Chromatic Roots of a Ring of Four Cliques

F.M. Dong*

Mathematics and Mathematics Education, National Institute of Education
Nanyang Technological University, Singapore

fengming.dong@nie.edu.sg

Gordon Royle

School of Mathematics & Statistics
University of Western Australia, Australia

gordon@maths.uwa.edu.au

Dave Wagner

Department of Combinatorics and Optimization
University of Waterloo, Canada

dgwagner@math.uwaterloo.ca

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Abstract

For any positive integers a, b, c, d, let $R_{a,b,c,d}$ be the graph obtained from the complete graphs K_a, K_b, K_c and K_d by adding edges joining every vertex in K_a and K_c to every vertex in K_b and K_d. This paper shows that for arbitrary positive integers a, b, c and d, every root of the chromatic polynomial of $R_{a,b,c,d}$ is either a real number or a non-real number with its real part equal to $(a + b + c + d - 1)/2$.

Keywords: graph, chromatic polynomial, chromatic root, ring of cliques

1 Introduction

A *ring of cliques* is a graph whose vertex set is the disjoint union of cliques, arranged in a cyclic order, such that the vertices of each clique are joined to all the vertices in the two neighbouring cliques. If the cliques have size $a_1, a_2, \ldots, a_n$ then we denote this graph by $R_{a_1,a_2,\ldots,a_n}$. Figure 1 shows the graph $R_{2,2,3,3}$.

*Corresponding author.

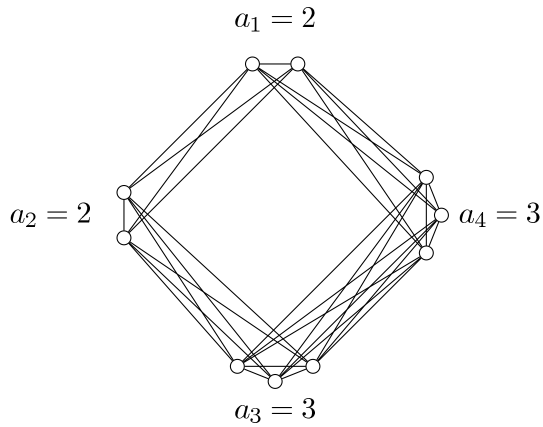

Figure 1: The graph $R_{2,2,3,3}$

Graphs with this structure have occurred several times previously in the study of chromatic polynomials and their roots. In particular, in proving that there are non-chordal graphs with integer chromatic roots, Read [6] considered the graphs in this family with $a_1 = 1$ (and he also used slightly different notation). Rings of cliques cropped up again recently in a preliminary investigation of the *algebraic properties* of chromatic roots (Cameron [1]) and in the course of this investigation, the chromatic roots of many of these graphs were computed. When the chromatic roots of the ring-of-clique graphs with *exactly four cliques* and a fixed number of vertices were plotted, an intriguing pattern was observed — all the non-real chromatic roots lie on a single vertical line. Figure 2 shows the union of the chromatic roots of the 12-vertex graphs of the form $R_{a,b,c,d}$.

Faced with such a striking empirically-observed pattern, we were led to explain it theoretically. This appears to require a surprisingly intricate argument, but eventually we obtain the following result:

Theorem 1 *For arbitrary non-negative integers a, b, c and d the chromatic roots of* $R_{a,b,c,d}$ *are either real, or non-real with real part equal to* $(a+b+c+d-1)/2$.

The overall structure of the argument is as follows: $P(R_{a,b,c,d}, \lambda)$, the chromatic polynomial of $R_{a,b,c,d}$, is first expressed as the product of linear factors and a factor $Q_{a,b,c,d}(\lambda)$. It then suffices to show that the non-real roots of $Q_{a,b,c,d}(\lambda)$ all lie on the vertical line $\Re(\lambda) = (a+b+c+d-1)/2$ in the complex λ-plane. Next the polynomial $F_{a,p,q,n}(z)$ is defined to be $Q_{a,b,c,d}(z + (a+b+c+d-1)/2)$ thus translating the vertical line supposed to contain the roots to the imaginary axis and also reparameterizing the problem (in a somewhat counterintuitive way). Then $F_{a,p,q,n}$ is shown to be an *even* polynomial and we define a fourth polynomial $W_{a,p,q,n}$ by $W_{a,p,q,n}(z^2) = F_{a,p,q,n}(z)$. The proof is completed by demonstrating that $W_{a,p,q,n}$ is real-rooted using polynomial interleaving techniques, and therefore $F_{a,p,q,n}$ has only real or pure imaginary roots as required.

