Title	DP color functions versus chromatic polynomials
Author(s)	Fengming Dong and Yan Yang

The final publication is available at: https://doi.org/10.1016/j.aam.2021.102301

DP color functions versus chromatic polynomials

Fengming Dong*

National Institute of Education, Nanyang Technological University, Singapore

Yan Yang†

School of Mathematics, Tianjin University, China

Abstract

For any graph G, the chromatic polynomial of G is the function $P(G,m)$ which counts the number of proper m-colorings of G for each positive integer m. The DP color function $P_{DP}(G,m)$ of G, introduced by Kaul and Mudrock in 2019, is a generalization of $P(G,m)$ with $P_{DP}(G,m) \leq P(G,m)$ for each positive integer m. Let $P_{DP}(G) \approx P(G)$ (resp. $P_{DP}(G) < P(G)$) denote the property that $P_{DP}(G,m) = P(G,m)$ (resp. $P_{DP}(G,m) < P(G,m)$) holds for sufficiently large integers m. It is an interesting problem of finding graphs G for which $P_{DP}(G) \approx P(G)$ (resp. $P_{DP}(G,m) < P(G,m)$) holds. Kaul and Mudrock showed that if G has an even girth, then $P_{DP}(G) < P(G)$ and Mudrock and Thomason recently proved that $P_{DP}(G) \approx P(G)$ holds for each graph G which has a dominating vertex. We shall generalize their results in this article. For each edge e in G, let $\ell(e) = \infty$ if e is a bridge of G, and let $\ell(e)$ be the length of a shortest cycle in G containing e otherwise. We first show that if $\ell(e)$ is even for some edge e in G, then $P_{DP}(G) < P(G)$ holds. However, the converse statement of this conclusion fails with infinitely many counterexamples. We then prove that $P_{DP}(G) \approx P(G)$ holds for every graph G that contains a spanning tree T such that for each $e \in E(G) \setminus E(T)$, $\ell(e)$ is odd and e is contained in a cycle C of length $\ell(e)$ with the property that $\ell(e') < \ell(e)$ for each $e' \in E(C) \setminus (E(T) \cup \{e\})$. Some open problems are proposed in this article.

Keywords: proper coloring; listing coloring; DP-coloring; chromatic polynomial; DP color function; spanning tree; cycle

Mathematics Subject Classification: 05C05, 05C30, 05C31

*Corresponding author. Email: fengming.dong@nie.edu.sg and donggraph@163.com.

†Email: yanyang@tju.edu.cn

1 Introduction

In this article, we consider simple graphs only, unless otherwise stated. For any graph G, let $V(G)$ and $E(G)$ be its vertex set and edge set respectively. For any nonempty subset S of $V(G)$, let $G[S]$ be the subgraph of G induced by S, i.e., the subgraph with vertex set S and edge set $\{uv \in E(G) : u, v \in S\}$, where uv denotes the edge joining u and v, and let $G - S$ be the subgraph $G[V(G) \setminus S]$ when $S \neq V(G)$. In particular, if $S = \{v\}$ for $v \in V(G)$, write $G - v$ for $G - S$. For $A \subseteq E(G)$, let $G\langle A\rangle$ denote the spanning subgraph of G with edge set A, and let $G - A$ be $G\langle E(G) \setminus A\rangle$. In particular, for $e \in E(G)$, $G - \{e\}$ is written as $G - e$. For two disjoint subsets S_1 and S_2 of $V(G)$, let $E_G(S_1, S_2)$ (or simply $E(S_1, S_2)$) denote the set $\{uv \in E(G) : u \in S_1, v \in S_2\}$. For any $u \in V(G)$, let $N_G(u)$ (or simply $N(u)$) be the set of neighbors of u in G and $d_G(u)$ (or simply $d(u)$) be the degree of u in G. The reader may refer to [5] for other terminology and notation.

1.1 Proper coloring, list coloring and DP coloring

Let $\mathbb{N}$ denote the set of positive integers. For any $n \in \mathbb{N}$, let $[\![n]\!] = \{1, 2, \cdots, n\}$. For any graph G and $k \in \mathbb{N}$, a *proper k-coloring* of G is a mapping of $f : V(G) \to [\![k]\!]$ such that $f(u) \neq f(v)$ for each edge $uv \in E(G)$. The *chromatic polynomial* $P(G, k)$ of G, introduced by Birkhoff [3] in 1912, is the function which counts the number of proper k-colorings of G for each $k \in \mathbb{N}$. Note that $P(G, k)$ is indeed a polynomial in k for each $k \in \mathbb{N}$ (see [4, 8, 17, 23]). The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number $k \in \mathbb{N}$ such that G admits a proper k-coloring. Obviously, $\chi(G)$ is the minimum number $k \in \mathbb{N}$ such that $P(G, k) > 0$. For more details on chromatic polynomials, we refer the readers to [3, 4, 8, 9, 13, 17, 18, 19, 23].

List coloring was introduced by Vizing [21] and Erdős, Rubin and Taylor [11] independently. A list coloring of G is associated with a *list assignment* L, where $L(v)$ is a subset of $\mathbb{N}$ for each $v \in V(G)$. Given a list assignment L of G, a *proper* L-coloring of G is a mapping $f : V(G) \to \mathbb{N}$ such that $f(v) \in L(v)$ for each $v \in V(G)$ and $f(u) \neq f(v)$ for each edge $uv \in E(G)$. If $L(v) = [\![k]\!]$ for each $v \in V(G)$, then a proper L-coloring of G is a proper k-coloring of G. If $|L(v)| = m$ for each $v \in V(G)$, then L is called an *m-assignment* . The *list chromatic number* of G, denoted by $\chi_l(G)$, is the smallest m such that G has a proper L-coloring for every m-assignment L of G. By definition, $\chi(G) \leq \chi_l(G)$. Due to Noel, Reed and Wu [16], $\chi(G) = \chi_l(G)$ holds whenever $\chi(G) \geq (|V(G)| - 1)/2$.

For each list-assignment L of G, let $P(G, L)$ be the number of proper L-colorings. For each $m \in \mathbb{N}$, let $P_l(G, m)$ be the minimum value of $P(G, L)$ among all m-assignments L. We call

$P_l(G,m)$ the *list color function* of G. By definition, $P_l(G,m) \le P(G,m)$ for each $m \in \mathbb{N}$. Wang, Qian and Yan [22] showed that $P_l(G,m) = P(G,m)$ holds when G is connected and $m > (|E(G)|-1)/\ln(1+\sqrt{2})$. The survey by Thomassen [20] provided some known results and open questions on the list color function.

DP-coloring was introduced by Dvorák and Postle [10] for the purpose of proving that every planar graph without cycles of lengths 4 to 8 is 3-choosable. DP-coloring is a generalization of list coloring, and a formal definition is given below. For a graph G, a *cover* of G is an ordered pair $\mathcal{H} = (L, H)$, where H is a graph and L is a mapping from $V(G)$ to the power set of $V(H)$ satisfying the four conditions below:

(i). the sets $\{L(u) : u \in V(G)\}$ is a partition of $V(H)$ of size $|V(G)|$;

(ii). for every $u \in V(G)$, $H[L(u)]$ is a complete graph;

(iii). if u and v are non-adjacent vertices in G, then $E_H(L(u), L(v)) = \emptyset$; and

(iv). for each edge $uv \in E(G)$, $E_H(L(u), L(v))$ is a matching.

An $\mathcal{H}$*-coloring* of G is an independent set I of H with $|I| = |V(G)|$. Clearly, for each $\mathcal{H}$-coloring I of G, $|I \cap L(u)| = 1$ holds for each $u \in V(G)$. A cover $\mathcal{H} = (L, H)$ of G is called an m*-fold* cover if $|L(u)| = m$ for each $u \in V(G)$. The *DP-chromatic number* of G, denoted by $\chi_{DP}(G)$, is the minimum integer m such that G has a $\mathcal{H}$-coloring for every m-fold cover $\mathcal{H} = (L, H)$. By definition, $\chi(G) \le \chi_l(G) \le \chi_{DP}(G)$. Bernshteyn, Kostochka and Zhu [2] showed that for any $n \in \mathbb{N}$, if $r(n)$ is the minimum number $r \in \mathbb{N}$ such that $\chi(G) = \chi_{DP}(G)$ holds for every n-vertex graph G with $\chi(G) \ge r$, then $n - r(n) = \Theta(\sqrt{n})$.

For any cover $\mathcal{H}$ of G, let $P_{DP}(G,\mathcal{H})$ be the number of $\mathcal{H}$-colorings of G. For each $m \in \mathbb{N}$, let $P_{DP}(G,m)$ be the minimum value of $P_{DP}(G,\mathcal{H})$ among all m-fold covers $\mathcal{H}$ of G. We call $P_{DP}(G,m)$ the *DP color function* of G, which was introduced by Kaul and Mudrock [14]. For any m-assignment L of G, $P(G,L) = P_{DP}(G,\mathcal{H})$ holds for the m-fold cover $\mathcal{H} = (L', H)$, where $L'(v) = \{(v,j) : j \in L(v)\}$ for each $v \in V(G)$ and for each edge $uv \in E(G)$, $E_H(L'(u), L'(v)) = \{(u,j)(v,j) : j \in \mathbb{N}, (u,j) \in L'(u), (v,j) \in L'(v)\}$. Thus, $P_{DP}(G,m) \le P_l(G,m) \le P(G,m)$ holds for each $m \in \mathbb{N}$.

1.2 Main results

For any graph G, by definition, $P_{DP}(G,m) \le P(G,m)$ holds for all integers $m \in N$. Thus, exactly one of the following three properties holds:

(i). there exists $N \in \mathbb{N}$ such that $P_{DP}(G, m) = P(G, m)$ for all integers $m \geq N$;

(ii). there exists $N \in \mathbb{N}$ such that $P_{DP}(G, m) < P(G, m)$ for all integers $m \geq N$; and

(iii). there exist two infinite sets $\{m_i \in \mathbb{N} : i \in \mathbb{N}\}$ and $\{n_i \in \mathbb{N} : i \in \mathbb{N}\}$ such that for all $i \in \mathbb{N}$, both $P_{DP}(G, m_i) = P(G, m_i)$ and $P_{DP}(G, n_i) < P(G, n_i)$ hold.

