A Metacognitive Approach to Support Heuristic Solution of Mathematical Problems

John TIONG Yeun Siew
Centre for Research in Pedagogy and Practice, National Institute of Education,
Nanyang Technological University, Singapore

John HEDBERG
Australian Centre for Educational Studies, Macquarie University, Australia

Luis Tirtasanjaya LIOE
Centre for Research in Pedagogy and Practice, National Institute of Education,
Nanyang Technological University, Singapore

Abstract

Singapore mathematics syllabuses identify eleven heuristics which are applicable to problem solving at the upper primary level (MOE, 2001a), and thirteen heuristics at lower secondary level (MOE, 2001b). This paper discusses some of the general characteristics of problem solving heuristics. This paper will also examines these heuristics, their functions and roles in mathematical problem solving and the relationships they have with each other. According to their different characteristics, how and when they can be used in mathematical problem solving, they can be classified into four categories: "representation heuristics", "simplification heuristics", "pathway heuristics", and "generic heuristics". Together they describe the process of mathematical problem solving.

Introduction

The Singapore mathematics syllabuses, developed by Curriculum Planning and Developing Division (CPDD), Ministry of Education Singapore (MOE), have identified 13 heuristics that are applicable to mathematical problem solving.

1. Act it out
2. Use a diagram/model
3. Use guess-and-check
4. Make a systematic list
5. Look for patterns
6. Work backwards
7. Use before-after concept
8. Make suppositions
9. Restate the problem in another way
10. Simplify the problem
11. Solve part of the problem
12. Think of a related problem
13. Use equations

    (Heuristics 12 and 13 are not in the primary syllabus.)

Though these heuristics are listed in the syllabus, the use of these heuristics are not fully reflected in Singapore published textbooks (Fan & Zhu, 2000), and according to Lee and Fan (2002), "it is by no means clear how these heuristics should be incorporated into teaching and when" (p. 5). This paper explores these heuristics from a functional point of view in the hope of finding the answer to the following questions.

1. What are appropriate problem solving heuristics?
2. Why, how, and when are they used to solve problems?

Based upon this exploration, we propose a framework for using, learning and teaching mathematical problem solving heuristics.

Heuristics

Heuristic methods, heuristic strategies, or simply heuristics, are rules of thumb for making progress on difficult problems (Polya, 1973). They are general suggestions on strategy that are designed to help when we solve problems (Schoenfeld, 1985). They can be explained as non-rigorous methods of achieving solutions to problems, For Bruner (1960) they are methods and strategies that can be helpful in problem solving. In sum, heuristics are ideas that have been useful in previous problem solving that we might want to apply when we solve our current problem.

Not all heuristics that can be used to solve problems are taught explicitly by teachers. In fact, sometimes the heuristics that we use are discovered from our own problem solving experience, or are identified when we observe other people solving problems. We can also learn them through examining and studying worked examples in textbooks. Schoenfeld (1985) summarizes these processes as:

> Occasionally the person solves a problem using a technique that was successful earlier, and some thing clicks. … If that method succeeds twice, the individual may use it when faced with another similar problem. In that way a method becomes a strategy. Over a period of years each individual problem solver comes to rely – quite possibly unconsciously – upon those methods that have proven useful for himself or herself. That is the individual develops a personal and idiosyncratic collection of problem solving strategies.(p. 70-71)
>
> By means of introspection (Polya's method) or by making systematic observations of experts solving large number of problems, it might be possible to identify and characterize the heuristic strategies that are used by expert problem solvers. (p. 71)

The idea of teaching heuristics explicitly is so that we can expedite the processes of discovering and identifying heuristics, and apply them in problem solving. Whether it is possible to simply expect the discovery and identification of processes, rather than explicitly teaching the heuristics, is questionable, it definitely requires more time and effort and is a less efficient approach to learning heuristics.

Heuristics versus algorithms

Heuristics are not algorithms. In this section, we examine some common characteristics of heuristics by looking at the differences between heuristics and algorithms. The arguments presented here are based heavily on Landa's (1976) work.

