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## Mathematical Problem Solving for Integrated Programme Students

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### Introduction

Recent changes in the Singapore Education System recognise the need for fresh approaches to realise the vision of *Thinking Schools, Learning Nation*. A broadening of the interpretation of merit and talent has also resulted in greater flexibility and diversity in the system. The introduction of integrated programmes (IP) in selected schools and junior colleges is one such recent innovation. Students admitted into these programmes are considered highly able in academic terms. Their academic programme is designed to be different from that of the mainstream students. There is scope for these students to be challenged through a curriculum which is not constrained by the O-Level Examination. While some subjects such as the different science subjects or different humanities are integrated in some IP curricula, mathematics is still taught on its own, perhaps due to the very nature of the discipline. Unconstrained by the need to meet the curricular and assessment demands of an O-level Examination, a new challenge arose to confront teachers and schools to design a mathematics curriculum to cater to the academically able students. How may schools meet this challenge? This paper reports on some preliminary findings from an ongoing project which aims to understand better the mathematical development of IP students from Secondary Three to Junior College Year 2 levels. Specifically, the project will examine how students' mathematical thinking and beliefs concerning mathematics develop through such learning experiences and whether the learning is sustainable and applied by the students after the learning programme when they are in junior college (JC). It intends to draw implications for the teaching of problem-solving and mathematical thinking to the students on an IP curriculum.

In line with global trends in mathematics education, mathematical problem solving has been established as the central theme of the Singapore primary and secondary mathematics curriculum since the 1990s and the primary aim of the curriculum is to develop students' ability to solve mathematics problems. Curriculum documents state that problem solving heuristics are to be taught in our primary and secondary school mathematics classes. Singapore teachers were either equipped with the knowledge to teach their students in the use of heuristics as problem solving strategies during their initial teacher training or via the professional development courses (e.g., in-service or school-based workshops). However, the resources used in these courses tended to emphasise the learning of heuristics and neglect to give due coverage to the mathematics content at a deep level or to the kind of mathematical thinking such as conjecturing and proving that are commonly used by mathematicians. Towards upper secondary and junior college levels, students here tend to concentrated on national exam-type mathematics problems. An over concern with doing well in the O-level and A-level examinations, which are high-stake events, often led to the heuristics learned at lower levels to be ignored by both students and possibly their

teachers, instead of being used more in their mathematical problem solving. This situation suggests that there is need for the development of materials and teaching resources for the explicit teaching of problem solving in the environment of deep mathematics content at the higher school levels.

On the research front, the 1980s and 1990s saw much research in mathematical problem solving in primary and secondary mathematics especially in USA, UK and Australia. However, the two decades or more of research on mathematical problem solving in schools is ambivalent in its findings (Schoenfeld, 1992; Lester, 1994). Interest in problem solving and problem posing research has also generated local research but these have largely concentrated on pupils in the primary and lower secondary levels (see for example, Wong & Lim-Teo (2002); Foong (1993)). There is little local research on the problem-solving strategies of *high-ability* students at the upper secondary levels. It is thus of interest and significance to study whether academically strong students can learn the problem solving strategies and develop mathematical thinking through a problem solving approach, and whether such learning will be internalised and used in a mathematical environment at college level and beyond. Research (see Schoenfeld, 1985; 1992) revealed that there is more to the teaching of heuristics if we wish to improve the problem solving performance of our students. Students' attitudes towards mathematics and their beliefs about mathematics, just to name two, are influences at play in determining success in problem solving. It is therefore important to also address the questions of what attitudes and beliefs do these IP students have towards mathematics. In what ways do these attitudes and beliefs influence their mathematical problem solving performance and practices? In what ways does explicit instruction in problem-solving techniques and strategies influence attitudes and beliefs towards mathematics? In this paper, we will present a first look at the attitudes and beliefs of the students and discuss the possible effect of explicit instruction on problem solving on performance in the "traditional" mathematics examination.