Two questions proposed by Kaul and Mudrock [14] are closed related to property (iii), and there would be no graphs having property (iii) if the answer to any one of them had been yes. Question 7 in [14] asks if, for any graph G, there always exist an $N \in \mathbb{N}$ and a polynomial $p(m)$ such that $P_{DP}(G, m) = p(m)$ whenever $m \geq N$. Halberg, Kaul, Liu, Mudrock, Shin and Thomason [12] showed that this question has a positive answer for each graph G with a vertex v such that $G - v$ is acyclic. Question 15 in [14] asks if $P_{DP}(G, m_0) = P(G, m_0)$ for some $m_0 \geq \chi(G)$ implies that $P_{DP}(G, m) = P(G, m)$ for all $m \geq m_0$. Unfortunately, Bui, Kaul, Maxfield, Mudrock, Shin and Thomason [6] found graphs with negative answer to the second question.

For any one of the above properties, it is an interesting problem of knowing which graphs have this property. For convenience purposes, let $P_{DP}(G) \approx P(G)$ (resp. $P_{DP}(G) < P(G)$) denote property (i) (resp. property (ii)) above for a graph G.

Problem 1. *Is it true that for each graph G, either $P_{DP}(G) \approx P(G)$ or $P_{DP}(G) < P(G)$?*

So far the comparison of DP color functions with chromatic polynomials focuses on following problem.

Problem 2. *Determine the set of graphs G such that $P_{DP}(G) \approx P(G)$ holds and the set of graphs G such that $P_{DP}(G) < P(G)$ holds.*

Kaul and Mudrock [14] obtained some important results on the study of Problem 2. For example, they showed that if there exists an edge e in G such that $P(G-e, m) < mP(G, m)/(m-1)$, then $P_{DP}(G, m) < P(G, m)$ holds (see Theorem 6).

For each edge e in G, if e is a bridge of G, let $\ell_G(e) = \infty$; otherwise, let $\ell_G(e)$ be the length of a shortest cycle containing e in G. Write $\ell_G(e)$ as $\ell(e)$ when G is clear from the context. Thus, the girth g of G is the minimum value of $\ell(e)$ among all edges e in G. Kaul and Mudrock [14] showed that if G has an even girth, then $P_{DP}(G) < P(G)$. We apply Theorem 6 to generalize this result below.

Theorem 1. *For any graph G, if $\ell(e)$ is even for some edge e in G, then $P_{DP}(G) < P(G)$.*

The converse statement of Theorem 1 fails, and counterexamples will be given in Section 3.

Theorem 2. *There exist infinitely many graphs G such that $P_{DP}(G) < P(G)$ and $\ell(e) = 3$ for each edge e in G.*

For a disconnected graph G, if $P_{DP}(G_i) < P(G_i)$ for some component G_i of G, then $P_{DP}(G) < P(G)$ obviously holds. This conclusion also holds for connected graphs.

Theorem 3. *For a connected graph G, if $P_{DP}(G_i) < P(G_i)$ for some block G_i of G, then $P_{DP}(G) < P(G)$ holds.*

Some results on the study of graphs with the property $P_{DP}(G) \approx P(G)$ have been obtained. Kaul and Mudrock [14] showed that $P_{DP}(G) \approx P(G)$ holds for the graph G obtained from any two odd cycle graphs C_{2k+1} and C_{2r+1} by identifying one edge in C_{2k+1} with one edge in C_{2r+1}. For two vertex-disjoint graphs G and G', let $G \vee G'$ denote the join of G and G', i.e., the graph obtained from G and G' by adding all edges in $\{uv : u \in V(G), v \in V(G')\}$. Kaul and Mudrock [14] asked that for every graph G, does there exist $p \in \mathbb{N}$ such that $P_{DP}(K_p \vee G) \approx P(K_p \vee G)$, where K_p is the complete graph with p vertices? Recently, Mudrock and Thomason [15] showed that the problem has a positive answer for $p = 1$. Obviously, a graph is isomorphic to $K_1 \vee G$ for some graph G if and only if it has a dominating vertex (i.e., a vertex which is adjacent to all other vertices in the graph).

For any graph G and any integer $m > 0$, there is a special m-fold cover of G which corresponds to proper m-colorings. Let $\mathcal{H}_0(G, m)$ denote the m-fold cover (L, H) of G, where $L(u) = \{(u, i) : i \in [\![m]\!]\}$ for each $u \in V(G)$ and $E_H(L(u), L(v)) = \{(u, i)(v, i) : i \in [\![m]\!]\}$ for each edge uv in G. The graph H in $\mathcal{H}_0(G, m) = (L, H)$ is denoted by $H_0(G, m)$ (or simply $H_0(m)$). Obviously, $P_{DP}(G, \mathcal{H}_0(G, m)) = P(G, m)$ holds for each $m \in \mathbb{N}$.

Let $\mathcal{DP}^*$ denote the set of graphs G for which there exists $M \in \mathbb{N}$ such that for every m-fold cover $\mathcal{H} = (L, H)$ of G, if $H \not\cong H_0(G, m)$, then $P_{DP}(G, \mathcal{H}) > P(G, m)$ holds for all integers $m \geq M$. By definition, $P_{DP}(G) \approx P(G)$ holds for each graph G in $\mathcal{DP}^*$. But it is unknown if the converse statement is also true.

Problem 3. *Is it true that if $P_{DP}(G) \approx P(G)$, then $G \in \mathcal{DP}^*$?*

Our next result provides a sufficient condition for a graph G to be in $\mathcal{DP}^*$ and therefore $P_{DP}(G) \approx P(G)$ holds.

Theorem 4. *If a graph G contains a spanning tree T such that for each edge e in $E(G) \setminus E(T)$, $\ell(e)$ is odd and e is contained in a cycle C of length $\ell(e)$ with the property that $\ell(e') < \ell(e)$ holds for each $e' \in E(C) \setminus (E(T) \cup \{e\})$, then $G \in \mathcal{DP}^*$ and hence $P_{DP}(G) \approx P(G)$.*

A vertex u in a graph G is called *simplicial* if either $d_G(u) = 0$ or $G[N(u)]$ is a complete graph. A graph G is called *chordal* if for each cycle C in G, $G[V(C)]$ contains 3-cycles. Due to Dirac [7], a graph G is chordal if and only if there exists an ordering $v_1, v_2, \cdots, v_n$ of its vertices, called a *perfect elimination ordering*, such that each v_i is simplicial in the subgraph of G induced by $\{v_j : j \in [\![i]\!]\}$. Due to Kaul and Mudrock [14], for any chordal graph G, $P_{DP}(G,m) = P(G,m)$ holds for all $m \in \mathbb{N}$, and hence $P_{DP}(G) \approx P(G)$ holds. We notice that this conclusion does not follow from Theorem 4. But the next result is its generalization.

Theorem 5. *For any graph G with a simplicial vertex u, if $P_{DP}(G-u) \approx P(G-u)$, then $P_{DP}(G) \approx P(G)$; also, if $G - u \in \mathcal{DP}^*$, then $G \in \mathcal{DP}^*$.*

Theorems 2 and 3 are proved in Section 3, while Theorems 1, 4 and 5 are proved in Sections 2, 4 and 5 respectively.

2 Proof of Theorem 1

The following result due to Kaul and Mudrock [14] will be applied to study graphs G with the property $P_{DP}(G) < P(G)$.

Theorem 6 ([14]). *Let G be a graph with an edge e. If $m \geq 2$ and $P(G-e,m) < \frac{m}{m-1}P(G,m)$, then $P_{DP}(G,m) < P(G,m)$.*

In this section, we shall apply two fundamental properties of the chromatic polynomial $P(G,x)$ of G. The variable x in $P(G,x)$ can be considered a real number. By the inclusion-exclusion principle, it can be proved that

$$P(G,x) = \sum_{A \subseteq E(G)} (-1)^{|A|} x^{c(A)}, \tag{1}$$

where $c_G(A)$ (or simply $c(A)$) is the number of components in the spanning subgraph $G\langle A \rangle$ of G (see [23]). Note that (1) holds even if G has parallel edges or loops.

The deletion-contraction theorem of chromatic polynomials (see [8, 17, 18]) states that for each edge e in a graph G,

$$P(G,x) = P(G-e,x) - P(G/e,x), \tag{2}$$

where G/e is the graph obtained by contracting edge e (i.e., the graph obtained from $G-e$ by identifying the two ends of e). Clearly, G/e may have parallel edges. By (2), for any $e \in E(G)$, when $x \neq 1$,

$$P(G-e,x) - \frac{x}{x-1}P(G,x) = P(G-e,x) - \frac{x}{x-1}\left(P(G-e,x) - P(G/e,x)\right)$$
$$= \frac{1}{x-1}\left(xP(G/e,x) - P(G-e,x)\right). \qquad (3)$$

For any edge e in G, let $\mathcal{C}(e)$ denote the set of cycles in G that contain e and are of length $\ell(e)$. Obviously, $\mathcal{C}(e) \neq \emptyset$ if e is not a bridge of G.

Proposition 7. *Let G be a simple graph and e be an edge in G with $\ell(e) < \infty$. Then, the leading term in the polynomial $xP(G/e,x) - P(G-e,x)$ is $(-1)^{\ell(e)-1}|\mathcal{C}(e)|x^{n-\ell(e)+2}$.*

Proof. Note that $G-e$ and G/e have the same edge set, i.e., $E(G)\setminus\{e\}$, and when $\ell(e)=3$, G/e has parallel edges. Applying (1) to both $G-e$ and G/e, we have

$$P(G-e,x) = \sum_{A\subseteq E(G)\setminus\{e\}} (-1)^{|A|} x^{c_G(A)} \qquad (4)$$

and

$$P(G/e,x) = \sum_{A\subseteq E(G)\setminus\{e\}} (-1)^{|A|} x^{c_{G/e}(A)}. \qquad (5)$$

Let u,v be the two ends of e, and let $\mathcal{E}_e$ be the set of subsets A of $E(G)\setminus\{e\}$, such that u and v are in the same component of the spanning subgraph $G\langle A\rangle$ of G. Let $\mathcal{E}'_e$ be the set of subsets A of $E(G)\setminus\{e\}$ with $A \notin \mathcal{E}_e$. If $A \in \mathcal{E}_e$, then $c_G(A) = c_{G/e}(A)$; and if $A \in \mathcal{E}'_e$, then $c_G(A) = c_{G/e}(A)+1$. Thus, (4) and (5) imply that

$$xP(G/e,x) - P(G-e,x) = \sum_{A\in\mathcal{E}_e} (-1)^{|A|} x^{c_G(A)}(x-1). \qquad (6)$$

For each $A \in \mathcal{E}_e$, let G_A denote the component of $G\langle A\rangle$ that contains both vertices u and v. Then G_A has a (u,v)-path P, implying that $|V(G_A)| \geq |V(P)| \geq \ell(e)$. If $|V(G_A)| = \ell(e)$, then $V(G_A) = V(P)$ and P must be a path $C-e$ for some cycle $C \in \mathcal{C}(e)$. As each cycle in G containing e must be of length at least $\ell(e)$, $|V(G_A)| = \ell(e)$ implies that G_A is a path $C-e$ for some cycle $C \in \mathcal{C}(e)$.