One main caution in teaching heuristics explicitly is that they are often taught as rigorous procedures for students to follow when solving procedural problems. As Polya (1973) wrote

> Heuristic reasoning is good in itself. What is bad is to mix up heuristic reasoning with rigorous proof. What is worse is to sell heuristic reasoning for rigorous proof. (p.113)

The most recognizable and immediate difference between heuristics and algorithms is that heuristics do not guarantee solution, while algorithms, if the procedures are followed exactly, guarantee a solution. A problem, by definition, does not come with known algorithms that can be used to solve it, therefore heuristics are used instead. Even if there do exist algorithms, heuristics usually enable us to find a solution with less time and effort than algorithms.

Another, and perhaps the more important, difference between the two is that algorithms come with very specific procedures, while heuristics only have non-rigorous suggestions about what we should do. Not all heuristics are the same in terms of specificity. Some heuristics, such as "make a systematic list", do come with more instructions and procedures than others, but these procedures still require decision making on the part of the student. In general, the more general a heuristic is, the less specific its procedures, and the more descriptive it tends to be. Contrastingly, the more specific a heuristic is, the more prescriptive it tends to be. Landa (1976) categorized strategies according to specificity into four categories: *algorithm*, *semi-algorithm*, *semi-heuristic*, and *heuristic*.

A very natural consequence of specificity is its applicability. Algorithms are usually applicable to a certain type of question in a specific topic area, while heuristics are generally applicable through types of question and topic areas. This raises another important characteristic of heuristics, that of *transferability*. The most important reason to learn heuristics is that because they can help solve problems in unfamiliar topic areas. Ultimately, what are truly transferable are the ideas about how we can identify and approach a problem and not merely employ the procedures. Thus the grasp of the idea is important for learning the heuristic, not the procedures or the name of the heuristic.

It should be emphasized that heuristics are only scaffolds that help to solve problems, and no heuristic can replace the role of subject matter knowledge, concepts and skills in problem solving. Even without the help of heuristics, it is still possible to solve problems; heuristics only increase the chance of finding the correct solution.

The role of heuristics in problem solving

Based on the characteristics of the thirteen heuristics identified in the syllabus, and the role each plays in problem solving, heuristics can be classified into four general categories, “representation heuristics”, “simplification heuristics”, “pathway heuristics” and “generic heuristics”.

Representation heuristics

One common mistake in problem solving is to start solving the problem without first understanding it (Eggen & Kauchack, 2003; Rubinstein, 1986). Representing the problem in various ways allows us to have a fuller understanding of the problem that will normally help us tackle the problem better. “Representation heuristics” are heuristics that guide the representation of problems in different forms. Among the thirteen listed, “act it out”, “use a diagram/model” and “use an equation” are “representation heuristics”, correspond to Bruner’s (1967) three forms of knowledge representation: *enactive*, *iconic*, and *symbolic representation*.

Representation plays an essential role in problem solving. In fact, mathematics to a big extent is all about these different forms of representation (mainly symbolic and some iconic representations) and ways in which we can manipulate them. Bruner (1967) pointed out two factors determining the use of representations.

Here economy refers to “the amount of information that must be kept in mind and processed to achieve comprehension” (Bruner, 1967, p. 45). For example, it is more economical to represent ten apples using 10a than to draw ten apples to represent them. From a problem solving point of view, an economical representation requires less working memory, allowing a focus on solving the problem. Bruner (1967) refers the power of a representation as “its capacity, in the hands of a leaner (problem solver in our case), to connect matters that, on the surface, seem quite separate” (p. 48). In general, it is more powerful to represent the direction to a certain place using a map than using a host of verbal or written instructions and descriptions about how to get there, though the two representational forms hold the similar amounts of information. Larkin and Simon (cited in Stylianou, 2004) have pointed out, spatial relations are more visually explicit. Usually powerful representations are generally more economical as well; however economical representations are not necessarily powerful (Bruner, 1967).

As Bruner (1967) pointed out, “modes (of representation), economy, and power vary in relation to different ages, to different “styles” among learners, and to different subject matter” (p.46). In problem solving, other than age, style and subject matter, the problem types and problem solver’s grasp of the problem also play important parts in the use of representations. Therefore when solving a problem, the encouragement of the student to view the problem using different representations becomes important, as it might help them gain a new perspective and see relationships not be seen otherwise.

