t),$$

$$p_y(t,B) = p_y(0)\cos(2\Omega_B t) + \frac{\sin(2\Omega_B t)}{\Omega_B}[p_x(0)B - \Delta p_z(0)],$$

$$p_z(t,B) = p_z(0)\frac{B^2 + \Delta^2\cos(2\Omega_B t)}{\Omega_B^2} + p_x(0)\frac{2\Delta B}{\Omega_B^2}\sin^2(\Omega_B t) + p_y(0)\frac{\Delta}{\Omega_B}\sin(2\Omega_B t). \tag{17}$$

To get this result, it is essential that the couplings g_k of Eq. (15) peak near their average value with finite standard deviation.

By rewriting the reduced density/matrix $\rho(t)$ in the form

$$\rho(t) = \lambda_1(t)|\psi_1(t)\rangle\langle\psi_1(t)| + \lambda_2(t)|\psi_2(t)\rangle\langle\psi_2(t)|, \tag{18}$$

we get the geometric phase [25] of the particle acquired at time τ,

$$\Phi_g'(\tau) = \arg\left(\sum_{i=1,2}\sqrt{\lambda_i(0)\lambda_i(\tau)}\langle\psi_i(0)|\psi_i(\tau)\rangle e^{-\int_0^\tau\langle\psi_i(t)|\partial/\partial t|\psi_i(t)\rangle dt}\right). \tag{19}$$

After simple manipulations, we arrive at

$$\Phi_g' = \arg\Bigg[\Lambda_1(\tau)\left(\cos\frac{\theta(0)}{2}\cos\frac{\theta(\tau)}{2}e^{-i[\phi(\tau)-\phi(0)]} + \sin\frac{\theta(0)}{2}\sin\frac{\theta(\tau)}{2}\right)e^{iC} + \Lambda_2(\tau) \times\left(\cos\frac{\theta(0)}{2}\cos\frac{\theta(\tau)}{2}e^{i[\phi(\tau)-\phi(0)]} + \sin\frac{\theta(0)}{2}\sin\frac{\theta(\tau)}{2}\right)e^{-iC}\Bigg]. \tag{20}$$

Here,

$$\Lambda_{1,2} = \frac{1}{2}\sqrt{(1 \pm |\vec{p}(0)|)(1 \pm |\vec{p}(t)|)},$$

$$\cos\theta(t) = \frac{p_z(t)}{\sqrt{p_x^2(t) + p_y^2(t) + p_z^2(t)}},$$

$$\tan\phi(t) = \frac{p_y(t)}{p_x(t)},$$

$$C = \frac{i}{2}\left(\int_0^\tau \frac{\partial\phi}{\partial t}dt + \int_0^\tau \cos\theta\frac{\partial\phi}{\partial t}dt\right).$$

The dependence of the geometric phase Φ_g' on the variance s_N and system free energy Δ is complicated. We discuss here two limiting cases: $s_N \gg \Delta$ and $s_N \ll \Delta$ with a specific initial state $p_x(0)=1$, $p_y(0)=p_z(0)=0$. In the limit $s_N \gg \Delta$, the dynamics of the spin-$\frac{1}{2}$ particle is so slow that its behavior should approach $p_x(t)=e^{-2t^2 s_N^2}$ and $p_y(t)=p_z(t)=0$, which yields $\Phi_g'=0$ because $\phi=0$ in this limit with the initial state. In the limit $s_N \ll \Delta$, Eq. (17) gives

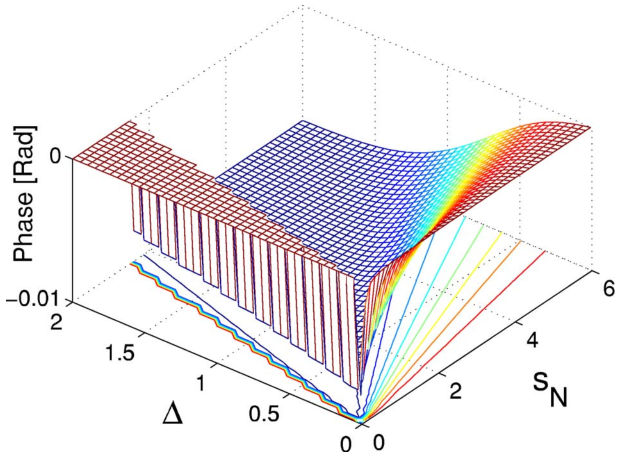

FIG. 2. (Color online) The geometric phase of the spin-$\frac{1}{2}$ particle vs s_N and Δ. The spin was dissipatively coupled to an environment and the phase was calculated at time $\tau=2\pi/\Delta$, depending on the time scale of free evolution of the particle. The units chosen are the same as in Fig. 1.