Consequently, for each $A \in \mathcal{E}_e$, $c_G(A) \leq n-\ell(e)+1$ holds, and $c_G(A) = n-\ell(e)+1$ if and only if $A\cup\{e\}$ is the edge set of some cycle C in $\mathcal{C}(e)$. Thus, $c_G(A) = n-\ell(e)+1$ holds for exactly $|\mathcal{C}(e)|$ subsets $A \in \mathcal{E}_e$, and for each of them, $|A| = \ell(e)-1$.

By (6) and the above conclusions, $xP(G/e,x) - P(G-e,x)$ is a polynomial of degree $n-\ell(e)+2$ and the coefficient of its leading term is $(-1)^{\ell(e)-1}|\mathcal{C}(e)|$.

Hence the result holds. □

We are now going to prove Theorem 1.

Proof of Theorem 1: Let e be an edge in G such that $\ell(e)$ is even. By the equality of (3) and Proposition 7, there exists $M \in \mathbb{N}$ such that $P(G-e,m) < \frac{m}{m-1}P(G,m)$ for all integer $m \geq M$. The result then follows from Theorem 6. □

3 Proof of Theorems 2 and 3

Let $\omega(G)$ denote the *clique number* of a graph G. For any vertex-disjoint graphs G_1 and G_2 and $k \in \mathbb{N}$ with $k \leq \min\{\omega(G_i) : i = 1, 2\}$, let $\mathscr{G}(G_1 \cup_k G_2)$ denote the set of graphs obtained from G_1 and G_2 by identifying a k-clique in G_1 with a k-clique in G_2. Due to Zykov [24], the following identity on chromatic polynomials holds for any $G \in \mathscr{G}(G_1 \cup_k G_2)$ and all $m \geq k$:

$$P(G,m) = \frac{P(G_1,m)P(G_2,m)}{m(m-1)\cdots(m-k+1)}. \tag{7}$$

If u is a simplicial vertex of a graph G, the following identity on chromatic polynomials follows from (7) (also see [8, 18]):

$$P(G,m) = (m - d_G(u))P(G-u,m), \qquad \forall m \in \mathbb{N}. \tag{8}$$

It is natural to ask if (8) holds for the DP color function.

Problem 4. *If u is a simplicial vertex of G, is it true that for all integers $m \geq d(u)$,*

$$P_{DP}(G,m) = (m - d(u))P_{DP}(G-u,m)? \tag{9}$$

As $P_{DP}(G,m) \geq (m-d(u))P_{DP}(G-u,m)$ by definition, to prove the equality of (9), it suffices to show that $P_{DP}(G,m) \leq (m-d(u))P_{DP}(G-u,m)$ for all integers $m \geq d(u)$. It is trivial that Problem 4 has a positive answer when $d(u) = 0$, and due to Theorem 10, it also has a positive answer when $d(u) = 1$. In this section, we show that it has a positive answer for $d(u) = 2$. Applying this conclusion, we are able to prove that the converse statement of Theorem 1 fails.

Proposition 8. *If u is a simplicial vertex of G with $d(u) = 2$, then for each integer $m \geq 2$,*

$$P_{DP}(G,m) = (m-2)P_{DP}(G-u,m). \tag{10}$$

Proof. Let $m \geq 2$. If $m < \chi_{DP}(G-u)$, then $m < \chi_{DP}(G-u) \leq \chi_{DP}(G)$, implying that $P_{DP}(G-u,m) = P_{DP}(G,m) = 0$. It follows that (10) holds in this case.

As u is a simplicial vertex of G with degree 2, $\chi_{DP}(G) \geq \chi(G) \geq 3$, implying that $P_{DP}(G,2) = 0$. Thus (10) also holds when $m = 2$.

Now let $m \geq \max\{3, \chi_{DP}(G-u)\}$ and let $\mathcal{H}' = (L', H')$ be an m-fold cover of $G-u$ such that $P_{DP}(G-u,\mathcal{H}') = P_{DP}(G-u,m)$ and $|E(H')|$ has the maximum value. It is clear that $E_{H'}(L'(v_1), L'(v_2))$ is a matching in H' of size m for each pair of adjacent vertices v_1 and v_2 in $G-u$.

Let $N_G(u) = \{u_1, u_2\}$. Assume that (u_1, j) and $(u_2, \pi(j))$ are adjacent in H' for each $j \in [\![m]\!]$, where π is a bijection from $[\![m]\!]$ to $[\![m]\!]$.

Let H be the graph obtained from H' and a complete graph with vertex set $\{(u,j) : j \in [\![m]\!]\}$ by adding edges joining (u,j) to both (u_1,j) and $(u_2,\pi(j))$ for each $j \in [\![m]\!]$. Let $\mathcal{H} = (L,H)$ be the m-fold cover of G, where $L(u) = \{(u,j) : j \in [\![m]\!]\}$ and $L(v) = L'(v)$ for all $v \in V(G) - \{u\}$.

Let I be any member in $\mathcal{I}(H')$. Assume that $(u_1, j_1) \in I \cap L(u_1)$ and $(u_2, \pi(j_2)) \in I \cap L(u_2)$. As (u_1, j_1) and $(u_2, \pi(j_1))$ are adjacent in H, $j_1 \neq j_2$. Then, I can be extended to exactly $(m-2)$ independent sets of H of the form $I \cup \{(u,j)\}$, where $j \in [\![m]\!] \setminus \{j_1, j_2\}$. Thus,

$$P_{DP}(G,\mathcal{H}) = (m-2)P_{DP}(G-u,\mathcal{H}') = (m-2)P_{DP}(G-u,m), \tag{11}$$

by which $P_{DP}(G,m) \leq (m-2)P_{DP}(G-u,m)$. On the other hand, it is obvious that $P_{DP}(G,m) \geq (m-2)P_{DP}(G-u,m)$. Thus, the result follows. □

For any graph Q with at least one edge, let $\Phi(Q)$ be the family of graphs defined below:

(i). $Q \in \Phi(Q)$; and

(ii). if $Q' \in \Phi(Q)$, then $\mathscr{G}(Q' \cup_2 K_3) \subseteq \Phi(Q)$.

For example, $G_1 \in \Phi(C_4)$ and $G_2 \in \Phi(C_6)$, where C_k is the cycle graph of length k, and G_1 and G_2 are graphs in Figure 1.

By (8) and Proposition 8, for any graph $G \in \Phi(Q)$ and any integer $m \geq 2$,

$$P(G,m) = (m-2)^{|V(G)|-|V(Q)|}P(Q,m), \quad P_{DP}(G,m) = (m-2)^{|V(G)|-|V(Q)|}P_{DP}(Q,m). \tag{12}$$

By (12), we have the following observation.

(a) G_1 (b) G_2

Figure 1: $G_1 \in \Phi(C_4)$ and $G_2 \in \Phi(C_6)$

Proposition 9. *For any graph Q with at least one edge and any $G \in \Phi(Q)$, if $P_{DP}(Q) \approx P(Q)$, then $P_{DP}(G) \approx P(G)$; also, if $P_{DP}(Q) < P(Q)$, then $P_{DP}(G) < P(G)$.*

We can now easily prove Theorem 2.

Proof of Theorem 2: Let Q be any graph with $P_{DP}(Q) < P(Q)$. Clearly, Q contains edges. By Proposition 9, $P_{DP}(G) < P(G)$ holds for every $G \in \Phi(Q)$. By the definition of $\Phi(Q)$, there are infinitely many graphs $G \in \Phi(Q)$ such that $\ell_G(e) = 3$ for each edge e in G. For example, for graph G_1 in Figure 1 (a), if G is a graph in $\Phi(G_1)$, then $P_{DP}(G) < P(G)$ and $\ell_G(e) = 3$ holds for each edge e in G.

Theorem 2 holds. □

Theorem 3 will be proved directly by applying the following result due to Becker, Hewitt, Kaul, Maxfield, Mudrock, Spivey, Thomason and Wagstrom [1].

Theorem 10 ([1])**.** *For any connected graph G with blocks $G_1, G_2, \cdots, G_r$, where $r \geq 2$,*

$$P_{DP}(G, m) \leq \frac{1}{m^{r-1}} \prod_{i=1}^{r} P_{DP}(G_i, m). \tag{13}$$

Proof of Theorem 3: Let $G_1, G_2, \cdots, G_r$ be the blocks of G. By identity (7) and Theorem 10, we have

$$P_{DP}(G, m) \leq \frac{1}{m^{r-1}} \prod_{i=1}^{r} P_{DP}(G_i, m) \leq \frac{1}{m^{r-1}} \prod_{i=1}^{r} P(G_i, m) = P(G, m). \tag{14}$$

By (14), if $P_{DP}(G_i) < P(G_i)$ for some i, then $P_{DP}(G) < P(G)$ holds. □

It is natural to ask the following problem.

Problem 5. *For a connected graph G, if $P_{DP}(G_i) \approx P(G_i)$ holds for each block G_i of G, is it true that $P_{DP}(G) \approx P(G)$?*

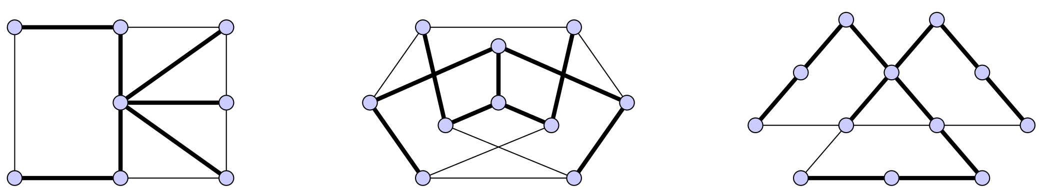
4 Proof of Theorem 4

4.1 A set of ordered pairs (G,T), where T is a spanning tree of G

Let $\mathcal{GT}$ be the set of ordered pairs (G,T), where G is a connected graph and T is a spanning tree of G such that for each edge e in $E(G)\setminus E(T)$, $\ell_G(e)$ is odd and e is contained in a cycle C of length $\ell_G(e)$ with the property that $\ell_G(e')<\ell_G(e)$ holds for each $e'\in E(C)\setminus(E(T)\cup\{e\})$.