Representations should be used together cooperatively to solve problems, each complementing the other. However the abilities to handle these different representations are in themselves skills and concepts that must be learned through years of practicing and experience. After all, mathematics, to a large extent, is a study of representations, their manipulations, their applications and more importantly, how to switch between them. Thus subject matter knowledge in the use of representations can contribute to problem solving as a whole.

One interesting observation is that the more abstract a representation, the more precise and objective are the mathematical rules and regulations it comes with. The more concrete the representation; the more subjective and open for interpretation it becomes.

Act it out

The heuristic “act it out” guides the learner towards enactive representations. This involves representing data with physical objects, manipulatives, and attaches conditions to actions. For example, four apples can be represented using four paper cuttings with apple shape, and eating half the apple can be represented by cutting and throwing away half the paper apple. Another more commonly known example of “act it out” would be young children doing arithmetic with their fingers.

Compared to iconic and symbolic representations, enactive representations seldom come with a strict set of concepts and rules, unless we are talking about performance such as dance. The implication is that the use of enactive representations will be difficult to incorporate into formal mathematics textbooks, but they might be effective in teaching and solving mathematical problems, especially for young children. Enactive representations make mathematical concepts and problems more concrete and physical. the approach can be useful when dealing with a problem that is dynamic, for example a problem that involves multiple moving objects or changing variables. However, enactive representations may require more time, resources and effort, which might not be always available when solving the problem individually.

As an adult, it is relatively easy for us to produce “moving images” in our heads without the aid of physical objects when dealing with a simple problem. Therefore, it reduces the need for creating an enactive representation. On the other hand, children may not have the same mental capacity as adults’, and the presence of an enactive representation might help them to solve the problem. However, further empirical studies are needed to better understand the effect of enactive representations on students’ problem solving.

Use a diagram/model

The heuristic “use a diagram/model” directs us to look at our problem visually. This involves transforming problems into visual representations, and representing data with diagrams or models. Iconic or visual representations come in various forms. There are those that have strict rules on how they should be drawn and standard ways of interpreting them, plot of an equation, statistical diagram such as histogram, maps etc. There are also those that have more flexibility in their interpretation, presentation, flow chart, tree diagram, roughly sketched map for direction. Finally, there are those that do not come with any rule and regulation, such as representing twelve apples with four boxes. Depending on which “level of precision” our representation is at and the problem types we are dealing with, they can be used directly to solve the problem or merely be used as visual support to better understand the problem from different perspectives.

Stylianou’s (2004) study on experts’ (with advanced degrees PhDs) and novices’ (undergraduate students) differences in using visual representations in problem solving suggested two functions of visual representations:

1. Visual representations as exploratory tools, (p. 369)
2. Visual representation within segment of the problem solving process. (p. 377)

These support the general argument, but it is interested to explore how upper primary and lower secondary students use iconic/visual representation in solving problems. An empirical research on this topic will need to be conducted to find out more about the use iconic/visual representation in solving problems among students.

Different people may interpret the same visual representation in various ways; students do not necessarily “see” what their teachers can “see”.

> In sum, many times our perceptions are conceptually driven, and seeing the unseen in this is not just producing/interpreting a “display that reveals” or a tool with which we can think, as in many examples above. Seen the unseen may refer, as in the examples above, to the development and use of an intervening conceptual structure which enables us to see through the same visual display, things similar to those seen by an expert. Moreover, it also implies the competence to disentangle contexts in which similar objects can mean very different things, even to the same expert. (Arcavi, 2003. p. 234).

Use equations

Equations, being a symbolic representation, are one of the key concepts in modern mathematics. The strength of equations is in their economy, the ability to represent relationships among elements with just a few symbols. By simplifying a complex problem into equations, solving the problem becomes a matter of solving the equations. For example, the problem “find two quantities whose sum is 78 and whose product is 1296” (Polya, 1973, p. 175), is just a matter of solving the simultaneous equations $x + y = 78$ and $xy = 1296$.