## Procedure

### *The study*

[Remark: The project we have embarked upon will track two cohorts of IP students over their four years of study in the IP. We will report on what we have learnt from the first cohort of students.]

We begin with an exploratory study which consists of a series of classroom lessons on problem solving techniques and strategies for one cohort (5 classes) of Secondary 3 students enrolled on the IP offered by a junior college in 2005. These lessons were taught by a member of the research team and the teachers of the JC, using resource materials designed by the research team. This exploratory phase enabled the research team to obtain a preliminary understanding of the challenges the students face in learning mathematical problem solving. The study will track the students through their JC years. At the beginning of their problem solving course, in the first year of study, the students completed two questionnaires to provide base-line information on their beliefs about mathematics and their attitudes towards mathematics.

The learning environment consists of a series of lessons, each of duration 1.5 hours within two school terms. One class (intact) was taught by the same researcher. The rest of the classes were taken by the participating teachers; each to a class. A model of mathematical problem solving is displayed throughout the lesson. Students work in pairs or individually on the specially designed problems, which are given out at the start of the lesson. The materials and teaching approaches emphasise the mathematical thinking processes such as making extensions, conjecturing and methods of proof which are fundamental to the discipline of mathematics. The teacher leads whole-class

discussions on solutions, nature of problem, heuristics and extensions to problems.

At the beginning of the term, two lectures of about an hour and a half each are conducted for all the five classes in the IP. These lectures introduce mathematical problem solving to the students and include Polya's (1954) model (see Fig. 1) and Schoenfeld's (1985) mathematical problem-solving framework which consists of four components, viz. Resources, Heuristics, Control and Beliefs. The researcher-lecturer also models some examples of problem solving. In the middle of the term, with more understanding of how the students are progressing, a third lecture is conducted to highlight key points and strategies of the problem solving process.

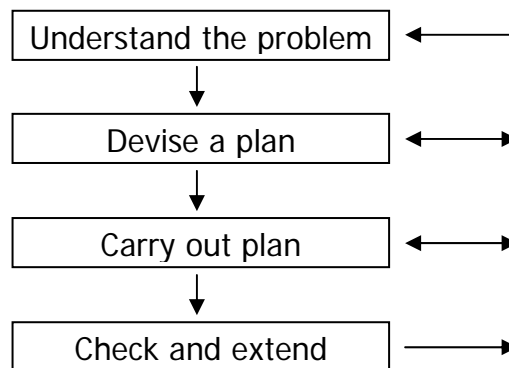


Figure 1: Polya's problem-solving model

At selected lessons, one of the researchers (not the teacher) observes or interacts with the students as they solve the problems. Some of these interactions will be audio-taped, with students' consent. Teachers will also be asked to audio-tape interesting discussions as and when they arise. Some students will be selected and their problem-solving work and discussions will be video-taped to track their development through the study.

Questionnaires will be administered at selected points in their 4-year programme to learn about the development of their attitudes towards mathematics and beliefs about mathematics. A year apart was chosen for each administration of the questionnaire, as it is recognised that attitudes and beliefs are relatively stable and changes are likely to take time. At this time of reporting, the students have completed two questionnaires. The same questionnaire was used to improve test-retest reliability. We also take it as a reasonable assumption to make that the students are unlikely to be able to remember their response to a questionnaire item which they gave a year ago.

### *Participants*

A Junior College which runs an Integrated Programme collaborated in the project. All students (about 130 per cohort) in the Junior College's five Integrated Programme classes took part in the project. An entry requirement to the IP is high academic ability. The average class size is 25. All the mathematics teachers of these classes participated in carrying out the problem-solving sessions. They were also involved in the data collection pertaining to student development for the duration that they were in the project.

### *Data Collection*

For this paper, we will present our findings and discussion of preliminary analyses of data from the following sources:

- Self-administered questionnaires on students' beliefs and attitudes towards mathematics and problem solving.
- Classroom observations and field notes of problem-solving lessons and students as they engage in solving problems.
- Final Examination results.