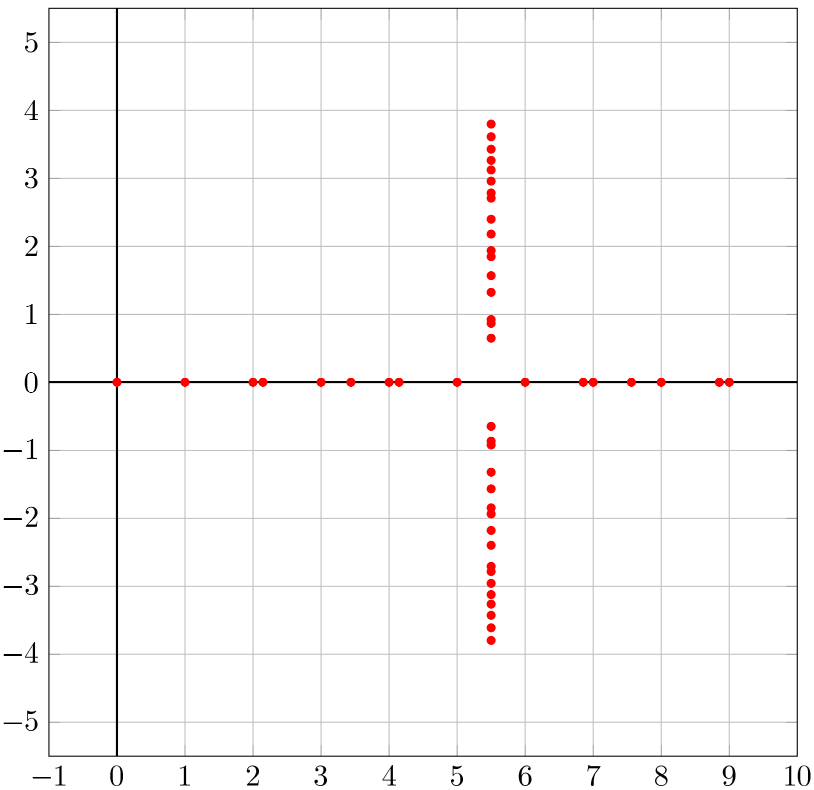

Figure 2: Chromatic roots of the graphs $R_{a,b,c,d}$ where $a + b + c + d = 12$.

2 Basics

For any graph G and any positive integer λ, let $P(G, \lambda)$ be the number of mappings ϕ from $V(G)$ to $\{1, 2, \ldots, \lambda\}$ such that $\phi(u) \neq \phi(v)$ for every two adjacent vertices u and v in G. It is well-known that $P(G, \lambda)$ is a polynomial in λ, called *the chromatic polynomial* of G.

The chromatic polynomial of a graph G has the following properties (see, for instance, [3, 5, 7, 9]), which we will apply later.

Proposition 1 *Let G be a simple graph of order at least* 2*.*

(i) *If u and v are two non-adjacent vertices in G, then*

$$P(G, \lambda) = P(G + uv, \lambda) + P(G/uv, \lambda), \tag{1}$$

where $G + uv$ is the graph obtained from G by adding the edge joining u and v, and G/uv is the graph obtained from G by identifying u and v and removing all parallel edges but one.

(ii) *If u is a vertex in G which is adjacent to all other vertices in G, then*

$$P(G, \lambda) = \lambda P(G - u, \lambda - 1), \tag{2}$$

where $G-u$ is the graph obtained from G by removing u.

If $a=0$, $R_{a,b,c,d}$ is a chordal graph and its chromatic polynomial is

$$P(R_{0,b,c,d},\lambda)=\frac{(\lambda)_{b+c}(\lambda)_{c+d}}{(\lambda)_c}, \tag{3}$$

and if $a\geq 1$ and $c\geq 1$, then applying Proposition 1 repeatedly yields that

$$P(R_{a,b,c,d},\lambda)=\lambda P(R_{a-1,b,c,d},\lambda-1)+c\lambda P(R_{a-1,b,c-1,d},\lambda-1). \tag{4}$$

For a non-negative integer a and real numbers b,c and d, define a polynomial $Q_{a,b,c,d}(z)$ in z as follows: $Q_{0,b,c,d}(z)=1$ and for $a\geq 1$,

$$Q_{a,b,c,d}(z)=(z-b-c)(z-c-d)Q_{a-1,b,c,d}(z-1)+c(z-a-c+1)Q_{a-1,b,c-1,d}(z-1). \tag{5}$$

It is clear that $Q_{a,b,c,d}(z)$ is a polynomial of order $2a$ in z.

Proposition 2 *Let a,b,c and d be any non-negative integers. Then*

$$P(R_{a,b,c,d},\lambda)=\frac{(\lambda)_{b+c}(\lambda)_{c+d}}{(\lambda)_{a+c}}Q_{a,b,c,d}(\lambda). \tag{6}$$