Note that $\mathcal{GT}$ contains a subfamily $\mathcal{GT}_0$ of ordered pairs (G,T), where T is a spanning tree of G such that for each $e\in E(G)\setminus E(T)$, $\ell(e)$ is odd and the fundamental cycle $C_T(e)$ of e with respect to T is of length $\ell(e)$.

Let $\mathcal{G}$ (resp. $\mathcal{G}_0$) be the set of graphs G such that $(G,T)\in\mathcal{GT}$ (resp. $(G,T)\in\mathcal{GT}_0$) for some spanning tree T of G. For example, for $i=1,2$, $G_i\in\mathcal{G}_0$, where G_1 and G_2 (i.e., the Petersen graph) are the graphs in Figure 2 (a) and (b) respectively. It is also obvious that $\mathcal{G}_0$ contains every graph that has a dominating vertex. But, it can be verified that G_3 in Figure 2 (c) belongs to $\mathcal{G}\setminus\mathcal{G}_0$.



(a) G_1 (b) G_2 (c) G_3

Figure 2: $G_i\in\mathcal{G}_0\subset\mathcal{G}$ for $i=1,2$ and $G_3\in\mathcal{G}\setminus\mathcal{G}_0$

Let $(G,T)\in\mathcal{GT}$. By definition, $\ell_G(e)$ is odd for each $e\in E(G)\setminus E(T)$. But, it does not guarantee directly that $\ell_G(e)$ is not even for any $e\in E(T)$. By Theorem 1, if $\ell_G(e)$ is even for some $e\in E(T)$, then Theorem 4 fails. Thus, before proving Theorem 4, it is necessary to show that $\ell_G(e)$ is not even for every $e\in E(T)$.

For $(G,T)\in\mathcal{GT}$, if $E(G)=E(T)$, let $\ell(G,T)=\infty$; otherwise, let $\ell(G,T)=\max\limits_{e\in E(G)\setminus E(T)}\ell_G(e)$.

Proposition 11. *Let $(G,T)\in\mathcal{GT}$. For each edge $e\in E(T)$, if e is not a bridge of G, then $\ell_G(e)$ is odd and $\ell_G(e)\le\ell(G,T)$.*

Proof. We prove the result by induction on $|E(G)|$. Note that $|E(G)|\ge|E(T)|$. The result is obvious when $|E(G)|\le|E(T)|+1$. Now assume that $|E(G)|\ge|E(T)|+2$ and the result holds for every ordered pair $(G',T')\in\mathcal{GT}$ with $|E(G')|\le|E(G)|-1$.

Choose an edge e_1 in $E(G) \setminus E(T)$ such that $\ell_G(e_1) = \ell(G,T) < \infty$. Clearly, T is a spanning tree of $G - e_1$. We first show that $(G - e_1, T) \in \mathcal{GT}$.

Let G' denote $G - e_1$ and let e be any edge in $E(G') \setminus E(T)$. As $e \in E(G) \setminus E(T)$, by definition, $\ell_G(e)$ is odd and e is contained in a cycle C in G of length $\ell_G(e)$ such that $\ell_G(e') < \ell_G(e)$ for each $e' \in E(C) \setminus (E(T) \cup \{e\})$. By the choice of e_1, $\ell_G(e_1) \geq \ell_G(e)$, implying that $e_1 \notin E(C)$. Thus, C is in G' and $\ell_G(e) = \ell_{G'}(e)$.

Hence, by definition, $(G', T) \in \mathcal{GT}$ and $\ell_{G'}(e) = \ell_G(e)$ for each $e \in E(G') \setminus E(T)$, implying that $\ell(G', T) \leq \ell(G, T)$.

By inductive assumption, the conclusion holds for $(G', T) \in \mathcal{GT}$. Now suppose $e_0 \in E(T)$ and e_0 is not a bridge in G. Then, either e_0 is a bridge of G' or $\ell_{G'}(e_0)$ is odd. Furthermore, if e_0 is not a bridge of G', then $\ell_{G'}(e_0) \leq \ell(G', T) \leq \ell(G, T) = \ell_G(e_1)$. We shall show that $\ell_G(e_0)$ is odd and $\ell_G(e_0) \leq \ell(G, T)$.

Case 1: e_0 is a bridge of G' (i.e., $G - e_1$).

In this case, for each cycle C in G, either $E(C) \cap \{e_0, e_1\} = \emptyset$ or $\{e_0, e_1\} \subseteq E(C)$, implying that $\ell_G(e_0) = \ell_G(e_1) = \ell(G, T)$ is odd.

Case 2: $\ell_{G'}(e_0)$ is odd.

In this case, $\ell_{G'}(e_0) \leq \ell(G', T) \leq \ell(G, T) = \ell_G(e_1)$. If $\ell_G(e_0) < \ell_{G'}(e_0)$, then e_0 is contained in a cycle C in G with $|E(C)| = \ell_G(e_0)$. Since $|E(C)| = \ell_G(e_0) < \ell_{G'}(e_0)$, C is not in G' and thus $e_1 \in E(C)$, implying that $\ell_G(e_1) \leq |E(C)|$. Hence

$$\ell_G(e_1) \leq |E(C)| = \ell_G(e_0) < \ell_{G'}(e_0) \leq \ell(G', T) \leq \ell_G(e_1),$$

a contradiction. Hence $\ell_G(e_0) = \ell_{G'}(e_0)$ is odd. Obviously, $\ell_G(e_0) = \ell_{G'}(e_0) \leq \ell(G', T) \leq \ell(G, T)$.

Hence the result holds. □

Remark: From the proof of Proposition 11, for any $(G, T) \in \mathcal{GT}$, G can be obtained from T by adding a sequence of edges. Actually, G is the last graph G_k in a sequence of graphs $G_0, G_1, G_2, \cdots, G_k$, where $k = |E(G)| - |V(G)| + 1$, $G_0 = T$ and each graph G_{i+1}, where $0 \leq i \leq k-1$, can be obtained from G_i by adding a new edge joining two nonadjacent vertices u and v in G_i in which there is a shortest (u, v)-path P such that $|E(P)| \geq \ell_{G_i}(e) - 1$ for each $e \in E(G_i) \setminus E(T)$ and $|E(P)| > \ell_{G_i}(e) - 1$ for each $e \in E(P) \setminus E(T)$.

4.2 Proof of Theorem 4

We are now going to prove Theorem 4.

Proof of Theorem 4: Let $G \in \mathcal{G}$ and $n = |V(G)|$. The result is trivial for $n = 1$. Now assume that $n \geq 2$. By definition, $(G, T) \in \mathcal{GT}$ for some spanning tree T of G. Thus, for each $e \in E(G) \setminus E(T)$, $\ell_G(e)$ is odd and e is contained in a cycle C of length $\ell_G(e)$ with the property that $\ell_G(e') < \ell_G(e)$ holds for each $e' \in E(C) \setminus (E(T) \cup \{e\})$.

Let $\mathcal{H} = (L, H)$ be any m-fold cover of G such that $H \ncong H_0(G, m)$. As T is a spanning tree of G, by Proposition 21 in [14], we may assume that $L(v) = \{(v, j) : j \in [\![m]\!]\}$ for each $v \in V(G)$ and $E_H(L(u), L(v)) \subseteq \{(u, j)(v, j) : j \in [\![m]\!]\}$ for each $uv \in E(T)$. Note that relabeling vertices in $L(u)$ for any $u \in V(G)$ does not affect the condition that $H \ncong H_0(G, m)$.

If $E_H(L(u), L(v)) \subseteq \{(u, j)(v, j) : j \in [\![m]\!]\}$ holds for each $uv \in E(G) \setminus E(T)$, then $H \ncong H_0(G, m)$ implies that H is a proper spanning subgraph of $H_0(G, m)$. Without loss of generality, assume that $(u, 1)(v, 1) \notin E_H(L(u), L(v))$ for some edge $uv \in E(G)$. Then, for $m \geq n - 2$,

$$P_{DP}(G, \mathcal{H}) - P(G, m) \geq P_{DP}(G - \{u, v\}, \mathcal{H}') > 0,$$

where $\mathcal{H}' = (L', H')$ is the $(m - 1)$-fold cover of $G - \{u, v\}$, $L'(w) = L(w) \setminus \{(w, 1)\}$ for all $w \in V(G) \setminus \{u, v\}$ and $H' = H[\cup_{w \in V(G) \setminus \{u,v\}} L'(w)]$. Thus, the result holds in this case.

Now assume that $E_H(L(u), L(v)) \nsubseteq \{(u, j)(v, j) : j \in [\![m]\!]\}$ for some $uv \in E(G) \setminus E(T)$. By definition, adding any possible edge to H does not increase the value of $P_{DP}(G, \mathcal{H})$. Thus, we can assume that $|E_H(L(u), L(v))| = m$ for each edge $uv \in E(G)$, and in particular, $E_H(L(u), L(v)) = \{(u, j)(v, j) : j \in [\![m]\!]\}$ for each $uv \in E(T)$.

For each $e = uv \in E(G)$, let $X_e = E_H(L(u), L(v)) \setminus \{(u, j)(v, j) : j \in [\![m]\!]\}$. As $|E_H(L(u), L(v))| = m$, $X_e = \emptyset$ if and only if $E_H(L(u), L(v)) = \{(u, j)(v, j) : j \in [\![m]\!]\}$. By the assumption above, $X_e = \emptyset$ for each $e \in E(T)$, but $X_e \neq \emptyset$ for some edge $e \in E(G) \setminus E(T)$. For $s \geq 3$, let

$$\mathscr{X}_s = \bigcup_{\substack{e \in E(G) \setminus E(T) \\ \ell_G(e) = s}} X_e. \tag{15}$$

By the given condition, $\ell_G(e) \geq 3$ is odd for each $e \in E(G) \setminus E(T)$, implying that $\mathscr{X}_s = \emptyset$ for each even $s \geq 4$. Now assume that r is the minimum integer such that $\mathscr{X}_r \neq \emptyset$. So $r \geq 3$ and r is odd. We will prove Theorem 4 by an approach similar to the proof of Theorem 7 in [15].

