UNDERSTANDING IN MATHEMATICS

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The aim of many teachers, even if not stated explicitly, is to enable their pupils to achieve *understanding*. The conscientious teacher strives to ensure that the pupils understand what she is saying, to understand the grammatical structure of particular language lessons, to understand the experiments and the laws associated with a scientific investigation, and to unravel mathematical problems with an understanding of the processes and procedures involved. What do educators mean, however, when they talk about *understanding*? The Nuffield Research Team in England quoted the ancient Chinese proverb in their guides for teachers and tried to use its implication in their recommendations for learning mathematics. The proverb purports to suggest that,

I hear and I forget

I see and I remember

I do and I understand.

In a mathematical context, how does the teacher really know when a child understands? How can "understanding" be measured or appraised? Indeed, how may "mathematical understanding" be defined in order to evaluate its existence?

The Oxford Dictionary explains that "to understand" is "to know the meaning or the nature of . . ." So, for example, "We may know Mandarin," or "We may understand the problem."

To reach "an understanding" suggests that something has importance, or has a purpose or a meaning, and to know what is "meant or intended" is to have in mind something which results from experience. One is also "informed" because one has "learned" something. In order, therefore, to examine "understanding in mathematics", it is necessary to appreciate something of the nature of mathematics itself. Lucienne Félix in her book on *Modern Mathematics and the Teacher* (p. 3), suggests that "mathematics is not just a ready-made succession of simple ideas, easily expressed in a few words. It is a human activity which arises from experience, is created in the mind and demands a special form of language."

R R Skemp in his introduction to *The Psychology of Learning Mathematics* warns against an exclusive presentation of mathematics as simply a logical development. This approach is laudable in that it aims to show that mathematics is sensible and not arbitrary, but is restrictive in that it confuses the logical and psychological approaches – "The main purpose of a logical approach is to convince doubters; that of a psychological one is to bring understanding." *(The Psychology of Learning Mathematics,* p. 13). Problems of learning and teaching are psychological problems and before we can make improvements in the teaching of mathematics we need to know more about how it is learnt. If possible we need to know more about how children come to understand mathematics. Most people think they know when they understand something or not, and most have a deep-rooted belief that "understanding" matters. Skemp presses the point, however, when he asks "But just what happens when we understand, which does not happen when we don't. Most of us have no idea! Sometimes we think we have understood something, only to find afterwards that we did not. So, until we have a better understanding of understanding itself, we shall be in a poorer position either to understand mathematics ourselves, or to help other people to do so," (p. 14).

One of the main problems facing investigators who attempt to find out about different ideas held by children, is the way in which the information is obtained. Ultimately, this problem is one of communication. Hughes and Rogers in *Conceptual Powers of Children: an Approach through Mathematics and Science* states (on p. 3), "In other words, seeking evidence of understanding (an internal realization) using words – questions and answers – (an external expression) – is not a particularly reliable method." However, research workers have attempted to eliminate verbal factors by carefully designing the tasks used to test children's understanding, but in the main these have not been successful, as the same difficulties emerge whatever form of communication is used between the questioner and the child.

The reader will appreciate that the topic of understanding is not as simple as first considerations might suggest. A review of the contents of most education texts and psychology manuals will show that the topic is very rarely mentioned. Skemp is one of the few educators to have dealt with it at length and who has placed it in a conceptual framework in his recent text, *Intelligence, Learning and Action: A Foundation for Theory and Practice in Education.*

One way of tackling this problem of "understanding in mathematics" is to look at it from a pedagogical perspective, especially when the teacher is faced with the task of teaching a topic for the first time. Consider the need to teach the operation of addition to young children. "What is really meant by addition?" "Can I say simply to myself what I really think it means, so that what I prepare and plan for the children will enable them to understand the fundamental ideas behind *addition*?" Someone at some time must have been the first person to recognise the "addition" activity and gave it a word label. Why was the label so chosen? Does it convey the real meaning today? More especially, does the English label "to add" convey meaning to a child who is much more familiar with a different mother tongue?

When it is realised that "addition" comes from the Latin word "addere" which means "to put together" we can perhaps more readily understand that the basic idea behind the operation of "adding" was also embodied in the label "addere" and this would much more readily convey the real meaning to the Roman boy or girl. A teacher introducing the idea of "addition" and understanding its real meaning can provide an appropriate mathematical experience for her pupils. For example, she could provide a collection of four toys and another collection of three toys and actually "put these together" to form a single collection of seven toys. A situation is therefore created in which the child operates with understanding. Lucienne Félix states on p. 2 of *Modern Mathematics and the Teacher,* "Nothing can be defined from nothing. What we must do is to begin by describing a mathematical situation." This situation is not presented in words from the beginning. An adequate verbal description is itself most difficult to make and definitions can only be formulated after much hard work. However, if the teacher can resolve in her own preliminary preparation what the meaning is behind a particular concept she may then provide suitable activities for her pupils. This situation allows the children to describe a real activity and therefore the language which is used, is applied in a meaningful way, i.e. in a way which will permit discussion and understanding.