Proof. If $a=0$, then (6) follows from (3) and the definition of $Q_{a,b,c,d}(\lambda)$. Now assume that $a\geq 1$. By (4) and induction, we have

$$\begin{aligned}
P(R_{a,b,c,d},\lambda) &= \lambda P(R_{a-1,b,c,d},\lambda-1)+c\lambda P(R_{a-1,b,c-1,d},\lambda-1)\\
&= \lambda\frac{(\lambda-1)_{b+c}(\lambda-1)_{c+d}}{(\lambda-1)_{a+c-1}}Q_{a-1,b,c,d}(\lambda-1)\\
&\quad +c\lambda\frac{(\lambda-1)_{b+c-1}(\lambda-1)_{c+d-1}}{(\lambda-1)_{a+c-2}}Q_{a-1,b,c-1,d}(\lambda-1)\\
&= \frac{(\lambda)_{b+c}(\lambda)_{c+d}}{(\lambda)_{a+c}}\left[(\lambda-b-c)(\lambda-c-d)Q_{a-1,b,c,d}(\lambda-1)\right.\\
&\quad \left.+c(\lambda-a-c+1)Q_{a-1,b,c-1,d}(\lambda-1)\right].
\end{aligned} \tag{7}$$

The result then follows. □

Define $\binom{x}{0}=1$ and $\binom{x}{r}=x(x-1)\dots(x-r+1)/r!$ for any positive integer r and any complex number x.

Proposition 3 *For any non-negative integer a and real numbers b,c and d,*

$$Q_{a,b,c,d}(\lambda)=a!\sum_{i=0}^{a}i!(a-i)!\binom{c}{i}\binom{\lambda-b-c}{a-i}\binom{\lambda-c-d}{a-i}\binom{\lambda-a-c+i}{i}. \tag{8}$$

Proof. It is trivial if $a = 0$ as $Q_{0,b,c,d}(z) = 1$. Now assume that $a \geq 1$. By (5) and induction,

$$
\begin{aligned}
& Q_{a,b,c,d}(\lambda) \\
= \; & (\lambda - b - c)(\lambda - c - d)Q_{a-1,b,c,d}(\lambda - 1) + c(\lambda - a - c + 1)Q_{a-1,b,c-1,d}(\lambda - 1) \\
= \; & (\lambda - b - c)(\lambda - c - d)(a-1)! \sum_{i=0}^{a-1} \left\{ i!(a-i-1)! \binom{c}{i} \binom{\lambda - b - c - 1}{a - i - 1} \right. \\
& \qquad \left. \binom{\lambda - c - d - 1}{a - i - 1} \binom{\lambda - a - c + i}{i} \right\} \\
& + c(\lambda - a - c + 1)(a-1)! \sum_{i=0}^{a-1} \left\{ i!(a-i-1)! \binom{c-1}{i} \binom{\lambda - b - c}{a - i - 1} \right. \\
& \qquad \left. \binom{\lambda - c - d}{a - i - 1} \binom{\lambda - a - c + i + 1}{i} \right\} \\
= \; & (a-1)! \sum_{i=0}^{a-1} i!(a-i)!(a-i) \binom{c}{i} \binom{\lambda - b - c}{a - i} \binom{\lambda - c - d}{a - i} \binom{\lambda - a - c + i}{i} \\
& + (a-1)! \sum_{i=0}^{a-1} \left\{ i!(a-i-1)!(i+1)^2 \binom{c}{i+1} \binom{\lambda - b - c}{a - i - 1} \right. \\
& \qquad \left. \binom{\lambda - c - d}{a - i - 1} \binom{\lambda - a - c + i + 1}{i + 1} \right\} \\
= \; & (a-1)! \sum_{i=0}^{a-1} i!(a-i)!(a-i) \binom{c}{i} \binom{\lambda - b - c}{a - i} \binom{\lambda - c - d}{a - i} \binom{\lambda - a - c + i}{i} \\
& + (a-1)! \sum_{i=1}^{a} (i-1)!(a-i)! i^2 \binom{c}{i} \binom{\lambda - b - c}{a - i} \binom{\lambda - c - d}{a - i} \binom{\lambda - a - c + i}{i} \\
= \; & a! \sum_{i=0}^{a} i!(a-i)! \binom{c}{i} \binom{\lambda - b - c}{a - i} \binom{\lambda - c - d}{a - i} \binom{\lambda - a - c + i}{i}.
\end{aligned}
\tag{9}
$$

The result then follows. □

For any non-negative integer a and real numbers p, q, n, define

$$
F_{a,p,q,n}(z) \;=\; a! \sum_{i=0}^{a} i!(a-i)! \binom{a+p+q-1}{i} \binom{z+n+i-1}{i} \binom{z-p}{a-i} \binom{z-q}{a-i}. \tag{10}
$$

Then (8) and (10) implies that $Q_{a,b,c,d}(z + (a+b+c+d-1)/2) = F_{a,p,q,n}(z)$, where

$$
\begin{cases}
p = (b + c - a - d + 1)/2 \\
q = (c + d - a - b + 1)/2 \\
n = (b + d - a - c + 1)/2.
\end{cases}
\tag{11}
$$