We first find an expression for $P_{DP}(G, \mathcal{H})$ which is similar to (1) for $P(G, m)$. Let $\mathscr{S}$ be the set of subsets S of $V(H)$ with $|S \cap L(v)| = 1$ for each $v \in V(G)$. For each edge $e = uv \in E(G)$, let $\mathscr{S}_e$ be the set of $S \in \mathscr{S}$ such that the two vertices in $S \cap (L(u) \cup L(v))$ are adjacent in H. For each $A \subseteq E(G)$, let

$$\mathscr{S}_A = \bigcap_{e \in A} \mathscr{S}_e. \tag{16}$$

As $P_{DP}(G, \mathcal{H}) = |\mathscr{S}| - | \cup_{e \in E(G)} \mathscr{S}_e|$, by the inclusion-exclusion principle, we have

$$P_{DP}(G, \mathcal{H}) = \sum_{A \subseteq E(G)} (-1)^{|A|} |\mathscr{S}_A|. \tag{17}$$

For each $U \subseteq V(G)$, let $\mathscr{S}|_U$ be the set of subsets S of $V(H)$ such that $|S \cap L(v)| = 1$ for each $v \in U$ and $S \cap L(v) = \emptyset$ for each $v \in V(G) \setminus U$. For any subgraph G_0 of G and $S \in \mathscr{S}|_{V(G_0)}$, let $H[S]|_{G_0}$ denote the spanning subgraph of $H[S]$ with edge set $\{(u, j_1)(v, j_2) \in E(H) : uv \in E(G_0), u, v \in V(G_0), (u, j_1), (v, j_2) \in S\}$. Equivalently, $H[S]|_{G_0}$ can be obtained from $H[S]$ by deleting all those edges $(u, j_1)(v, j_2)$ in $H[S]$ with $uv \notin E(G_0)$. Clearly, $H[S]|_{G_0}$ is $H[S]$ when G_0 is a subgraph of G induced by $V(G_0)$. For any $S \in \mathscr{S}|_{V(G_0)}$, $|E(H[S]|_{G_0})| \leq |E(G_0)|$ holds, and the following statements are equivalent:

(a) $H[S]|_{G_0} \cong G_0$;

(b) $|E(H[S]|_{G_0})| = |E(G_0)|$; and

(c) for each $uv \in E(G_0)$, the two vertices in $S \cap (L(u) \cup L(v))$ are adjacent in H.

Let $\mathscr{H}(G_0)$ be the set of subgraphs $H[S]|_{G_0}$ of H, where $S \in \mathscr{S}|_{V(G_0)}$, such that $H[S]|_{G_0} \cong G_0$.

Recall that for $A \subseteq E(G)$, $G\langle A \rangle$ is the spanning subgraph of G with edge set A, and $c(A)$ is the number of components of $G\langle A \rangle$. By the definition of $\mathscr{S}_A$, the following claim holds.

Claim 1. *For any $A \subseteq E(G)$, if $G_1, G_2, \cdots, G_{c(A)}$ are the components of $G\langle A \rangle$, then*

$$|\mathscr{S}_A| = \prod_{i=1}^{c(A)} |\mathscr{H}(G_i)|.$$

Claim 2. *Let G_0 be a connected subgraph of G. If $H[S_1]|_{G_0}, H[S_2]|_{G_0} \in \mathscr{H}(G_0)$, where $S_1, S_2 \in \mathscr{S}|_{V(G_0)}$, then either $S_1 = S_2$ or $S_1 \cap S_2 = \emptyset$. Hence $|\mathscr{H}(G_0)| \leq m$, where the equality holds if $X_e = \emptyset$ holds for each edge $e \in E(G_0)$.*

Proof. Suppose that $H[S_1]|_{G_0}, H[S_2]|_{G_0} \in \mathscr{H}(G_0)$. Then, $H[S_1]|_{G_0} \cong H[S_2]|_{G_0} \cong G_0$, implying that whenever $uv \in E(G_0)$, the two vertices in $S_i \cap (L(u) \cup L(v))$ are adjacent in H for $i = 1, 2$. Let uv be any edge in G_0. As $E_H(L(u), L(v))$ is a matching of H of size m, each vertex in $L(u)$ is only adjacent to one vertex in $L(v)$. If $H[S_1]|_{G_0}$ and $H[S_2]|_{G_0}$ have a common vertex in $L(u)$, then $H[S_1]|_{G_0}$ and $H[S_2]|_{G_0}$ must have a common vertex in $L(v)$. As G_0 is connected, we conclude that either $S_1 \cap S_2 = \emptyset$ or $S_1 = S_2$. Thus, $|\mathscr{H}(G_0)| \le m$ holds.

If $X_e = \emptyset$ holds for each edge $e \in E(G_0)$, then $H[S_j]|_{G_0} \in \mathscr{H}(G_0)$ for each $j \in [\![m]\!]$, where $S_j = \{(u, j) : u \in V(G_0)\}$. Thus, $|\mathscr{H}(G_0)| = m$ and Claim 2 holds. ♮

Claim 3. *Let G_0 be a connected subgraph of G. If $X_e = \emptyset$ holds for each $e \in E(G_0)$ that is not a bridge of G_0, then $|\mathscr{H}(G_0)| = m$.*

Proof. Assume that $X_e = \emptyset$ holds for each $e \in E(G_0)$ that is not a bridge of G_0. Let B be any block of G_0. If B is trivial (i.e., it consists of a bridge $e = uv$ of G_0 only), then, it is clear that $\mathscr{H}(B)$ has exactly m members which correspond to the m edges in $E_H(L(u), L(v))$. If B is an non-trivial block of G_0, we have $X_e = \emptyset$ for each $e \in E(B)$, and $\mathscr{H}(B)$ has exactly m members $H[S_j]|_B$ for $j \in [\![m]\!]$, where $S_j = \{(v, j) : v \in V(B)\}$. Thus, $|\mathscr{H}(G_0)| = m$ if G_0 has only one block.

Suppose that G_0 has at least two blocks and B_0 is a block of G_0 which has only one vertex u shared by other blocks of G_0. Let G' denote $G_0 - (V(B_0) \setminus \{u\})$. Assume that both $\mathscr{H}(G')$ and $\mathscr{H}(B_0)$ have exactly m members. Each member $H[S']|_{G'}$ of $\mathscr{H}(G')$ can be extended to exactly one member of $\mathscr{H}(G_0)$ by combining $H[S']|_{G'}$ with the member in $\mathscr{H}(B_0)$ which shares a vertex in $L(u)$ with $H[S']|_{G'}$. Hence $|\mathscr{H}(G_0)| = m$.

Claim 3 holds. ♮

The next claim follows from Claims 1, 2 and 3 directly.

Claim 4. *For each $A \subseteq E(G)$, $|\mathscr{S}_A| \le m^{c(A)}$ holds. If $X_e = \emptyset$ holds for each $e \in A$ that is not a bridge of $G\langle A\rangle$, then $|\mathscr{S}_A| = m^{c(A)}$.*

By Claim 4, the next claim follows.

Claim 5. *For any $A \subseteq E(G)$, if $|A|$ is odd, $(-1)^{|A|}\left(|\mathscr{S}_A| - m^{c(A)}\right) = m^{c(A)} - |\mathscr{S}_A| \ge 0$.*

Let $\mathscr{E}$ be the set of subsets A of $E(G)$ such that $X_e \neq \emptyset$ holds for some $e \in A$ that is not a bridge of $G\langle A\rangle$. Note that such an edge e may be not unique. By (1), (17) and Claim 4, we have

$$P_{DP}(G, \mathcal{H}) - P(G, m) = \sum_{A \in \mathscr{E}} (-1)^{|A|}\left(|\mathscr{S}_A| - m^{c(A)}\right). \tag{18}$$

The following claim presents some properties of members in $\mathscr{E}$.

Claim 6. *For each $A \in \mathscr{E}$, $G\langle A\rangle$ has a component G_1 and an edge e in some cycle of G_1 with $X_e \neq \emptyset$. Furthermore, $|V(G_1)| \ge r$ and $c(A) \le n - r + 1$, and $|A| = r$ whenever $c(A) = n - r + 1$.*

Proof. As $A \in \mathscr{E}$, $G\langle A\rangle$ has an edge e that is not a bridge of $G\langle A\rangle$ such that $X_e \neq \emptyset$. Let G_1 be the component of $G\langle A\rangle$ containing e. As $X_e \neq \emptyset$, we have $\ell_G(e) \geq r$. Thus, each cycle in G_1 containing e has at least r edges, implying that $|V(G_1)| \geq r$, and hence $c(A) \leq n - r + 1$.

Assume that $c(A) = n - r + 1$. Then $|V(G_1)| \geq r$ implies that $|V(G_1)| = r$ and all other components of $G\langle A\rangle$ are isolated vertices. As e is in a cycle C of length r in G_1 and each cycle containing e is of length at least r, $|V(G_1)| = r$ implies that $G_1 \cong C$ and $|A| = r$.

Claim 6 holds. $\natural$

Assume that $\{e_1, e_2, \cdots, e_s\}$ is the set of edges in $E(G) \setminus E(T)$ with $\ell_G(e_i) = r$. By the given condition, for each $i \in [\![s]\!]$, e_i is contained in a cycle, denoted by C_i, such that $|V(C_i)| = r$ and $\ell_G(e') < r$ for each $e' \in E(C_i) \setminus (E(T) \cup \{e_i\})$. Thus, $E(C_i) \cap \{e_j : j \in [\![s]\!]\} = \{e_i\}$ for each $i \in [\![s]\!]$, implying that $C_1, C_2, \cdots, C_s$ are pairwise distinct.

Claim 7. *For each $i \in [\![s]\!]$, $|\mathscr{H}(C_i)| = m - |X_{e_i}|$.*

Proof. Without loss of generality, let $V(C_i) = \{v_1, v_2, \cdots, v_r\}$ and let $v_1v_2\cdots v_r$ be the path $C_i - e_i$ in G. Obviously, e_i is the edge v_1v_r. By the definition of r, $X_{e'} = \emptyset$ holds for each $e' \in E(G) \setminus E(T)$ with $\ell_G(e') < r$. By the given condition on C_i, $X_{e'} = \emptyset$ holds for each edge e' in the path $v_1v_2\cdots v_r$, implying that the subgraph obtained from $H[S]$, where $S = \{(v_q, j) : q \in [r], j \in [\![m]\!]\}$, by removing all edges in $E_H(L(v_1), L(v_r))$, consists of m disjoint paths $(v_1, j)(v_2, j)\cdots(v_r, j)$ for $j = 1, 2, \cdots, m$. Assume that

$$E_H(L(v_1), L(v_r)) \setminus X_{e_i} = \{(v_1, j)(v_r, j) : 1 \leq j \leq m - |X_{e_i}|\}. \tag{19}$$

Then $(v_1, j)(v_r, j) \notin E(H)$ for each j with $m - |X_{e_i}| < j \leq m$. Let $S_j = \{(v_q, j) : q \in [r]\}$ for each $j \in [\![m]\!]$. Clearly, $H[S_j]|_{C_i} \cong C_i$ if and only if $1 \leq j \leq m - |X_{e_i}|$. On the other hand, for any $S' \in \mathscr{S}|_{V(C_i)}$, if $H[S']|_{C_i} \cong C_i$, then $H[S']|_{C_i}$ must contain a path $(v_1, j)(v_2, j)\cdots(v_r, j)$ for some $j \in [\![m]\!]$, implying that $S' = S_j$ for some $j \in [\![m]\!]$. Thus, $|\mathscr{H}(C_i)| = m - |X_{e_i}|$. $\natural$

Now we are going to apply Claims 5 and 6 to prove the next claim.