## Findings and Discussions

### *Quantitative analyses on 'Attitude towards mathematical problem solving' and 'Beliefs about mathematics'*

We adapted the items from a questionnaire (source unknown), replacing “test item” with “maths problem”) to get an idea of how the IP students approach and manage problem solving. (For the purpose of this exploratory study we label it Problem Solving Questionnaire (PS).) The questionnaire was administered at the beginning of the first year (PS pre) and the same questionnaire at the beginning of the second year (PS post). The questionnaire consists of 20 statements and the respondent has to write down a number, ranging from Not at all = 1 to Very much = 6, which best describes how he thinks when solving mathematics problems. The students were informed that there are no right or wrong answers and asked not spend too much time on any one statement. Examples of the questionnaire items are:

- *I attempted to discover the main ideas in the maths problem.*
- *I was aware of which thinking technique or strategy to use and when to use it.*
- *I asked myself how the maths problem is related to what I already knew.*
- *I thought through the meaning of the maths problem before I began to answer it.*
- *I kept track of my progress and, if necessary, I changed my techniques or strategies.*
- *I used multiple thinking techniques or strategies to solve maths problems.*

Although the data obtained is Likert scale and not ratio, the similarity of the statements emphasizing thinking skills over routine and memorization allows us to total the 20 responses as a rough score of the attitude towards mathematical problem solving. A pre- and post-comparisons of the scores on this questionnaire reveals shifts in the students' attitude towards mathematical problem solving. Changes in either direction, favourable to unfavourable, and vice-versa, were found. In one extreme, a student reported a change from a score of 95 to 74 on the PS scale (maximum score = 120; minimum score = 20) in the favourable to unfavourable direction. At the other extreme, a student moved from a PSA score of 46 (unfavourable) to 87 (favourable).

The cohort also completed a questionnaire for a measure of their beliefs towards mathematics at the beginning of the year (B pre) and the same questionnaire at the end of the year (B post). The questionnaire was adapted from the mathematics belief scales developed by Kloosterman and Stage (1992) for use with secondary or college level students, and to which the Fennema-Sherman (1976; cited in Kloosterman & Stage) usefulness of mathematics scale was appended. The items were minimally re-worded to make more suitable for our students (e.g., replacing “word problem” with “maths problem”). The questionnaire consists of 36 statements and the respondent has to write down a number, ranging from Not at all = 1 to Very much = 6, which best describes how his views about mathematics. We used a 6-point Likert scale instead of the original 5-point. The students were reminded that there are no right or wrong answers and asked not spend too much time on any one statement. Examples of the questionnaire items (\* indicates negatively worded items) are:

- *It's not important to understand why a mathematical procedure works as long as it gives a correct answer.\**
- *I can get smarter in maths by trying hard.*
- *I study mathematics because I know how useful it is.*
- *Learning to do maths problems is mostly a matter of memorizing the right steps to follow.\**
- *There are maths problems that just can't be solved by following a predetermined sequence of steps.*
- *If I can't do a maths problem in a few minutes, I probably can't do it at all.\**

Again although the data obtained is Likert scale and not ratio, we have taken the liberty at this preliminary stage to total the 36 responses (with appropriate items reversed) as a rough score of positive beliefs about mathematics. Changes were found. Extreme cases were (a) a “negative” change from a score of 148 to 118 on the belief scale for a student who obtained a score of 59 for the routine problems and 0 for the non-routine problems in the final examination and (b) a “positive” change from a score of 120 to 177 for a student who obtained a score of 20 for the routine problems and 0 for the non-routine problems in the final examination. The final examination had embedded in it the following non-routine problems which accounted for 10% of the marks.