In the next section, we shall show that $F_{a,p,q,n}(z)$ is an even polynomial in z, and the polynomial obtained from $F_{a,p,q,n}(z)$ by replacing z^2 by z (i.e., $W_{a,p,q,n}(z)$ defined on Page 9) has only real roots for an arbitrary positive integer a and arbitrary real numbers p, q, n satisfying the condition that $p+q, p+n$ and $q+n$ are all non-negative (see Proposition 10). This result implies that every root of $F_{a,p,q,n}(z)$ is either a real number or a non-real number with its real part equal to 0 if a is a positive integer and $p+q, p+n$ and $q+n$ are all non-negative real numbers. For arbitrary positive integers a, b, c, d, if $a \leq \min\{b, c, d\}$ and p, q and n are given in (11), then $p+q = c-a+1 > 0$, $p+n = b-a+1 > 0$ and $q+n = d-a+1 > 0$. Since $Q_{a,b,c,d}(z+(a+b+c+d-1)/2) = F_{a,p,q,n}(z)$, where p, q and n are given in (11), the following result is obtained.

Proposition 4 *For arbitrary positive integers a, b, c and d, if $a \leq \min\{b, c, d\}$, then every root of $Q_{a,b,c,d}(z)$ is either a real number or a non-real number with its real part equal to $(a+b+c+d-1)/2$. Therefore, for arbitrary non-negative integers a, b, c and d, every root of $P(R_{a,b,c,d}, \lambda)$ is either a real number or a non-real number with its real part equal to $(a+b+c+d-1)/2$.* □

3 The polynomial $F_{a,p,q,n}(z)$

From the definition of $F_{a,p,q,n}(z)$, we have $F_{0,p,q,n}(z) = 1$ and $F_{1,p,q,n}(z) = z^2+pq+pn+qn$. We shall show that $F_{a,p,q,n}(z)$ has a recursive expression in terms of $F_{a-1,p,q,n}(z)$ and $F_{a-2,p,q,n}(z)$. We first prove two properties of $F_{a,p,q,n}(z)$.

Proposition 5 *For any integer $a \geq 1$ and arbitrary real numbers p, q, n, if $p+q = 0$, then*

$$F_{a,p,q,n}(z) = (z-p)(z-q)F_{a-1,p+1,q+1,n}(z). \tag{12}$$

Proof. For $a \geq 1$,

$$\begin{aligned} F_{a,p,q,n}(z) &= a! \sum_{i=0}^{a} i!(a-i)! \binom{a-1}{i} \binom{z+n+i-1}{i} \binom{z-p}{a-i} \binom{z-q}{a-i} \\ &= (z-p)(z-q)(a-1)! \\ &\quad \sum_{i=0}^{a-1} i!(a-1-i)! \binom{a}{i} \binom{z+n+i-1}{i} \binom{z-p-1}{a-1-i} \binom{z-q-1}{a-1-i} \\ &= (z-p)(z-q)F_{a-1,p+1,q+1,n}(z). \end{aligned} \tag{13}$$

□

Proposition 6 *For any integer $a \geq 1$ and arbitrary real numbers p, q, n,*

$$F_{a,p+1,q,n}(z) - F_{a,p,q,n}(z) = a(a+n+q-1)F_{a-1,p+1,q,n}(z). \tag{14}$$

Proof. For $a \geq 1$,

$$\begin{aligned}
& F_{a,p+1,q,n}(z) - F_{a,p,q,n}(z) \\
= \;& a! \sum_{i=0}^{a} \left\{ i!(a-i)! \binom{z+n+i-1}{i} \binom{z-q}{a-i} \right. \\
& \left. \left(\binom{a+p+q}{i} \binom{z-p-1}{a-i} - \binom{a+p+q-1}{i} \binom{z-p}{a-i} \right) \right\} \\
= \;& a! \sum_{i=0}^{a} \left\{ i!(a-i)! \binom{z+n+i-1}{i} \binom{z-q}{a-i} \right. \\
& \left. \left(\binom{a+p+q-1}{i-1} \binom{z-p-1}{a-i} - \binom{a+p+q-1}{i} \binom{z-p-1}{a-i-1} \right) \right\} \\
= \;& a! \sum_{i=1}^{a} i!(a-i)! \binom{z+n+i-1}{i} \binom{z-q}{a-i} \binom{a+p+q-1}{i-1} \binom{z-p-1}{a-i} \\
& -a! \sum_{i=0}^{a-1} i!(a-i)! \binom{z+n+i-1}{i} \binom{z-q}{a-i} \binom{a+p+q-1}{i} \binom{z-p-1}{a-i-1} \\
= \;& a! \sum_{i=0}^{a-1} (i+1)!(a-i-1)! \binom{z+n+i}{i+1} \binom{z-q}{a-i-1} \binom{a+p+q-1}{i} \binom{z-p-1}{a-i-1} \\
& -a! \sum_{i=0}^{a-1} i!(a-i)! \binom{z+n+i-1}{i} \binom{z-q}{a-i} \binom{a+p+q-1}{i} \binom{z-p-1}{a-i-1} \\
= \;& (n+q+a-1)a! \\
& \sum_{i=0}^{a-1} i!(a-1-i)! \binom{z+n+i-1}{i} \binom{z-q}{a-1-i} \binom{a+p+q-1}{i} \binom{z-p-1}{a-i-1} \\
= \;& a(a+n+q-1) F_{a-1,p+1,q,n}(z). \qquad (15)
\end{aligned}$$

□

Now we can prove that $F_{a,p,q,n}(z)$ can be expressed in terms of $F_{a-1,p,q,n}(z)$ and $F_{a-2,p,q,n}(z)$.