Solving problems using equations involves two skills: setting up and solving equations. In fact, most of the time, the generation of the equations can be a problem in itself. Polya (1973) explained the process of setting up equations as “translation from one language into another” (p. 174), and “the difficulties which we may have in setting up equations are difficulties of translation” (p. 174). When students have difficulties in using equations, it might be as much a language (normal language, verbal and written) problem, as a mathematical one. The same concern also applies to enactive and iconic representations, but it is more evident in this case. Mathematical "language", like normal language, is precise and definite so that the slightest "translation error" could lead to a totally different meaning, hence a different outcome. Solving the equations might also be a difficult task to achieve, since not all equations can be solved easily, and solving some of these equations might involve other heuristics. For example, Keiran (2004) suggested three heuristics activities in algebra: *generational activities*, *transformational (rule based) activities*, and *mathematical activities (global, meta-level)*.

Simplification heuristics

Rubinstein (1986) made the following statement,

> The complexity of a problem often be traced to the number of relationships between the elements. If we can remove relationships that are of minor importance, we may reduce a complex problem to a number of independent simpler problems. (p. 53)

Not all information given in the problems is important or necessary to find solutions. Dealing with those details might not help us in solving them. Hence, it is important to identify the necessary and sufficient information from the problem statements. "Simplification heuristics" are heuristics that guide in making this decision. The heuristics may help us deal with problems' complexities and may help us focus on the important relationships and connections. They may increase the chance of solving the problem successfully. However, in order to start simplifying the problems, the problem must be fully understood.

"Simplification heuristics" are often problem-specific. One useful heuristic may not be helpful in solving other types of problems. Among the thirteen heuristics discussed in this paper, "restate the problem in another way", "make supposition", "solve part of the problem", and "look for pattern" might be identified as "simplification heuristics". Heuristic "simplify the problem" is viewed as synonymous with "simplification heuristics", which is a general term of the four heuristics listed here.

Restate the problem in another way

"Restate the problem in another way" tries to view problems from different angles and perspectives. Quite often by looking at problems differently, it is possible to find a solution or better and easier ways to solve the problem. As Polya (1973) suggested, "varying the problem, we bring in new points, and so we create new contacts, new possibilities of contacting elements relevant to our problem" (p. 210).

Make suppositions

The heuristic "make suppositions" involves adding in or removing elements to or from the problem in order to see how the other elements are affected by it. It can help to find the relationships among the elements, and expose the differences between essential relationships and the remaining components. Adding or removing elements might make the problems be easier to understand. It is also a useful heuristic when the data and conditions given in the problem are not enough to solve the problem. Heuristic "make suppositions" may help solve open-ended problems and construct proofs.

Look for patterns

Pattern is one of the most essential concepts of mathematics. In fact, many view mathematics as science of patterns. One main goal of mathematics is to be able to explain the world by identifying patterns making it easier to explain and predict. Each problem comes with elements that interact with each other, and the change of one will affect the others. With the heuristic "look for patterns" the emphasis is to explore the general relationships among the elements in our problem. Once we found the patterns that describe these elements, we can then use them to solve our problem. The procedures involved in "look for patterns" are not easy to describe. To a large extent, it is a matter of perception and observation. However, by arranging our data in particular ways, such as numbers in the form of a table, or diagrams side by side, we have better chance to find the underlying patterns. When we have found a pattern, we have done more than just solve our problem. Potentially a tool has been uncovered which might be applied to solve other problems under the same conditions as the first problem.

Solve part of the problem

Usually, albeit is not possible to find the unknowns directly from the data and conditions given. In this situation, new unknowns need to be created as mediators of our existing data and conditions. The heuristic "solve part of the problem" breaks down problem into successive sub-problems that might be easier to solve.