1(a) A man travels a total distance of 130 km by car, from Town A to Town B. For the first 120 km, his average speed is  $x$  km/h but for the last 10 km, his average speed is 40 km/h less than for the earlier part. If the total journey takes 2 hours 30 minutes, write down an equation which  $x$  must satisfy, and show that it reduces to  $x^2 - 92x + 1920 = 0$ . [4]

2(b)(ii) A sphere with diameter 1 cm is enclosed within a cube of side 1 cm. Find the shortest distance between the sphere and a vertex of the cube, expressing your answer in surd form. [2]

3(c) The numbers 1 to 625 are arranged in 25 equal rows of 25 in the following way:

Row 1	1	2	3	...	25
Row 2	26	27	28	...	50
Row 3	51	52	53	...	75
:	:	:	:	:	:
:	:	:	:	:	:
:	:	:	:	:	:
Row 25	601	602	603	...	625

Find the total sum of all numbers with at least 2 digits of “2”. [4]

To what extent were there changes in beliefs within the cohort? The results for paired-samples *t*-tests for change in problem solving attitudes and change in beliefs are shown below.

Table 1: Paired Samples Test

	Mean	S.D.	<i>t</i>	df	Sig. (2-tailed)
PS pre – PS post	-1.82	10.9	-1.591	90	0.115
B pre – B post	2.86	15.5	1.779	92	0.079

The results indicate weak significant differences in both the problem solving attitudes ( $P = 0.115$ ) and beliefs ( $P = 0.079$ ). Our earlier investigation into the extreme values suggest that perhaps more significant changes can be found if we divide the students according to how they fared in the non-routine and routine sections of the final examination, and then look at their beliefs about mathematics and attitudes towards problem solving.

We split the cohort according to the median score of the Routine portion of the final examination (median = 62.5) and the median score of the Non-routine portion (median = 5) obtaining four groups, namely:

- High TR High NR - High Score on Routine Problems and High Score on Non-routine Problems
- High TR Low NR - High Score on Routine Problems and Low Score on Non-routine Problems
- Low TR High NR - Low Score on Routine Problems and High Score on Non-routine Problems
- Low TR Low NR - Low Score on Routine Problems and Low Score on Non-routine Problems

Table 2 shows the ANOVA summary table for this variable, Student Class.

Table 2: ANOVA for Student Class

		df	F	Sig.
B pre	Between Groups	3	1.848	.144
	Within Groups	90		
	Total	93		
B post	Between Groups	3	3.289	.024
	Within Groups	89		
	Total	92		
PS pre	Between Groups	3	2.891	.040
	Within Groups	90		
	Total	93		
PS post	Between Groups	3	4.230	.008
	Within Groups	87		
	Total	90		

ANOVA shows significant differences between the groups for B post, PS pre and PS post. To determine where the differences lie, we conducted a post-hoc Bonferroni analysis. We show only the significant results in Table 3 below.

Table 3

Groups compared	Variable	Mean difference	Sig.
Low TR Low NR - High TR High NR	PS pre	-7.79	.037
Low TR Low NR - High TR High NR	PS post	-8.27	.015

Low TR Low NR - Low TR High NR	B post	-10.98	.083
Low TR Low NR - High TR High NR	B post	-8.36	.074

The group Low TR Low NR showed significantly lower scores in PS pre, PS post and B post than the group High TR High NR indicating less favourable attitudes and beliefs. In a way, this is to be expected because the first group are the lowest achieving students while the latter group are the highest achieving students. Of greater interest are the significantly higher scores in B post of the group Low TR High NR over the group Low TR Low NR. In other words, students that did just as badly in the routine problems but fared much better in the non-routine problems seemed to have more positive beliefs about mathematics by the beginning of the second year. One possible explanation is that the non-routine problems in the final examination did not require, in the terminology of Schoenfeld, many resources (mathematical knowledge and formulas). Positive beliefs about mathematics might then be a factor causing these students to try harder (and thus succeed) with such non-routine problems. Since there were no significant differences in Beliefs between the groups at the beginning of the first year (B pre), it could also be that success in non-routine during the problem solving tutorials of the first year had contributed to the positive shift in beliefs. Interviews with these students may reveal much more.