Claim 8. *The following result holds:*

$$\sum_{\substack{A \in \mathscr{E} \\ c(A) = n-r+1}} (-1)^{|A|} \left(|\mathscr{S}_A| - m^{c(A)}\right) \geq |\mathscr{X}_r| m^{n-r}. \tag{20}$$

Proof. By Claim 6, $|A| = r$ for each $A \in \mathscr{E}$ with $c(A) = n - r + 1$. As r is odd, by Claim 5, for any $\mathscr{E}_0 \subseteq \{A \in \mathscr{E} : c(A) = n - r + 1\}$, we have

$$\sum_{\substack{A \in \mathscr{E} \\ c(A)=n-r+1}} (-1)^{|A|} \left(|\mathscr{S}_A| - m^{c(A)} \right) \geq \sum_{A \in \mathscr{E}_0} \left(m^{n-r+1} - |\mathscr{S}_A| \right). \tag{21}$$

For each $i \in [\![s]\!]$, $G\langle E(C_i)\rangle$ consists of exactly $n-r+1$ components, i.e., C_i and $n-r$ isolated vertices in $V(G) \setminus V(C_i)$. By Claim 7, $|\mathscr{H}(C_i)| = m - |X_{e_i}|$. Thus, by Claim 1,

$$|\mathscr{S}_{E(C_i)}| = |\mathscr{H}(C_i)| m^{n-r} = (m - |X_{e_i}|) m^{n-r} = m^{n-r+1} - |X_{e_i}| m^{n-r}. \tag{22}$$

Let $\mathscr{E}_0 = \{E(C_i) : i \in [\![s]\!]\}$. By (21) and (22),

$$\sum_{\substack{A \in \mathscr{E} \\ c(A)=n-r+1}} (-1)^{|A|} \left(|\mathscr{S}_A| - m^{c(A)} \right) \geq \sum_{i=1}^{s} \left(m^{n-r+1} - |\mathscr{S}_{E(C_i)}| \right) = \sum_{i=1}^{s} |X_{e_i}| m^{n-r} = |\mathscr{X}_r| m^{n-r}. \tag{23}$$

Claim 8 holds. ♮

Claim 9. *For any subgraph G_1 of G, if $\ell_G(e) \leq r$ for each edge $e \in E(G_1)$, then $|\mathscr{H}(G_1)| \geq m - 2|\mathscr{X}_r|$.*

Proof. For each $j \in [\![m]\!]$, let $S_j = \{(u, j) : u \in V(G_1)\}$ and $Q_j = H[S_j]|_{G_1}$. By the definition of $H[S_j]|_{G_1}$, $Q_j \in \mathscr{H}(G_1)$ if and only if $(u, j)(v, j) \in E(H)$ for each $uv \in E(G_1)$.

Let $S = \cup_{j \in [\![m]\!]} S_j$, and let $\psi : S \to \{0, 1\}$ be the mapping defined below:

$$\psi((u, j)) = \begin{cases} 1, & \text{if } (u, j)(v, j') \in E(H) \text{ for some } v \in N_{G_1}(u) \text{ and } j' \neq j; \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

If $\psi((u, j)) = 1$, by definition, (u, j) is one end of some edge $(u, j)(v, j')$ of X_e, where $e = uv \in E(G_1)$. Thus,

$$\begin{aligned} \sum_{(u,j) \in S} \psi((u, j)) &\leq \sum_{e \in E(G_1)} \sum_{(u,j)(v,j') \in X_e} (\psi((u, j)) + \psi((v, j'))) \\ &= 2 \sum_{e \in E(G_1)} |X_e| \\ &\leq 2|\mathscr{X}_r|, \end{aligned} \tag{25}$$

where the last inequality follows from the facts that for each $e \in E(G_1)$, $\ell_G(e) \leq r$ holds, and $\ell_G(e) < r$ implies that $X_e = \emptyset$.

By the definition of ψ, $Q_j \ncong G_1$ if and only if $\psi((u,j)) = 1$ for some $u \in V(G_1)$. Then, by (25), there are at most $2|\mathscr{X}_r|$ numbers $j \in [\![m]\!]$ such that $Q_j \ncong G_1$, implying that

$$|\mathscr{H}(G_1)| \geq m - 2|\mathscr{X}_r|. \tag{26}$$

Thus, Claim 9 holds. ♮

Claim 10. *For any $A \in \mathscr{E}$ with $c(A) = n - r$, we have $|\mathscr{S}_A| \geq (m - 2|\mathscr{X}_r|)m^{m-r-1}$.*

Proof. Let $A \in \mathscr{E}$ with $c(A) = n - r$. By Claim 6, $G\langle A\rangle$ has a component G_1 with $|V(G_1)| \geq r$. Let $G_2, \cdots, G_{n-r}$ be the components of $G\langle A\rangle$ different from G_1 with $|V(G_2)| \geq \cdots \geq |V(G_{n-r})|$. As $c(A) = n - r$, one of the two cases below happens:

(i). $|V(G_1)| = r$, $|V(G_2)| = 2$ and $|V(G_i)| = 1$ for all $3 \leq i \leq n - r$, or

(ii). $|V(G_1)| = r + 1$ and $|V(G_i)| = 1$ for all $2 \leq i \leq n - r$.

In both Cases (i) and (ii) above, by Claim 3, $|\mathscr{H}(G_i)| = m$ holds for all $i = 2, 3, \cdots, n - r$. By Claim 1, it remains to show that $|\mathscr{H}(G_1)| \geq m - 2|\mathscr{X}_r|$.

In both cases above, by Claim 6, there is an edge e with $X_e \neq \emptyset$ which is in some cycle of G_1. Such an edge may be not unique. As $X_e \neq \emptyset$, we have $\ell_G(e) \geq r$. Thus, each cycle in G_1 containing e must be of length at least r. In Case (i), G_1 can only be a cycle of length r. In Case (ii), it can be verified that G_1 is one of the graphs in Figure 3.

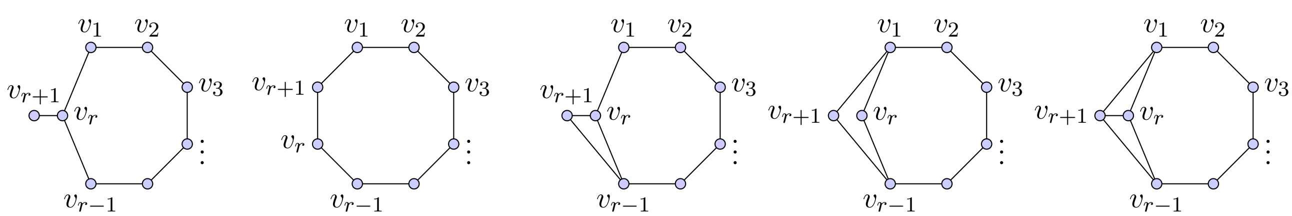


Figure 3: Possible structures of G_1 when $|V(G_1)| = r + 1$

As each cycle in G_1 is of length at most $r + 1$, for each edge e' in cycles of G_1, we have $\ell_G(e') \leq r + 1$. As $\ell_G(e')$ is odd, we have $\ell_G(e') \leq r$ for such edges e'. Thus, if G_1 contains an edge e' with $\ell_G(e') \geq r + 2$, then e' must be a bridge of G_1.

If G_1 has no bridge, then $\ell_G(e) \leq r$ for each $e \in E(G_1)$. By Claim 9, $\mathscr{H}(G_1) \geq m - 2|\mathscr{X}_r|$.

If G_1 has bridges, then G_1 is the graph in Figure 3 (a), where $G_1 - v_{r+1}$ is a cycle. By Claim 9 again, $\mathscr{H}(G_1 - v_{r+1}) \geq m - 2|\mathscr{X}_r|$. Clearly, each member of $\mathscr{H}(G_1 - v_{r+1})$ can be extended to a member of $\mathscr{H}(G_1)$, even when $X_{v_r v_{r+1}} \neq \emptyset$. Thus, Claim 10 also holds in this case.

Claim 10 is proved. ♮

For any $k \in [\![n-r]\!]$, let ϕ_k be the number of elements A of $\mathscr{E}$ such that $c(A) = k$ and $|A|$ is even.

Claim 11. *The following inequality holds:*

$$\sum_{\substack{A\in\mathscr{E}\\ c(A)=n-r}} (-1)^{|A|}\left(|\mathscr{S}_A| - m^{c(A)}\right) \geq -2\phi_{n-r}|\mathscr{X}_r|m^{n-r-1}. \tag{27}$$

Proof. By Claim 5,

$$\sum_{\substack{A\in\mathscr{E}\\ c(A)=n-r}} (-1)^{|A|}\left(|\mathscr{S}_A| - m^{c(A)}\right) \geq \sum_{\substack{A\in\mathscr{E},c(A)=n-r\\ |A| \text{ is even}}} \left(|\mathscr{S}_A| - m^{c(A)}\right). \tag{28}$$

For each $A \in \mathscr{E}$ with $c(A) = n - r$, by Claim 10,

$$|\mathscr{S}_A| - m^{c(A)} \geq (m - 2|\mathscr{X}_r|)m^{n-r-1} - m^{n-r} = -2|\mathscr{X}_r|m^{n-r-1}. \tag{29}$$

Then Claim 11 follows from the definition of ϕ_{n-r}. ♮

Claim 12. *For each $k \in [\![n-r-1]\!]$, we have*

$$\sum_{\substack{A\in\mathscr{E}\\ c(A)=k}} (-1)^{|A|}\left(|\mathscr{S}_A| - m^{c(A)}\right) \geq -\phi_k m^k. \tag{30}$$

Proof. For each $A \in \mathscr{E}$ with $c(A) = k$, if $|A|$ is even,

$$(-1)^{|A|}\left(|\mathscr{S}_A| - m^{c(A)}\right) = |\mathscr{S}_A| - m^{c(A)} \geq -m^k. \tag{31}$$

Thus Claim 12 follows from Claim 5 and the definition of ϕ_k. ♮

Let ϕ'_k be the number of subsets $A \subseteq E(G)$ such that $c(A) = k$, $G\langle A\rangle$ is not a forest and $|A|$ is even. Obviously, $\phi'_k \geq \phi_k$. By the expression of (18) and Claims 6, 8, 11 and 12,

$$P_{DP}(G,\mathcal{H}) - P(G,m) \ \ \geq \ \ |\mathscr{X}_r|m^{n-r} - 2\phi_{n-r}|\mathscr{X}_r|m^{n-r-1} - \sum_{k=1}^{n-r-1} \phi_k m^k$$

$$
\begin{aligned}
&\geq m^{n-r} - 2\phi_{n-r}m^{n-r-1} - \sum_{k=1}^{n-r-1} \phi_k m^k \\
&\geq m^{n-r} - 2\phi'_{n-r}m^{n-r-1} - \sum_{k=1}^{n-r-1} \phi'_k m^k, \qquad (32)
\end{aligned}
$$

where the second inequality holds when $m \geq 2\phi_{n-r}$. As ϕ'_k is independent of the value of m, by (32), there must be a number $M_r \in \mathbb{N}$ such that $P_{DP}(G, \mathcal{H}) - P(G, m) > 0$ for all $m \geq M_r$.