Depending on the problems' conditions, it might be an efficient strategy to simplify them and make them easier to solve. The general steps in "solving part of the problem" are essentially breaking down the problem into few parts and defining what the sub-goals might be. More than one sub-goal might be required to construct the solution, and the time and effort required for solving the problem through each of the sub-goals might vary

Pathway heuristics

Pathway heuristics refer to the way the solution moves from the initial state of to the required outcome state. Normally one would start from the initial state and work towards to the required state, which is termed "working forwards". However, this is not always the simplest approach with which to solve a problem. "Pathway heuristics" are essentially the task of choosing suitable and effective paths to solve the problem. They may be combined with another heuristic such as "solve part of the problem" where the initial and required state may be replaced by the current state of our problem.

Work backwards

The heuristic "work backwards" points to starting to solving the problem from the required state, and then work backward to the initial state. It involves the process of reversing conditions given. However, it is important to note that not all processes are two-way processes; in other words, getting from A to B does not necessary means that we can get from B to A. "Work backwards" is often used to solve two types of problems, problems where a required state is given and we are required to find the current state, or problems that require us to find paths reaching from one set of data to another, such as proofs.

Use before-after concept

The heuristic "use before-after" suggests the search for a solution should begin at both the initial state and the required state, with the goal of meeting somewhere in the middle. It can be viewed as a combination of working forwards and working backwards.

Generic heuristics

"Generic heuristics" are heuristics that do not suggest an immediate solution strategy with which to solve a problem, instead they suggest where a solution approach might be "found". It may be the final solution itself or a process for solving a problem like a heuristic, which might be "modified" it to fit the problem. In other words, we can view "generic heuristics" as heuristics for choosing heuristics. They may help solve the problem directly, or help in finding suitable representations, simplifications and pathways to solve the problem, as "a certain formerly solved problem influences our conception of the present problem" (Polya, 1973, p. 111). They are effective, though not necessarily efficient, and frequently used way of approaching a problem.

"Generic heuristics" are general rules-of-thumb that can be used almost across many topics and many types of problem, and are not necessarily limited to the mathematical domain. These heuristics are often learned without the explicit teaching, they may to generalizations from experience.

Think of a related problem

"Think of a related problem" involves relating our current problem with problems that have been solved before, in order to find a suitable way to solve the problems. Polya (1973) suggested that "a great discovery solves a great problem but there is a grain of discovery in the solution of any problem". The procedures taken to solve the related problem may not exactly be the same as those required to solve the current problem, but by comparing the two problems and looking at how the related problem was solved, it might be possible to initiate new ideas and perceptions to solve the current problem. The process of heuristic "think of a related problem" can be divided into two major phases:

1. finding a related problem,
2. making connections between our current problem with the related problem.

The challenge in the application of the heuristic "think of a related problem" is that students usually have difficulties in finding a suitable problem to relate. It is also possible to relate the current problem to an inappropriate problem that leads the solver in a wrong direction (Eggen, 2003, p. 322). Secondly, although it might be possible to find a related problem, students might not know how to make connections between the two. One possible way to overcome this difficulty might be the provision of a related problem and guiding the solver to make connections between the two problems.

Use guess-and-check

The heuristic "use guess-and-check", sometimes called "trial and error", is one of the first heuristics applied when solving an unfamiliar problem (Eggen, 2003, p. 322). Its mechanism can be divided into three phases:

1. make an initial guess,
2. check and verify the guess, and
3. make a next guess.

Phases two and three iterate until a solution is found. The choice of the initial guess and next guess is usually the key challenge for the heuristic "use guess-and-check", while the checking phase is usually predetermined by the problem.

There are various ways of using this heuristic. For example, random guesses might be made which hopefully find the solution by luck. A random initial guess might be made and then next guess might be adjusted according to how well the solution fits the problem. An alternative would be to make an "educated" or "plausible" initial guess, based on past experience in dealing with similar problems or problems in the same field and topic area, and the solution might then "fall out" from the first guess. However, it is also possible to use this to test how the elements in the problem react to each guess and how it is adjusted to gain new perspectives and an understanding of the problem.

Make a systematic list

Heuristic "make a systematic list" might be viewed as a special case of heuristic "use guess-and-check". It is similar to heuristic "use guess-and-check" except that its decision on how to make the next "guess" is made before starting to search for a solution. All these "guessed" solutions are generated in a systematic ways, for example by an increment of one, from the "initial guess", which is usually given in the problem, until reaching a solution. "Make a systematic list" is the algorithmic heuristic among the thirteen heuristics.