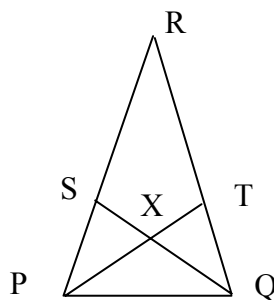
*Utilization of resources and impact of beliefs*

We shall move from the quantitative aspect of our preliminary findings to report qualitatively on the problem solving process of the students observed while in the classroom.

The controlling influence of beliefs is evident in the problem solving efforts of two students. One student, working on a proof that the product of four consecutive numbers is divisible by 4, began by writing down the expression  $(x+1)(x+2)(x+3)(x+4)$ . He was quick to seek another approach – listing chains of four consecutive numbers and checking that their products were divisible by 4. When asked for his reason for abandoning the approach, he said that it would be “too messy” to expand the expression, that a solution in mathematics is unlikely to be messy; and so that it couldn’t be the right approach. His belief (or intuition, if we wish to call it that) prevented him from reaching the solution. When questioned, the student revealed that he was aware that his listing and verification of a number of numerical cases did not constitute an acceptable proof. With his resources apparently depleted, he could only wait for the lecturer to discuss the proof.

In another observation, the resources available to the student enabled him to proceed to solve the following problem.

In the following diagram  $PR = QR = 12$  and  $RS = RT = 8$ . If the area of  $RSXT$  is 8, what is the area of triangle  $PQR$ ?



The student, working silently, started by introducing lines and, having drawn the lines, began labelling the unknown lengths as shown below.

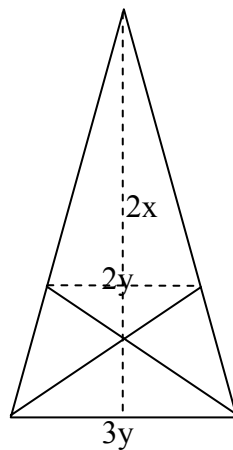


Figure 2

Upon the introduction of  $x$ 's and  $y$ 's, this algebraic tack appeared to “control” the subsequent search for solution. His command of algebraic skills allowed him to work with  $3y$  and  $2y$  and  $2x$  with ease (Fig. 2). Reliance on hunches or the belief that it “has to be like that” provided the confidence to forge ahead. When asked “Why  $3y$ ? How do you know?” he replied, pointing to the diagram in which he had put in the given information, “Because this is 8 to 12 so it will be  $2y$  to  $3y$ .” His facility with ratios was evident in the ease with which he worked from  $8 : 12$  to  $2y : 3y$ . That the ratios apply appeared to be intuitive to him.

He could call on his command of algebraic resources to write out the solution:

$$PQ = PR = 12, \quad PS = RT = 8, \quad RSXT = 8$$

$$\frac{1}{2} (2y)(2x + 2x/5) = y(12x/5) = 8$$

...

...

$$xy = 10/3$$

$$\frac{1}{2} (3y)(3x) = 9xy/2 = (9/2)(10/3) = 15$$

Notice that he was confident and comfortable in leaving an interim answer of  $xy = 10/3$  as it was. It suggests a certain level of mathematical maturity and reflects his beliefs on how mathematics work. The two students held different views about what is “messy” in mathematics. The second student did not seem to see the algebra involved as “messy”. It would have been enlightening if we could ask the first student for his views on the solution of the second student.

### Conclusion

More questions than answers are generated by the preliminary analyses. It appears that beliefs and attitudes did shift for some of the students. What brought this about? Also, the shift was different for different students. Again, what were the precipitating events? Some students did well on the routine problems but not the non-routine problems, and vice-versa. Were these students affected by the explicit instruction in problem solving, and if so, how were they affected? The availability of resources (here, knowledge and skills in algebra for that particular student observed) can be a potent factor in

successful problem solving (although there were gaps in the student's solution and there exists a more efficient way). The beliefs a student hold about mathematics and about mathematical problem solving can also affect decision making during problem solving. Here, the first student's decision to abandon an approach prevented the student from reaching a solution, whereas the second student somehow plodded on, perhaps, because of the confidence he had in his ability.

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