Proposition 7 *Let p, q, n be arbitrary real numbers. For any integer $a \geq 2$,*

$$\begin{aligned}
F_{a,p,q,n}(z) &= (z^2 + (a-1)(2p+2q+2n+2a-3) + pq + pn + qn) F_{a-1,p,q,n}(z) \\
& \quad -(a-1)(p+q+a-2)(q+n+a-2)(p+n+a-2) F_{a-2,p,q,n}(z). \qquad (16)
\end{aligned}$$

Proof. By the definition of $F_{a,p,q,n}(z)$, we have $F_{0,p,q,n}(z) = 1$, $F_{1,p,q,n}(z) = z^2+pq+pn+qn$ and

$$\begin{aligned}
F_{2,p,q,n}(z) &= z^4 + (2q+2pq+1+2pn+2p+2qn+2n)z^2 + pq^2 + pq \\
& \quad +qn + q^2n + p^2q^2 + p^2n^2 + p^2q + 4pqn + pn^2 + 2p^2qn
\end{aligned}$$

$$+pn+2pq^2n+2pqn^2+qn^2+q^2n^2+p^2n. \quad (17)$$

Thus it can be verified that (16) holds when $a = 2$.

Assume that (16) holds for every integer $2 \le a < k$, where $k \ge 3$. Now consider the case that $a = k$.

By the definition of $F_{a,p,q,n}(z)$, $F_{a,p,q,n}(z)$ is also a polynomial of order a in p. Let q, n, z be any fixed real numbers. If (16) holds for all numbers p in the set $\{-q + r : r = 0, 1, 2, \ldots\}$, then the result is proven.

By assumption on a, (16) holds for $F_{a-1,-q+1,q+1,n}(z)$ and thus

$$\begin{aligned} & F_{a-1,-q+1,q+1,n}(z) \\ = \; & (z^2 - 5a + 2an + 2a^2 + 3 - 2n - q^2)F_{a-2,-q+1,q+1,n}(z) \\ & -(a-2)(a-1)(-q-2+n+a)(q-2+n+a)F_{a-3,-q+1,q+1,n}(z). \end{aligned}$$

By Proposition 5, for any integer $m \ge 1$,

$$F_{m,-q,q,n}(z) = (z^2 - q^2)F_{m-1,-q+1,q+1,n}(z).$$

Hence

$$\begin{aligned} F_{a,-q,q,n}(z) \; = \; & (z^2 - 5a + 2an + 2a^2 + 3 - 2n - q^2)F_{a-1,-q,q,n}(z) \\ & -(a-2)(a-1)(-q-2+n+a)(q-2+n+a)F_{a-2,-q,q,n}(z), \end{aligned}$$

implying that (16) holds for $F_{a,-q,q,n}(z)$.

In the remaining part of this proof, we shall show that if (16) holds for $F_{a,p,q,n}(z)$, then (16) holds for $F_{a,p+1,q,n}(z)$. Assume (16) holds for $F_{a,p,q,n}(z)$, and so

$$\begin{aligned} F_{a,p,q,n}(z) \; = \; & (z^2 + (a-1)(2p+2q+2n+2a-3) + pq + pn + qn)F_{a-1,p,q,n}(z) \\ & -(a-1)(p+q+a-2)(q+n+a-2)(p+n+a-2)F_{a-2,p,q,n}(z). \quad (18) \end{aligned}$$

By assumption on a, (16) holds for $F_{a-1,p+1,q,n}(z)$ and so

$$\begin{aligned} & F_{a-1,p+1,q,n}(z) \\ = \; & (z^2 + (a-2)(2p+2q+2n+2a-3) + (p+1)(n+q) + qn)F_{a-2,p+1,q,n}(z) \\ & -(a-2)(p+q+a-2)(q+n+a-3)(p+n+a-2)F_{a-3,p+1,q,n}(z). \quad (19) \end{aligned}$$