The caution in using this heuristic is that the solution being sought may fall between the "cracks" of our systematically generated "guessed" solutions. The solution might also lie on the "other side" of the "initial guess", for example, instead of increments, the solution might require decrements. Therefore, as in the case of "use guess-and-check", there is a need to monitor "guessed" solutions and make changes as the solution proceeds. In fact, how the systematic generation of solutions is approached becomes the key to finding the solution.

By combining this heuristic with the computing power of computers, it can become a powerful and easy way to solve a problem. However if no computer is available, the heuristic "make a systematic list" will probably be best used together with heuristic "guess-and-check". In using this heuristic, there is high chance for students might make careless mistakes, since it involves many calculations and tracking of each answer.

Toward a model for problem solving in Mathematics

From the above discussion, problem solving in mathematics may be viewed as a process that involves the following processes.

1. Finding representations to represent our problem, based upon the economy and power of the representations (representation heuristics).
2. Finding ways to simplify our problem so that we can focus the more essential elements in the problem (simplification heuristics).
3. Finding suitable and simplest paths to approach to our problems (pathway heuristics).

Or we can find previously known or guessed solution to either solve our problem, or help us choose our representations, simplifications, and pathways (generic heuristics).

Unlike Polya's four phases model - *understanding*, *planning*, *executing*, and *reflecting*, (Polya, 1973) which describes the problem solving process as four phases, this exploration has taken a functional point of view, and examined three parallel components for solving mathematical problems. Thus solving mathematical problems can be viewed as a matter of finding the right representations, simplifications and pathways. However representation, simplification and pathway are not independent. Often, though not always, the representations that we might choose will determine the simplifications and pathways that can be chosen, and vice versa. This model suggests aspects of problem solving we can seek to better understand the problem, and devise a plan to solve it. The model simplifies the thirteen heuristics suggested in the syllabus into four ideas of representation, simplification, pathways, and relating, which might be more accessible for lower primary and upper secondary students to grasp. The model also provides a structure on which students can expand their personal heuristics, making learning and discovering new heuristics easier and more meaningful.

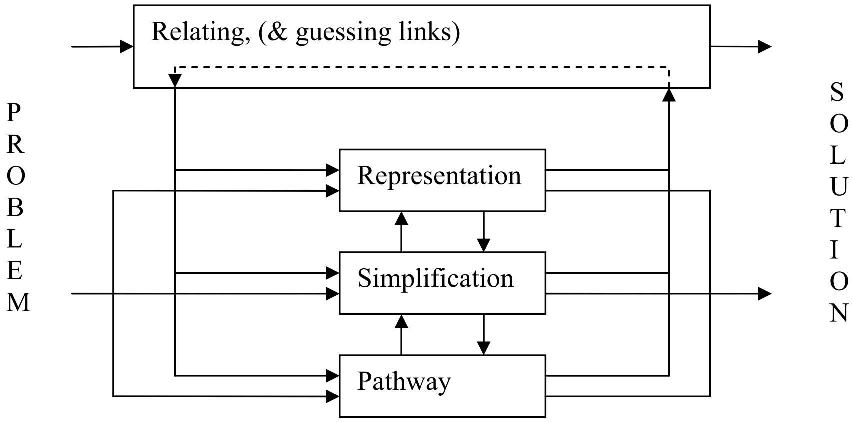

Figure 1: Model for problem solving in mathematics

The model provides a conceptual basis for further empirical research to support the arguments presented. Hopefully, this exploration has established the understanding about heuristics required for these research studies to proceed. However care needs to be taken when we examine heuristics from an empirical point of view. In conducting empirical studies on heuristics, it is difficult to define what it means by "learning heuristics", and "learned heuristics" or grasping the ideas. Students' understanding of the ideas of heuristics can be very subjective, and they are open to various interpretation. It is also important to ascertain that the student holds the ideas, or whether they are merely following procedures or imitating what their teacher is doing. Distinguishing between whether they are really solving problems or simply completing an exercise remains elusive.

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