At this point, it is appropriate to consider what happens in the human process of "thinking". In the school situation the teacher is continually encouraging the pupils "to think". What does this involve? Robert Thomson in *The Psychology of Thinking* reminds us that Aristotle selected rationality, the capacity "to think", as the defining attribute of man. Descartes sought to distinguish mind from matter characterizing the former as "that which thinks". But what is thinking? This might seem a pointless question, since everyone knows by acquaintance what thinking is from his own first-hand experience of doing it. However, very few people think about thinking. It is one thing to practise an activity and quite another to stand back and try to observe, describe and account for that activity.

Much thinking may be labelled "autistic" that is "day dreaming" and fantasy. This activity though leads to imaginative expression. In a different sense we use the phrase, "I am trying to think when I last used my bus pass." The idea here of course is "remembering" or "recall". Between uncontrolled autistic thought and recall is "imagination," which is closely allied with reasoning. Sometimes teachers use the phrase "Think what you are doing" meaning to heed or pay attention to. Professor Ryle has pointed out that "thinking what one is doing" when engaged on a practical task is like adopting a frame of mind as in mountaineering.

When a man tells us in conversation what "he thinks of a football team", he is expressing his opinion or belief. This leads us to the concept of thinking in the sense of "reasoning", "reflecting" or "pondering". This idea, although not precise, enables us to see that "thinking" is what Ryle calls a polymorphous term like "farming". Many activities are involved. Sowing, reaping, ploughing, etc.

Our attempt as teachers is to get pupils to think more effectively. Therefore, if we provide real mathematical situations upon which to reflect and reason, there is a greater likelihood of more understanding taking place. Reflection on elements of a situation will lead to a greater understanding of the ideas involved. The fundamental underlying ideas may be referred to as *concepts.* Skemp considers that a thorough grasp of concepts is nowhere more important than in mathematics. The higher order concepts in this subject are particularly abstract and there is a possibility that in many instances children may learn the words without understanding the underlying concepts.

We can tell someone a fact and they can use it without necessarily understanding it, i.e. "The capital of Zambia is Lusaka." This fact may simply be learned as a fact. Skemp contrasts the understanding of a concept with the learning of a fact with this example: if I say, "The density of iron is greater than that of water" it will not imply that the hearer will be able to use it, or even to understand it. Being able to use the idea in this instance depends on being able to understand it. Understanding the concepts behind this principle permitted men to make iron ships which would not sink.

How may the understanding of the concept be conveyed to someone, if we cannot do so simply by telling them?

Imagine a man blind from birth, who by a miracle of modern surgery is given his sight in adult life. He then enters a further field of experience for the first time. He may ask, "What does *red* mean?"

It would convey little to reply, "*Red* is the colour we experience from light of wave lengths in the region of 6,500 Angstrom units." This is an accurate definition, which he could learn by heart and use for an answer to an examination question. He would get full marks, but would this indicate that the man who was previously blind now understood what was really meant by "red"? An effective method for conveying the idea of red would be to point to a number of real objects saying "This is a red dress", "This is red book", etc, so that the real experience with suitable meaningful verbal expressions (to use Ausubel's phrase) would impart the basic idea (concept) of red.

Reinforcement could be achieved by a further series of experiences, "exemplars" of the concept. Emphasis could be achieved by showing non-examples, i.e. "This is a yellow book", "This is a green leaf." (non-exemplars). How may we test for "understanding"? By showing him various objects and asking if he can identify the red ones. This kind of enquiry tests whether or not someone has grasped the meaning of a concept. It is whether or not he can use it correctly (i.e. with understanding) not whether he can give a verbal definition.

Not all concepts can be communicated in this way. If the person is now asked the question "What is colour?" we cannot convey this only by pointing to various coloured objects. We could say "Red, blue, green . . .", these are colours. The examples from which the concept of colour is formed are themselves concepts.

Skemp introduces this distinction between the kind of concepts formed by sensory experience of the "outside world, such as red" and he calls these *primary concepts,* and those formed from other concepts (i.e. colour) which he calls *secondary concepts.* These may be extended to form a hierarchy of concepts.

Mathematics is a complex hierarchy of concepts and has to be communicated in a different way from simply telling *facts* if it is to be fully *understood*. Failure on the part of teachers to appreciate this may lead to intelligent children and adults saying "I cannot understand mathematics".