Let $M = \max\{M_r : 3 \leq r \leq n, r \text{ is odd}\}$. Then, we conclude that for any $m \geq M$ and any m-fold cover $\mathcal{H} = (L, H)$ of G, if $H \not\cong H_0(G, m)$, then $P_{DP}(G, \mathcal{H}) - P(G, m) > 0$ holds.

Hence Theorem 4 holds. □

We end this section with an application of Theorem 4 to the generalized θ-graphs. For any k numbers $a_1, a_2, \cdots, a_k \in \mathbb{N}$, where $k \geq 2$, let $G = \Theta_{a_1,a_2,\cdots,a_k}$ denote the generalized θ-graph obtained by connecting two distinct vertices with k internally disjoint paths of lengths $a_1, a_2, \cdots, a_k$ respectively.

Assume that $a_1 \leq a_2 \leq \cdots \leq a_k$ and $a_1 + a_2 \geq 3$. Halberg, Kaul, Liu, Mudrock, Shin and Thomason [12] showed that $P_{DP}(\Theta_{a_1,a_2,\cdots,a_k}) \approx P(\Theta_{a_1,a_2,\cdots,a_k})$ if $a_1 + a_i$ is odd for each $i \in [\![k]\!] \setminus \{1\}$, and $P_{DP}(\Theta_{a_1,a_2,\cdots,a_k}) < P(\Theta_{a_1,a_2,\cdots,a_k})$ otherwise.

In the case that $a_1 + a_i$ is odd for each $i \in [\![k]\!] \setminus \{1\}$, $\Theta_{a_1,a_2,\cdots,a_k}$ belongs to the set $\mathcal{G}_0$, and thus $\Theta_{a_1,a_2,\cdots,a_k} \in \mathcal{DP}^*$ by Theorem 4, implying that $P_{DP}(\Theta_{a_1,a_2,\cdots,a_k}) \approx P(\Theta_{a_1,a_2,\cdots,a_k})$.

5 Proof of Theorem 5

For a chordal graph G, $P_{DP}(G, m) = P(G, m)$ for all $m \in \mathbb{N}$ (see [14]), and thus, $P_{DP}(G) \approx P(G)$ holds. In the following, we first generalize this conclusion to some non-chordal graphs containing simplicial vertices.

Proposition 12. *Let u be a simplicial vertex of G. For each $m \in \mathbb{N}$ with $m \geq d(u) + 1$, if $P_{DP}(G - u, m) = P(G - u, m)$, then $P_{DP}(G, m) = P(G, m)$.*

Proof. Assume that $P_{DP}(G - u, m) = P(G - u, m)$. For any m-fold cover $\mathcal{H} = (L, H)$ of G,

$$
P_{DP}(G, \mathcal{H}) \geq (m - d(u))P_{DP}(G - u, \mathcal{H}'_u) \geq (m - d(u))P_{DP}(G - u, m) = (m - d(u))P(G - u, m), \qquad (33)
$$

where $\mathcal{H}'_u = (L', H')$ is the m-fold cover of $G - u$ with $L'(w) = L(w)$ for each $w \in V(G) \setminus \{u\}$

and $H' = H - L(u)$. Thus, $P_{DP}(G, m) \geq (m - d(u))P(G - u, m) = P(G, m)$ by (8). On the other hand, $P_{DP}(G, m) \leq P(G, m)$. Thus, the result follows. □

The first part of Theorem 5 follows from Proposition 12 directly. In order to prove the second part of Theorem 5, we need to introduce some preliminary results.

For any cover $\mathcal{H} = (L, H)$ of G, let $\mathcal{I}(H)$ denote the set of independent sets I in H with $|I| = |V(G)|$. Thus, $P_{DP}(G, \mathcal{H}) = |\mathcal{I}(H)|$. The *coloring number* of G, denoted by $col(G)$, is the smallest integer d for which there exists an ordering, $v_1, v_2, \cdots, v_n$ of the elements in $V(G)$, where $n = |V(G)|$, such that $|N_G(v_i) \cap \{v_1, v_2, \cdots, v_{i-1}\}| < d$ for each $i \in [\![n]\!]$. Obviously, $\chi_{DP}(G) \leq col(G) \leq n$. If $|L(v)| \geq col(G)$ for all $v \in V(G)$, then $\mathcal{I}(H) \neq \emptyset$.

The following fundamental property is important for the study of DP coloring.

Proposition 13. *Let $\mathcal{H} = (L, H)$ be a cover of G with $|L(v)| \geq |V(G)|$ for each $v \in V(G)$. Then, each independent set A of H is a subset of some set I in $\mathcal{I}(H)$.*

Proof. If $A = \emptyset$, then the conclusion follows from the the fact that $|V(G)| \geq col(G)$.

Now assume that $A = \{(v_i, \pi_i) : i \in [\![k]\!]\}$, where $k \geq 1$. Clearly, $v_1, v_2, \cdots, v_k$ are pairwise distinct. Let $\mathcal{H}' = (L', H')$ be the cover of the subgraph $G' = G - \{v_i : i \in [\![k]\!]\}$, where $L'(v) = L(v) \setminus N_H(A)$ for each $v \in V(G')$ and H' is the subgraph of H induced by $\bigcup_{v \in V(G')} L'(v)$.

Observe that $|L'(v)| \geq |L(v)| - k \geq |V(G')|$ for each $v \in V(G')$. By the conclusion for $A = \emptyset$, there exists $I' \in \mathcal{I}(H')$, implying that $I = A \cup I' \in \mathcal{I}(H)$. □

By Proposition 13, the following corollary is obtained.

Corollary 14. *For any cover $\mathcal{H} = (L, H)$ of G with $|L(v)| \geq |V(G)|$ for each $v \in V(G)$, if $\mathcal{H}' = (L, H')$ is a cover of G, where H' is obtained from H by removing any edge in some set $E_H(L(v_1), L(v_2))$, where $v_1 \neq v_2$, then $P_{DP}(G, \mathcal{H}') > P_{DP}(G, \mathcal{H})$.*

For any $u \in V(G)$ and an m-fold cover $\mathcal{H} = (L, H)$ of G, let $\mathcal{H}'_u = (L', H')$ be the cover of $G - u$, where $H' = H - L(u)$ and $L'(v) = L(v)$ for each $v \in V(G) \setminus \{u\}$. For any $I' \in \mathcal{I}(H')$, let

$$\mathcal{I}_H(I') = \{I' \cup \{(u, i)\} \in \mathcal{I}(H) : (u, i) \in L(u)\}.$$

Obviously, for $m \geq d(u)$ and $I' \in \mathcal{I}(H')$, $|\mathcal{I}_H(I')| \geq (m - d(u))$ holds, implying that for $m > d(u)$,

$$\begin{aligned} P_{DP}(G, \mathcal{H}) &= |\mathcal{I}(H)| = \sum_{I' \in \mathcal{I}(H')} |\mathcal{I}_H(I')| \geq \sum_{I' \in \mathcal{I}(H')} (m - d(u)) \\ &= (m - d(u))|\mathcal{I}(H')| = (m - d(u))P_{DP}(G - u, \mathcal{H}'_u), \end{aligned} \quad (34)$$

where $P_{DP}(G,\mathcal{H}) > (m-d(u))P_{DP}(G-u,\mathcal{H}'_u)$ if $|\mathcal{I}_H(I')| > m-d(u)$ for some $I' \in \mathcal{I}(H')$.

Proposition 15. *Let $\mathcal{H}=(L,H)$ be an m-fold cover of G, where $m \geq |V(G)|$, and $u \in V(G)$. Then $P_{DP}(G,\mathcal{H}) \geq (m-d(u))P_{DP}(G-u,\mathcal{H}'_u)$, where the inequality is strict under each of the following conditions:*

(i). $|E_H(L(u),L(v))| \leq m-1$ *for some* $v \in N_G(u)$; *or*

(ii). $N_H((u,i)) \setminus L(u)$ *is not a clique of* H *for some vertex* $(u,i) \in L(u)$.

Proof. By (34), $P_{DP}(G,\mathcal{H}) \geq (m-d(u))P_{DP}(G-u,\mathcal{H}'_u)$ holds. We need to prove that $P_{DP}(G,\mathcal{H}) > (m-d(u))P_{DP}(G-u,\mathcal{H}'_u)$ if either condition (i) or (ii) is satisfied.

Assume that condition (i) holds, i.e., $|E_H(L(u),L(v))| \leq m-1$ for some $v \in N_G(u)$. Then, there exists a m-fold cover $\mathcal{H}^*=(L,H^*)$ of G, where H^* is obtained from H by adding a new edge joining some vertex in $L(u)$ to some vertex in $L(v)$. By Corollary 14,

$$P_{DP}(G,\mathcal{H}) > P_{DP}(G,\mathcal{H}^*) \geq (m-d(u))P_{DP}(G-u,\mathcal{H}'_u). \tag{35}$$

Now assume that condition (ii) holds. Without loss of generality, assume that $N_H((u,1)) \setminus L(u)$ is not a clique of H. Let (v_1,i_1) and (v_2,i_2) be non-adjacent vertices in $N_H((u,1)) \setminus L(u)$. Clearly, $v_1 \neq v_2$.