By Proposition 6, (19) and (19), we have

$$\begin{aligned} & F_{a,p+1,q,n}(z) \\ = \; & F_{a,p,q,n}(z) + a(a+q+n-1)F_{a-1,p+1,q,n}(z) \\ = \; & (z^2 + (a-1)(2p+2q+2n+2a-3) + pq + pn + qn)F_{a-1,p,q,n}(z) \\ & -(a-1)(p+q+a-2)(q+n+a-2)(p+n+a-2)F_{a-2,p,q,n}(z) \\ & +a(a+q+n-1)F_{a-1,p+1,q,n}(z) \\ = \; & (z^2 + (a-1)(2p+2q+2n+2a-3) + pq + pn + qn) \end{aligned}$$

$$
\begin{aligned}
& (F_{a-1,p+1,q,n}(z) - (a-1)(a+q+n-2)F_{a-2,p+1,q,n}(z)) \\
& -(a-1)(p+q+a-2)(q+n+a-2)(p+n+a-2) \\
& (F_{a-2,p+1,q,n}(z) - (a-2)(a+q+n-3)F_{a-3,p+1,q,n}(z)) \\
& +a(a+q+n-1)F_{a-1,p+1,q,n}(z) \\
= \ & (z^2 + (a-1)(2p+2q+2n+2a-3) + pq + pn + qn) \\
& (F_{a-1,p+1,q,n}(z) - (a-1)(a+q+n-2)F_{a-2,p+1,q,n}(z)) \\
& -(a-1)(p+q+a-2)(q+n+a-2)(p+n+a-2)F_{a-2,p+1,q,n}(z) \\
& +(a-1)(a+q+n-2)(-F_{a-1,p+1,q,n}(z) + \\
& (z^2 + (a-2)(2p+2q+2n+2a-3) + (p+1)(n+q) + qn)F_{a-2,p+1,q,n}(z)) \\
& +a(a+q+n-1)F_{a-1,p+1,q,n}(z) \\
= \ & (z^2 + (a-1)(2p+2q+2n+2a-1) + (p+1)(n+q) + qn)F_{a-1,p+1,q,n}(z) \\
& -(a-1)(p+q+a-1)(q+n+a-2)(p+n+a-1)F_{a-2,p+1,q,n}(z).
\end{aligned}
$$

Thus (16) holds for $F_{a,p+1,q,n}(z)$. Hence (16) holds for $F_{a,p,q,n}(z)$ for all numbers p in the set $\{q+r : r = 0, 1, 2, \ldots\}$ and therefore the result is proved. □

Since $F_{0,p,q,n}(z) = 1$ and $F_{1,p,q,n}(z) = z^2 + pq + pn + qn$, Proposition 7 implies that $F_{a,p,q,n}(z)$ is an even polynomial in z. For any non-negative integer a and real numbers p, q, n, let $W_{a,p,q,n}(z)$ be the polynomial in z defined as follows: $W_{0,p,q,n}(z) = 1$, $W_{1,p,q,n}(z) = z + pq + pn + qn$ and for $a \geq 2$,

$$
\begin{aligned}
W_{a,p,q,n}(z) = \quad & (z + (a-1)(2p+2q+2n+2a-3) + pq + pn + qn)W_{a-1,p,q,n}(z) \\
& -(a-1)(p+q+a-2)(q+n+a-2)(p+n+a-2)W_{a-2,p,q,n}(z). \quad (20)
\end{aligned}
$$

Thus it is clear that $F_{a,p,q,n}(z) = W_{a,p,q,n}(z^2)$.

For two non-increasing sequences $(a_1, a_2, \ldots, a_m)$ and $(b_1, b_2, \ldots, b_n)$ of real numbers, we say the first *interleaves* the second if $m = n + 1$ and $(a_1, b_1, a_2, b_2, \ldots, a_n, b_n, a_{n+1})$ is an non-increasing sequence, or $m = n$ and $(a_1, b_1, a_2, b_2, \ldots, a_n, b_n)$ is an non-increasing sequence. If both polynomials $f(x)$ and $g(x)$ in x with real coefficients have only real roots and the non-increasing sequence formed by all roots of $f(x)$ interleaves the non-increasing sequence formed by all roots of $g(x)$, then we say $f(x)$ *interleaves* $g(x)$. We need to apply the following result (Proposition 8) given in Section 1.3 of [4]. Note that paper [8] has a result (Theorem 2.3 in that paper) stronger than Proposition 8. More details on polynomials with only real roots can be found in [2, 4, 8].

Proposition 8 ([4]) *Let $f(x)$ and $g(x)$ be polynomials with real coefficients and with positive leading coefficients and u and v be any real numbers. If $f(x)$ interleaves $g(x)$ and $v \leq 0$, then $(x-u)f(x) + vg(x)$ interleaves $f(x)$.* □

Applying Proposition 8 or Theorem 2.3 in [8], we can get the following result.

Proposition 9 *Let a be any positive integer and p, q, n be any real numbers.*

(i) *If $(p+q)(n+q)(n+p) \geq 0$, then $W_{2,p,q,n}(z)$ interleaves $W_{1,p,q,n}(z)$.*

(ii) *If $a \geq 3$, $(p+q+a-2)(q+n+a-2)(p+n+a-2) \geq 0$ and $W_{a-1,p,q,n}(z)$ interleaves $W_{a-2,p,q,n}(z)$, then $W_{a,p,q,n}(z)$ interleaves $W_{a-1,p,q,n}(z)$.*

Proof. By the definition of $W_{a,p,q,n}(z)$, $W_{1,p,q,n}(z) = z + pq + pn + qn$ and

$$\begin{aligned} W_{2,p,q,n}(z) &= (z + 2p + 2q + 2n + 1 + pq + pn + qn)(z + pq + pn + qn) \\ &\quad -(p+q)(q+n)(p+n). \end{aligned} \tag{21}$$

As the only root of $W_{1,p,q,n}(z)$ is $-pq - pn - qn$ and $W_{2,p,q,n}(-pq - pn - qn) = -(p+q)(n+q)(n+p) \leq 0$, $W_{2,p,q,n}(z)$ interleaves $W_{1,p,q,n}(z)$. So (i) holds.