$$p_x(t) \simeq p_x(0)\left[\gamma\left(\frac{\Delta}{\sqrt{2}s_N}\right) + \frac{\cos\left(2\Delta t + \frac{3\pi}{4}\right)}{\sqrt{8\Delta s_N^2 t^3}}\right],$$

$$p_y(t) = p_z(t) = 0, \tag{21}$$

where $\gamma(x)=\sqrt{\pi}xe^{x^2}[1-\mathrm{Erf}(x)]$, $\mathrm{Erf}(x)$ is the error function, and $\gamma\left(\frac{\Delta}{\sqrt{2}s_N}\right)\ll 1$ in this limit. By Eq. (20), it is clear that $\phi_g'=0$ in this limit, too. The numerical result for the geometric phase as a function of Δ and s_N is shown in Fig. 2. To plot this figure, we assume that the system has evolved for time $\tau=2\pi/\Delta$, which is the characterized time for the system undergoing a free evolution. Figure 2 shows that the geometric phase is zero in the two limiting cases $\Delta \gg s_N$ and $\Delta \ll s_N$ as expected. There are sharp changes among the line $\Delta \sim s_N$, indicating a crossover from the limit $\Delta \ll s_N$ to $\Delta \gg s_N$.

V. SUMMARY AND DISCUSSION

We have presented a general calculation for the geometric phase in open systems subject to dephasing and dissipation; the calculations are beyond the quantum Markov approximation and the weak-coupling limit. For the dephasing system, it acquires no geometric phase with the decoherence rate $\gamma \to \infty$; this can be explained as an effect of decoherence on the geometric phase, i.e., the quantum system could not maintain its phase information after the decoherence time. There is a sharp change along the line $\theta=\pi/2$ as Fig. 1 shows; this can be understood in terms of the Bloch sphere that represents the state of the system. The geometric phase increases due to decoherence when initial states fall onto the upper semisphere, but it decreases when the initial states are on the lower semisphere. These results are similar to the prediction given by the quantum trajectory analysis for dissipative systems. The geometric phase Φ_g' in dissipative systems is always zero as long as $p_x(t)/p_y(t)=$constant; this is exactly the case when $\Delta/s_N \to \infty$ or $\Delta/s_N \to 0$. $\Delta/s_N \to \infty$ implies that the self-energy Δ of the particle is much larger than the cumulative variance s_N of the coupling constants g_k. For g_k taking the value $+g$ or $-g$ (g arbitrary) with equal probability, $s_N^2 = \Sigma_k g_k^2$. This tells us that the geometric phase is zero when the self-Hamiltonian dominates. On the other hand, when $\Delta/s_N \to 0$ the interaction Hamiltonian dominates; pointer states in this situation coincide with the eigenstates of the interaction Hamiltonian and thus the spin-$\frac{1}{2}$ particle could not acquire geometric phase. In the crossover regime $\Delta \sim s_N$, the geometric phase changes sharply due to the interplay between the self-Hamiltonian and the interaction with the environment.

These results constitute the basis of a framework to analyze errors in the holonomic quantum computation, where two kinds of errors are believed to affect its performance. This first error would take the system out of the degenerate computation subspace, while the second takes place within the subspace. The first kind of error can be eliminated by working in the ground states and by having a system where the energy gap with the first excited state is very large. The second kind of error falls to the regime analyzed in Sec. II and III, since there is no dissipation but dephasing in the system, while the first belongs to the regime discussed in Sec. IV. The calculation presented here in principle allows one to study the geometric phase at any time scale, and hence it has advantages with respect to any treatment with approximations in most literature.

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