As $\mathcal{H}'_u=(L',H')$ is an m-fold cover of $G-u$ and $m \geq |V(G)|$, by Proposition 13, there exists $I' \in \mathcal{I}(H')$ such that $\{(v_1,i_1),(v_2,i_2)\} \subseteq I'$.

Note that $|I' \cap L(v)| = 1$ for each $v \in N_G(u)$ and $\{(v_1,i_1),(v_2,i_2)\} \subseteq I' \cap N_H((u,1))$. Assume that $N_G(u)=\{v_1,v_2,\cdots,v_r\}$, where $r=d(u)$, and $I' \cap L(v_j) = \{(v_j,\pi_j)\}$ for all $j \in [r]$. Then

$$\begin{aligned}\left|L(u) \cap \bigcup_{j\in[r]} N_H((v_j,\pi_j))\right| &\leq \left|L(u) \cap \bigcup_{j\in[\![2]\!]} N_H((v_j,\pi_j))\right| + \left|L(u) \cap \bigcup_{3\leq j\leq r} N_H((v_j,\pi_j))\right| \\ &\leq |\{(u,1)\}| + (r-2) = d(u)-1, \end{aligned} \tag{36}$$

implying that

$$|\mathcal{I}_H(I')| = m - \left|L(u) \cap \bigcup_{j\in[r]} N_H((v_j,\pi_j))\right| \geq m-d(u)+1.$$

By (34), $P_{DP}(G,\mathcal{H}) > (m-d(u))P_{DP}(G-u,\mathcal{H}'_u)$ holds. The result is proven. □

We are now ready to prove Theorem 5 by applying (8) and Propositions 12 and 15.

Proof of Theorem 5: If $P_{DP}(G-u) \approx P(G-u)$, then $P_{DP}(G) \approx P(G)$ due to Proposition 12.

Now assume that $G-u \in \mathcal{DP}^*$. Then, there exists $M \in \mathbb{N}$ such that $P_{DP}(G-u, \mathcal{H}') > P(G-u, m)$ for each integer $m \geq M$ and every m-fold cover $\mathcal{H}' = (L', H')$ of $G-u$ with $H' \ncong H_0(G-u, m)$.

Let $\mathcal{H} = (L, H)$ be any m-fold cover of G such that $H \ncong H_0(G, m)$. We may assume that $L(v) = \{(v, i) : i \in [\![m]\!]\}$ for each $v \in V(G)$. If $|E_H(L(v_1), L(v_2))| < m$ for some edge $v_1v_2 \in E(G)$, then, by Corollary 14, $P_{DP}(G, \mathcal{H}) > P_{DP}(G, \mathcal{H}^*)$ for $m \geq |V(G)|$, where $\mathcal{H}^*$ is the m-fold cover (L, H^*) obtained from $\mathcal{H}$ by adding a new edge joining a vertex in $L(v_1)$ to a vertex in $L(v_2)$. Therefore, we can assume that $|E_H(L(v_1), L(v_2))| = m$ for each edge $v_1v_2 \in E(G)$ and $H \ncong H_0(G, m)$.

Consider the m-fold cover $\mathcal{H}'_u = (L', H')$ of $G-u$.

Case 1: $H' \ncong H_0(G-u, m)$.

By the assumption in the beginning of the proof, $P_{DP}(G-u, \mathcal{H}'_u) > P(G-u, m)$ for each integer $m \geq M$. By (8) and Proposition 15, for $m \geq \max\{M, |V(G)|\}$,

$$P_{DP}(G, \mathcal{H}) \geq (m - d(u))P_{DP}(G-u, \mathcal{H}') > (m - d(u))P(G-u, m) = P(G, m). \tag{37}$$

Case 2: $H' \cong H_0(G-u, m)$.

We can assume that $H' = H_0(G-u, m)$. Since $H \ncong H_0(G, m)$, there must be some vertex $(u, i) \in L(u)$ that is adjacent to two vertices (v_1, i_1) and (v_2, i_2) with $v_1 \neq v_2$ and $i_1 \neq i_2$. Since $H' = H_0(G-u, m)$ and $i_1 \neq i_2$, (v_1, i_1) and (v_2, i_2) are not adjacent in H, implying that $N_H((u, i)) \setminus L(u)$ is not a clique of H. By Proposition 15 again, $P_{DP}(G, \mathcal{H}) > (m - d(u))P(G, m)$ for $m \geq |V(G)|$.

Thus Theorem 5 holds. □

By Theorem 5, we have the following consequence, which generalizes the known conclusion that $P_{DP}(G) \approx P(G)$ holds for every chordal graph G.

Corollary 16. *Let G_1 and G_2 be vertex-disjoint graphs and $k \in \mathbb{N}$, where $k \leq \min\{\omega(G_i) : i = 1, 2\}$. Assume that G_1 is chordal and $G \in \mathscr{G}(G_1 \cup_k G_2)$. If $P_{DP}(G_2) \approx P(G_2)$, then $P_{DP}(G) \approx P(G)$; also, if $G_2 \in \mathcal{DP}^*$, then $G \in \mathcal{DP}^*$.*

Proof. As G_1 is chordal, there must be an ordering $v_1, v_2, \cdots, v_r$ of vertices in $V(G) \setminus V(G_2)$, where $r = |V(G_1)| - k$, such that v_i is a simplicial vertex in $G - \{v_j : j \in [\![i-1]\!]\}$ for each

$i \in [\![r]\!]$. Then, the result follows from Theorem 5. $\square$

We wonder if Corollary 16 holds without the condition that G_1 is chordal.

Problem 6. *For any vertex-disjoint graphs G_1 and G_2 and $k \in \mathbb{N}$, where $k \leq \min\{\omega(G_i) : i = 1, 2\}$, is it true that if $P_{DP}(G_i) \approx P(G_i)$ for $i = 1, 2$, then $P_{DP}(G) \approx P(G)$ for every graph $G \in \mathscr{G}(G_1 \cup_k G_2)$; also, if $G_1, G_2 \in \mathcal{DP}^*$, then $\mathscr{G}(G_1 \cup_k G_2) \subseteq \mathcal{DP}^*$?*

Acknowledgement

The authors would like to thank the referees for their very helpful comments and suggestions. The second author, Yan Yang, is supported by NSFC (No. 11971346).

References

[1] J. Becker, J. Hewitt, H. Kaul, M. Maxfield, J. A. Mudrock, D. Spivey, S. Thomason and T. Wagstrom, The DP color function of joins and vertex-gluings of graphs, https://arxiv.org/abs/2104.12268.

[2] A. Bernshteyn, A. Kostochka and X. Zhu, DP-colorings of graphs with high chromatic number, *Eur. J. Combin.* **65** (2017), 122–129.

[3] G.D. Birkhoff, A determinant formula for the number of ways of coloring a map, *Annal. Math.* **14** (1912), 42–46.

[4] G.D. Birkhoff and D.C. Lewis, Chromatic polynomials, *Trans. Amer. Math. Soc.* **60** (1946), 355–351.

[5] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, American Elsevier, New York, 1976.

[6] M. V. Bui, H. Kaul, M. Maxfield, J. A. Mudrock, P. Shin, and S. Thomason, Non-chromatic-adherence of the DP color function via generalized theta graphs, https://arxiv.org/abs/2110.04058.

[7] G.A. Dirac, On rigid circuit graphs, *Abh. Math. Sem. Univ. Hamburg* **25** (1961), 71-76.

[8] F.M. Dong, K.M. Koh and K.L. Teo, *Chromatic Polynomials and Chromaticity of Graphs*, World Scientific, Singapore, 2005.

[9] F.M. Dong and K.M. Koh, "Foundations of the chromatic polynomial," in *the Handbook on the Tutte Polynomial and Related Topics*, Jo Ellis-Monaghan and Iain Moffatt (ed.), pp 232–266, CRC press, 2021.

[10] Z. Dvořák and L. Postle, Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8, *J. Combin. Theory Ser. B* **129** (2018), 38–54.

[11] P. Erdős, A.L. Rubin and H. Taylor, Choosability in graphs, *Congr. Numer.* **26** (1979), 125–127.

[12] C. Halberg, H. Kaul, A. Liu, J. A. Mudrock, P. Shin, and S. Thomason, On polynomial representations of the DP color function: theta graphs and their generalizations, https://arxiv.org/abs/2012.12897.

[13] B. Jackson, Chromatic polynomials, L.W. Beineke, R.J. Wilson (Eds.), *Topics in Chromatic Graph Theory*, vol. 156, Cambridge University Press, 2015, pp. 56–72.

[14] H. Kaul and J. A. Mudrock, On the chromatic polynomial and counting DP-colorings of graphs, *Advances in Applied Mathematics* **123** (2021), article 103121. Also in https://arxiv.org/abs/1904.07697, 2019.

[15] J. A. Mudrock and S. Thomason, Answers to two questions on the DP color function, *Electron. J. Combin.* **28**(2) (2021), #P2.24.

[16] J.A. Noel, B.A. Reed and H. Wu, A proof of a conjecture of Ohba, *J. Graph Theory* **79** (2015), 86–102.

[17] R.C. Read, An introduction to chromatic polynomials, *J. Combin. Theory* **4** (1968), 52–71.

[18] R.C. Read and W.T. Tutte, Chromatic polynomials, in *Selected Topics in Graph Theory 3*, Academic Press (1988), 15–42.

[19] G. Royle, Recent results on chromatic and flow roots of graphs and matroids, in: *Surveys in combinatorics*, London Math. Soc. Lecture Note Ser., vol. 365, Cambridge Univ. Press, Cambridge, 2009, pp. 289–327.

[20] C. Thomassen, The chromatic polynomial and list colorings, *J. Combin. Theory Ser. B* **99** (2009), 474–479.

[21] V.G. Vizing, Coloring the vertices of a graph in prescribed colors, *Diskret. Analiz.* no. 29, *Metody Diskret. Anal. v Teorii Kodovi Skhem* **101** (1976), 3–10.

[22] W. Wang, J. Qian, Z. Yan, When does the list-coloring function of a graph equal its chromatic polynomial, *J. Combin. Theory Ser. B* **122** (2017), 543–549.

[23] H. Whitney, A logical expansion in mathematics, *Bull. Amer. Math. Soc.* **38** (1932), 572–579.

[24] A.A. Zykov, On some properties of linear complexes, (Russian) *Mat. Sbornik N.S.* **24** (66) (1949), 163–188.