By Proposition 7,

$$\begin{aligned} F_{a,p,q,n}(z) &= (z^2 + (a-1)(2p + 2q + 2n + 2a - 3) + pq + pn + qn)F_{a-1,p,q,n}(z) \\ &\quad -(a-1)(p+q+a-2)(q+n+a-2)(p+n+a-2)F_{a-2,p,q,n}(z). \end{aligned} \tag{22}$$

Since $-(a-1)(p+q+a-2)(q+n+a-2)(p+n+a-2) \leq 0$ and $W_{a-1,p,q,n}(z)$ interleaves $W_{a-2,p,q,n}(z)$, Proposition 8 implies that $W_{a,p,q,n}(z)$ interleaves $W_{a-1,p,q,n}(z)$. Hence (ii) holds. □

Notice that $W_{a,p,q,n}(z) = W_{a,q,p,n}(z) = W_{a,n,q,p}(z)$ holds for arbitrary real numbers p, q, n and non-negative integer a, we assume that $p \leq q \leq n$ in the following.

Proposition 10 *Let p, q, n be arbitrary real numbers with $p \leq q \leq n$ and $p + q \geq 0$. Then, for every integer $a \geq 2$, $W_{a,p,q,n}(z)$ interleaves $W_{a-1,p,q,n}(z)$. Therefore, for every positive integer a, $W_{a,p,q,n}(z)$ has only real roots and every root of $F_{a,p,q,n}(z)$ is either a real number or a non-real number with its real part equal to 0.*

Proof. Since $p + q \geq 0$ and $p \leq q \leq n$, we have $q + n \geq p + n \geq 0$ and so Proposition 9 (i) implies that $W_{2,p,q,n}(z)$ interleaves $W_{1,p,q,n}(z)$. Then, by Proposition 9 (ii), $W_{a,p,q,n}(z)$ interleaves $W_{a-1,p,q,n}(z)$ for every integer $a \geq 3$. □

By the discussion immediately preceding Proposition 4, it follows that for all positive integers a, b, c, d with $a \leq \min\{b, c, d\}$, the hypotheses of Proposition 10 are satisfied and hence we have proved Theorem 1.

Remark: There is another way to obtain the result of Proposition 10 by showing that all roots of $W_{a,p,q,n}(z)$ are actually the eigenvalues of a symmetric matrix with real entries only. Assume that p, q, n are arbitrary real numbers with $p \leq q \leq n$ and $p + q \geq 0$. For any positive integer a, let $B_a = (b_{i,j})$ be the $a \times a$ symmetric matrix whose non-zero entries are $b_{i,i}, b_{i,i-1}, b_{i-1,i}$ given below:

$$b_{i,i} = -((i-1)(2p + 2q + 2n + 2i - 3) + pq + pn + qn)$$

for all $i = 1, 2, \cdots, a$ and

$$b_{i-1,i} = b_{i,i-1} = ((i-1)(p+q+i-2)(q+n+i-2)(p+n+i-2))^{1/2}$$

for all $i = 2, \cdots, a$. It is not difficult to show that $\det(zI_a - B_a) = W_{a,p,q,n}(z)$ for all $a \geq 1$, where I_a is the identity matrix of size a. Since B_a is a symmetric matrix with real entries only, all roots of $\det(zI_a - B_a)$ are real and thus Proposition 10 follows.

4 Further properties of $F_{a,p,q,n}$ and $W_{a,p,q,n}$

Even if $p+q<0$, there are still some situations in which $W_{a,p,q,n}(z)$ has only real roots. In this section we consider these, although they do not correspond to values of the parameters a, p, q and n that arise from rings of cliques. We need to apply the following result on the factorization of $F_{a,p,q,n}(z)$ when $a+p+n=1$ or $a+p+n=2$.