To ensure an understanding of mathematical ideas,

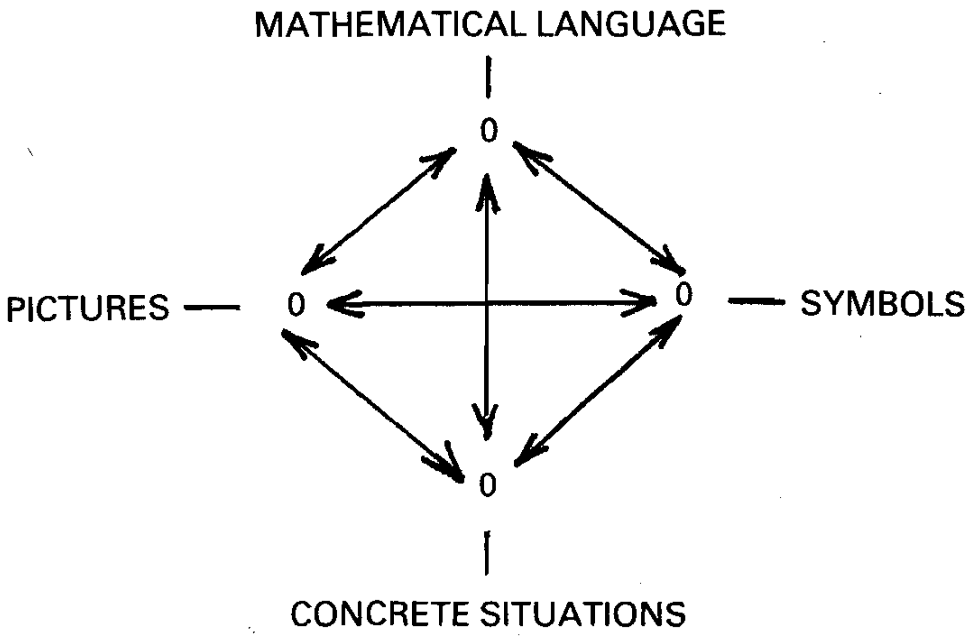
i) We must make sure that all contributory lower order concepts are fully understood.

ii) New concepts are formed in the minds of pupils by building carefully on the primary concepts and choosing suitable situations and examples.

A recent proposal for a model with which to consider understanding in mathematics is that by Derek W. Haylock in the 98th edition of *Mathematics Teaching* (March 1982, pp. 54–56). He suggests that to understand something means to make *(cognitive)* connections. Piaget and Skemp would regard this as mapping a situation into an existing mental scheme.

Haylock suggests that the more connections the learner can make between the new experience and previous experiences, the greater, and consequently, the more useful the understanding. Sometimes, the new experience enables the learner to connect previously unconnected experiences; when this occurs a spectacular advance in understanding can be achieved. If understanding means making connections, then in relation to any mathematical concept or principle there must be a wide spectrum of different levels or degrees of understanding.

Haylock continues, "One way of identifying some important connections in mathematics is to consider the four components of mathematical experience which a pupil encounters in doing mathematics: concrete situations, pictures, words, symbols. A pupil can demonstrate some degree of understanding by showing that he can make a suitable connection between two of these categories of experience. Any one of the twelve arrows in the diagram may suggest a means of assessing an aspect of understanding a given mathematical idea."



JANE PUT OUT SOME COUNTERS LIKE THIS.

0 0 0 0

0 0 0 0

0 0 0 0

Try to think of as many mathematical statements as you can for this situation, e.g. $6 + 6 = 12$, $4 \times 3 = 12$.

This was given recently to 250 eleven and twelve year old English children.

88% included multiplication examples.

e.g. $3 \times 4 = 12$, $6 \times 2 = 12$, etc.

42% only used division.

e.g. $12 \div 4 = 3$, $12 \div 2 = 6$, etc.

74% provided addition examples.

e.g. $3 + 9 = 12$, $4 + 8 = 12$, etc.

36% only made the connection with subtractions.

e.g. $12 - 9 = 3$, $12 - 8 = 4$, etc.

Two girls gave the following responses:–

Tessa

12 − 2 = 6	1/12 of 12 = 1
3 x 4 = 12	1.5 x 8 = 12
6 x 2 = 12	8 x 1.5 = 12
2 x 6 = 12	2/6 of 12 = 4
4 x 3 = 12	3/4 of 12 = 9
1 x 12 = 12	12 − 1 = 11
12 x 1 = 12	12 − 0 = 12
12 − 6 = 2	1 + 11 = 12
2 + 10 = 12	10 + 2 = 12
3 + 9 = 12	11 + 1 = 12
4 + 8 = 12	12 + 0 = 12
5 + 7 = 12	12 − 6 = 6
6 + 6 = 12	12 − 11 = 1
7 + 5 = 12	12 − 10 = 2
8 + 4 = 12	12 − 9 = 3
9 + 3 = 12	12 − 8 = 4

and so on.

Rachel

2 x 6 = 12	9 + 3 = 12
6 x 2 = 12	2 + 10 = 12
4 x 3 = 12	10 + 2 = 12
3 x 4 = 12	4 + 8 = 12
12 − 6 = 6	8 + 4 = 12
3 + 9 = 12	5 + 7 = 12
7 + 5 = 12	12 − 7 = 5
12 − 8 = 4	12 − 10 = 2
12 − 4 = 8	12 − 2 = 10
12 − 9 = 3	12 − 1 = 11
12 − 3 = 9	12 − 11 = 1
12 − 5 = 7	11 + 1 = 12

Comparisons can be made between the two sets of responses in terms of the number of connections which each of the two girls has made.

We thus have a model which I would suggest gives a framework in which the extent of a pupil's understanding and the depth of mathematical insight might be considered.

Tessa's inclusion of 1/12 of 12 = 1 and 1.5 x 8 = 12 illustrates an extension of connections when compared with Rachel's responses.

The assessment of pupil's "understanding in mathematics" is an area which requires monitoring and analysis by teachers and researchers.

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