Proposition 11 *Let a be an integer with $a \ge 1$ and p,q,n be arbitrary real numbers.*

(i) If $a+p+n=1$, then

$$F_{a,p,q,n}(z) = \prod_{j=0}^{a-1}(z^2-(n+j)^2). \tag{23}$$

(ii) If $a+p+n=2$, then

$$F_{a,p,q,n}(z) = \big(z^2+(p-1)(n-1)+aq\big)\prod_{j=0}^{a-2}(z^2-(n+j)^2). \tag{24}$$

Proof. (i) If $a+p+n=1$, then

$$i!(a-i)!\binom{z-p}{a-i}\binom{z+n+i-1}{i} = \prod_{j=0}^{a-1}(z+n+j).$$

Thus

$$\begin{aligned} F_{a,p,q,n}(z) &= a!\sum_{i=0}^{a} i!(a-i)!\binom{a+p+q-1}{i}\binom{z-p}{a-i}\binom{z-q}{a-i}\binom{z+n+i-1}{i} \\ &= a!\prod_{j=0}^{a-1}(z+n+j)\sum_{i=0}^{a}\binom{a+p+q-1}{i}\binom{z-q}{a-i} \\ &= a!\prod_{j=0}^{a-1}(z+n+j)\binom{a+p+q-1+z-q}{a} \\ &= a!\prod_{j=0}^{a-1}(z+n+j)\binom{z-n}{a} \\ &= \prod_{j=0}^{a-1}(z^2-(n+j)^2). \end{aligned} \tag{25}$$

Thus (i) holds.

(ii) Now let $a+p+n=2$. Since $F_{1,p,q,n}(z)=z^2+pq+pn+qn$, it is easy to verify that (ii) holds when $a=1$. Assume that (ii) holds for any integer $1\le a<k$, where $k\ge 2$. Now let $a=k$.

Since $a+p+n=2$, by Proposition 7,

$$F_{a,p,q,n}(z)=(z^2+(a-1)(2p+2q+2n+2a-3)+pq+pn+qn)F_{a-1,p,q,n}(z).$$

As $a-1+p+n=1$, by (i) of this result, we have

$$F_{a-1,p,q,n}(z)=\prod_{j=0}^{a-2}(z^2-(n+j)^2).$$

Since $p+n+a=2$, it can be verified that

$$(a-1)(2p+2q+2n+2a-3)+pq+pn+qn=(p-1)(n-1)+aq.$$

Hence (ii) also holds. □

Proposition 12 *Let p,q,n be arbitrary real numbers with $p\le q\le n$.*

(i) *If $p+q$ is a negative integer, then for every integer a with $a\ge 2-p-q$, $W_{a,p,q,n}(z)$ interleaves $W_{a-1,p,q,n}(z)$.*

(ii) *If $q+n$ is an integer, then for every integer a with $\max\{2,2-q-n\}\le a\le 2-p-n$, $W_{a,p,q,n}(z)$ interleaves $W_{a-1,p,q,n}(z)$.*

Proof. (i) First consider the case that $a=2-p-q$. Since $p+q\le -1$, we have $a\ge 3$. Proposition 11 implies that $W_{a,p,q,n}(z)$ interleaves $W_{a-1,p,q,n}(z)$.

Now assume that $a>2-p-q$ and $W_{a-1,p,q,n}(z)$ interleaves $W_{a-2,p,q,n}(z)$. Since $a>2-p-q$, we have $a+p+q-2\ge 1$ and so $a+q+n-2\ge a+p+n-2\ge a+p+q-2\ge 1$. Thus Proposition 9 (ii) implies that $W_{a,p,q,n}(z)$ interleaves $W_{a-1,p,q,n}(z)$. Therefore (i) holds.

(ii) The result is trivial if $\max\{2,2-q-n\}>2-p-n$. Now assume that $\max\{2,2-q-n\}\le 2-p-n$.

Let $a=\max\{2,2-q-n\}$. Then $a\ge 2-q-n$, implying that $a+q+n-2\ge 0$. We also have $a\le 2-p-n$, implying that $a+p+n-2\le 0$ and so $a+p+q-2\le 0$. If $a=\max\{2,2-q-n\}=2$, then Proposition 9 (i) implies that $W_{2,p,q,n}(z)$ interleaves $W_{1,p,q,n}(z)$, i.e., $W_{a,p,q,n}(z)$ interleaves $W_{a-1,p,q,n}(z)$. If $a=\max\{2,2-q-n\}=2-q-n$, then Proposition 11 implies that $W_{a,p,q,n}(z)$ interleaves $W_{a-1,p,q,n}(z)$.

Now assume that $\max\{2,2-q-n\}<a\le 2-p-n$ and $W_{a-1,p,q,n}(z)$ interleaves $W_{a-2,p,q,n}(z)$. Note that $\max\{2,2-q-n\}<a\le 2-p-n$ implies that $a+q+n-2>0$ and $a+p+q-2\le a+p+n-2\le 0$. Thus Proposition 9 (ii) implies that $W_{a,p,q,n}(z)$ interleaves $W_{a-1,p,q,n}(z)$. Therefore (ii) holds. □

By Proposition 12, the following result is obtained.

Proposition 13 *Let a be a positive integer and p,q,n be arbitrary real numbers with $p\le q\le n$. If one of the following conditions holds, then $W_{a,p,q,n}(z)$ has only real roots and therefore every root of $F_{a,p,q,n}(z)$ is either a real number or a non-real number with its real part equal to 0:*

(i) *$p+q$ is a negative integer and $a\ge 1-p-q$; and*

(ii) *$q+n$ is an integer and $\max\{1,1-q-n\}\le a\le 2-p-n$.* □

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